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Mohammed Taha Khalladi* and Abdelkader Rahmani

(ω, c) - Pseudo almost periodic distributions

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Abstract: The paper is a study of the (ω, c) –pseudo almost periodicity in the setting of Sobolev-Schwartz distributions. We introduce the space of (ω, c) –pseudo almost periodic distributions and give their principal properties. Some results about the existence of distributional (ω, c) –pseudo almost periodic solutions of linear differential systems are proposed.

Keywords: (ω, c) –Almost periodic functions, (ω, c) –Pseudo almost periodic functions, (ω, c) –Almost periodic distributions, (ω, c) –Pseudo almost periodic distributions, Linear differential systems.

MSC: Primary 46F05, 46E30, 28A20; Secondary 34C25, 42A75

Dedicated to the Memory of Prof. Constantin Corduneanu

1 Introduction

The theory of uniformly almost periodic functions was introduced and studied by H. Bohr in the beginning of the last century [6], since then, many authors contributed to the development of this theory in different directions, see [5], [8] and [9].

The concept of (ω, c) –almost periodicity introduced in [10] is a generalization of (ω, c) –periodicity which motivated by some known results regarding the qualitative properties of solutions to the Mathieu linear second-order differential equation

$$y''(t) + [a - 2q \cos 2t]y(t) = 0,$$

arising in seasonally forced population dynamics, see [1].

In [14], C. Zhang introduced an extension of the almost periodic functions, the so-called pseudo almost periodic functions. These pseudo almost periodic functions are related to many applications in the theory of differential equations.

The classes of (ω, c) –pseudo almost periodic functions were introduced and studied in [11], by taking into consideration the concept of pseudo ergodicity introduced by C. Zhang [14]. These classes generalize the concept of (ω, c) –pseudo periodicity introduced by Alvarez, Gómez and Pinto in [2].

The theory of Sobolev-Schwartz distributions is a very powerful tool in mathematics and their applications. Almost periodic distributions extending the classical Bohr and Stepanoff almost periodic functions are due to L. Schwartz, see [13]. The paper [12] introduces and investigates the (ω, c) –almost periodicity in the setting of Sobolev-Schwartz distributions.

As mentioned in the abstract, the main aim of this paper is to introduce a new space of (ω, c) –pseudo almost periodic distributions containing (ω, c) –pseudo almost periodic functions as well as (ω, c) –almost periodic Schwartz distributions. The paper is briefly described as follows: In section 2, we recall the basic definitions and results of the concept of (ω, c) –pseudo almost periodic functions. The main results of this

*Corresponding Author: Mohammed Taha Khalladi: Department of Mathematics and Computer Sciences, University of Adrar, Adrar, Algeria, E-mail: ktaha2007@yahoo.fr

Abdelkader Rahmani: Laboratory of Mathematics, Modeling and Applications, University of Adrar, Adrar, Algeria, E-mail: vuralrahmani@gmail.com

work are given in the section 3 and 4. In the first one, we introduce and study the (w, c) -pseudo almost periodicity in the setting of Sobolev-Schwartz distributions, by recalling some new basic spaces of functions and distributions in which we can study this concept of (w, c) -pseudo almost periodicity. The second one is devoted to some interesting applications to linear differential equations, we propose results about the existence of distributional (w, c) -pseudo almost periodic solutions of linear differential systems.

2 (w, c) -Pseudo almost periodic functions

Unless specified otherwise, throughout the paper, we will assume that $c \in \mathbb{C} \setminus \{0\}$, $w > 0$ and we will use the following notations:

$$\varphi_{w,c}(\cdot) = c^{-\frac{(\cdot)}{w}} \varphi(\cdot), \quad \forall \varphi \in \mathcal{C}^\infty \text{ or } L^p, \quad 1 \leq p \leq +\infty, \quad (2.1)$$

and

$$T_{w,c} = c^{-\frac{(\cdot)}{w}} T, \quad \forall T \in \mathcal{D}', \quad (2.2)$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

First, let us recall $(\mathcal{C}_b, \|\cdot\|_{L^\infty})$ the Banach algebra of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\|\cdot\|_{L^\infty}$ of uniform convergence on $\mathbb{R}$. Denote by AP the well-known space of Bohr almost periodic functions on $\mathbb{R}$; is the closed subalgebra of $(\mathcal{C}_b, \|\cdot\|_{L^\infty})$ that contains all the continuous functions $f: \mathbb{R} \rightarrow \mathbb{C}$, satisfying: the following structural property: For any $\varepsilon > 0$, the set of ε -almost periods of f , defined by

$$E\{\varepsilon, f\} = \left\{ \tau \in \mathbb{R} : \sup_{x \in \mathbb{R}} |f(x + \tau) - f(x)| < \varepsilon \right\},$$

is relatively dense in $\mathbb{R}$.

We recall also the space $AP_{w,c}$ of (w, c) -almost periodic functions which has been introduced in [10].

Definition 1. Let $c \in \mathbb{C} \setminus \{0\}$ and $w > 0$. A complex-valued function f defined and continuous on $\mathbb{R}$ is called (w, c) -almost periodic, if and only if, $f_{w,c} \in AP$. Denote by $AP_{w,c}$ the set of all such functions.

When $c = 1$ and $w > 0$ arbitrary, we obtain $AP_{w,c} := AP$. The space $AP_{w,c}$ is a vector space together with the usual operations of addition and pointwise multiplication with scalars. Some properties of (w, c) -almost periodic functions are summarized in the following proposition.

Proposition 1. (i) The space $AP_{w,c}$ equipped with the (w, c) -norm

$$\|f\|_{w,c} := \sup_{t \in \mathbb{R}} |f_{w,c}(t)|,$$

is a Banach space.

(ii) If $f \in AP_{w,c}$, then $\tilde{f}(\cdot) = f(-\cdot) \in AP_{w,1/c}$.

(iii) If $w > 0$, $c \in \mathbb{C} \setminus \{0\}$ such that $|c| = 1$ and if $f \in AP_{w,c}$, such that $\inf_{x \in \mathbb{R}} |f(x)| > 0$, then $1/f \in AP_{w,1/c}$.

(iv) If $f \in AP_{w,c}$ and $g_{w,c} \in L^1$, then $f * g \in AP_{w,c}$.

Proof. See [10]. □

Now, we recall the space of pseudo almost periodic functions, introduced by C. Zhang, see [14] and [15]. Set

$$PAP_0 = \left\{ f \in \mathcal{C}_b : \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |f(x)| dx = 0 \right\}.$$

Definition 2. A function $f \in \mathcal{C}_b$ is called pseudo almost periodic if it can be expressed as

$$f = g + h \quad (2.3)$$

where $g \in AP$ and $h \in PAP_0$.

Denote by PAP the set of all such functions. The decomposition (2.3) is unique, so, the functions g and h are respectively called the almost periodic component and the ergodic perturbation of the pseudo almost periodic function f . We have

$$AP \subset PAP \subset \mathcal{C}_b.$$

The (w, c) -pseudo almost periodicity of continuous functions and their Stepanov generalizations is recently introduced by M. T. Khalladi, M. Kostic, A. Rahmani and D. Velinov, see [11]. We recall the definition and some basic properties of (w, c) -pseudo almost periodic functions. For the proofs and more details see [10] and [11].

Consider the (w, c) -mean of a function $h : \mathbb{R} \rightarrow \mathbb{C}$, given by

$$\mathcal{M}_{w,c}(h) := \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t c^{-\frac{s}{w}} h(s) ds,$$

whenever the limit exists. Let us define the space

$$PAP_{0;w,c} := \{h \in \mathcal{C}(\mathbb{R}, \mathbb{C}) : \mathcal{M}_{w,c}(|h|) = 0\}.$$

A function h is said to be c -ergodic if and only if belongs to this space, i.e., if and only if $c^{-\frac{\cdot}{w}} h(\cdot) \in PAP_0$. Therefore, the ergodic space of C. Zhang [12] can be recovered by plugging $c = 1$ in the above definition.

Definition 3. A function $f \in \mathcal{C}$ is said to be (w, c) -pseudo almost periodic if it can be expressed as

$$f = g + h \tag{2.4}$$

where $g \in AP_{w,c}$ and $h \in PAP_{0;w,c}$. The space of all such functions will be denoted by $PAP_{w,c}$.

We can easily show that the decomposition (2.4) is unique.

Theorem 1. A function $f \in \mathcal{C}$ is (w, c) -pseudo almost periodic, if and only if

$$f(\cdot) = c^{\frac{\cdot}{w}} F(\cdot), \text{ with } F \in PAP.$$

Proof. See [11]. □

It can be simply shown that $f + g \in PAP_{w,c}$, $\lambda f \in PAP_{w,c}$ and $f_\tau(\cdot) = f(\cdot + \tau) \in PAP_{w,c}$, provided $f, g \in PAP_{w,c}$, $\lambda \in \mathbb{C}$ and $\tau \in \mathbb{R}$. Moreover, we have the following.

Proposition 2. The space $PAP_{w,c}$, equipped with the (w, c) -norm $\|\cdot\|_{w,c}$ is a Banach space.

Proof. See [11]. □

3 (w, c) -Pseudo almost periodic distributions

In this section, we introduce and study a new generalization of Schwartz (w, c) -almost periodic distributions introduced in [12]. To this end, we first recall that the L^p -distributions denoted by $\mathcal{D}'_{L^p}$ are introduced by L. Schwartz in [13], further developed by J. Barros-Neto in [3]. The space $\mathcal{D}'_{L^p}$, $1 < p \leq +\infty$, is defined to be the topological dual of the differential Fréchet space

$$\mathcal{D}_{L^q} := \left\{ \varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^q, \forall j \in \mathbb{Z}_+ \right\}, \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,q} := \sum_{j \leq k} \|\varphi^{(j)}\|_{L^q}, \quad k \in \mathbb{Z}_+.$$

The topological dual of $\mathcal{D}_{L^1}$, denoted by $\mathcal{D}'_{L^\infty}$, is called the space of bounded distributions. The space

$$\mathcal{D}_{L^\infty} := \left\{ \varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^\infty, \forall j \in \mathbb{Z}_+ \right\},$$

is endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,\infty} := \sup_{j \leq k} \|\varphi^{(j)}\|_{L^\infty}, \quad k \in \mathbb{Z}_+.$$

The space

$$\dot{\mathcal{D}}_{L^\infty} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \lim_{|x| \rightarrow \infty} \varphi^{(j)} = 0, \forall j \in \mathbb{Z}_+ \right\},$$

is the closure of the space of test functions $\mathcal{D}$ in the topology defined by $|\cdot|_{k,\infty}$.

The topological dual of $\dot{\mathcal{D}}_{L^\infty}$, denoted by $\dot{\mathcal{D}}'_{L^1}$, is called the space of integrable distributions.

We recall the following characterizations of L^p -distributions, $1 \leq p \leq +\infty$.

Theorem 2. Let $T \in \mathcal{D}'$. Then the following statements are equivalent:

- (i) $T \in \mathcal{D}'_{L^p}$.
- (ii) $T \star \varphi \in L^p$ for all $\varphi \in \mathcal{D}$.
- (iii) There exists $k \in \mathbb{Z}_+$ and $(f_j)_{j \leq k} \subset L^p$ such that $T = \sum_{j \leq k} f_j^{(j)}$.

Proof. See [13]. □

L. Schwartz has also introduced the space $\mathcal{B}'_{ap}$ of almost periodic distributions. This space is based on the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h , denoted by $\tau_h T$, is defined as:

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h} \varphi \rangle, \quad \varphi \in \mathcal{D},$$

where $\tau_{-h} \varphi(x) = \varphi(x+h)$.

The definition and characterizations of Schwartz almost periodic distributions are given in the following result.

Theorem 3. For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent:

- (i) The set $\{\tau_h T, h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.
- (ii) $T \star \varphi \in AP, \forall \varphi \in \mathcal{D}$.
- (iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{0 \leq j \leq k} \subset AP : T = \sum_{j=0}^k f_j^{(j)}$.

Proof. See [13]. □

The authors in [12] have introduced and studied a new weighted spaces of distributions $\mathcal{D}'_{L^p_{w,c}}$ and their test spaces $\mathcal{D}_{L^q_{w,c}}$, $\frac{1}{p} + \frac{1}{q} = 1$. To recall the concept of (w, c) -almost periodicity in the setting of Sobolev-Schwartz distributions, we need to present the space $\mathcal{D}_{L^1_{w,c}}$ and their dual space. Let $q = 1, \infty$. We denote by $L^q_{w,c}$, the set of (w, c) -Lebesgue functions with exponent q , i.e.

$$L^q_{w,c} = \{f : \mathbb{R} \rightarrow \mathbb{C} \text{ measurable} : f_{w,c} \in L^q\}.$$

When $c = 1$, $L^q_{w,c} = L^q$ the classical Lebesgue space over $\mathbb{R}$. The space $L^1_{w,c}$ (resp. $L^\infty_{w,c}$) endowed with the (w, c) -norm

$$\|f\|_{L^1_{w,c}} := \|f_{w,c}\|_{L^1} \quad (\text{resp. } \|f\|_{L^\infty_{w,c}} := \|f\|_{w,c}),$$

is a Banach space.

The space

$$\mathcal{D}_{L_{w,c}^1} := \{ \varphi \in \mathcal{C}^\infty : \varphi_{w,c} \in \mathcal{D}_{L^1} \},$$

endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,1;w,c} := |\varphi_{w,c}|_{k,1}, \quad k \in \mathbb{Z}_+,$$

is a Fréchet subspace of $\mathcal{C}^\infty$. When $c = 1$, we get $\mathcal{D}_{L_{w,c}^1} = \mathcal{D}_{L^1}$.

Proposition 3. *The space $\mathcal{D}$ is dense in $\mathcal{D}_{L_{w,c}^1}$.*

Proof. Since $\mathcal{D}$ is dense in the space $\mathcal{C}_c$ of continuous functions with compact support it suffices to show that $\mathcal{C}_c$ is dense in $L_{w,c}^1$. Let S be the set of all simple measurable functions s , with complex values, defined on $\mathbb{R}$ and such that

$$\text{mes} \{ t : s(t) \neq 0 \} < \infty.$$

First it is clear that S is dense in $L_{w,c}^1$. Indeed, as $c^{-\frac{t}{w}} s \in L^1$, then $S \subset L_{w,c}^1$. Suppose $f \in L_{w,c}^1$ is positive and define the sequence $(s_n)_n$ such that

$$0 \leq s_1 \leq s_2 \leq \dots \leq f \text{ and for all } t \in \mathbb{R}, s_n(t) \rightarrow f(t) \text{ when } n \rightarrow +\infty.$$

Then $(f - s_n)_{w,c} = c^{-\frac{t}{w}} (f - s_n) \in L^1$, hence $s_n \in S$. Furthermore, since

$$\left| c^{-\frac{t}{w}} (f - s_n) \right| \leq f,$$

Lebesgue's dominated convergence theorem shows that

$$\| (f - s_n)_{w,c} \|_{L^1} = \left\| c^{-\frac{t}{w}} (f - s_n) \right\|_{L^1} \rightarrow 0 \text{ when } n \rightarrow +\infty,$$

hence $\|f - s_n\|_{L_{w,c}^1} \rightarrow 0$ when $n \rightarrow +\infty$. On the other hand, by Lusin's theorem, for $s \in S$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}_c$ such that $g(t) = s(t)$, except on a set of measure less than ε , and $|g| \leq \|s\|_\infty$, and since s takes only a finite number of values, there exists a constant $C > 0$ which depends on c and w such that

$$\| (g - s)_{w,c} \|_{L^1} = \int_{\mathbb{R}} \left| c^{-\frac{t}{w}} (g(t) - s(t)) \right| dt \leq 2C\varepsilon \|s\|_\infty.$$

The density of S in $L_{w,c}^1$ completes the proof. $\square$

Definition 4. *The space of (w, c) -bounded distributions denoted by $\mathcal{B}'_{w,c}$ (or $\mathcal{D}'_{L_{w,c}^\infty}$), is the topological dual of $\mathcal{D}_{L_{w,c}^1}$.*

We have the following characterizations of $\mathcal{B}'_{w,c}$ distributions.

Theorem 4. *Let $T \in \mathcal{D}'$, the following statements are equivalent:*

- (i) $T \in \mathcal{B}'_{w,c}$.
- (ii) $c^{\frac{t}{w}} (T_{w,c} * \varphi) \in L_{w,c}^\infty, \forall \varphi \in \mathcal{D}$.
- (iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{0 \leq j \leq k} \subset L_{w,c}^\infty : T = c^{\frac{t}{w}} \sum_{j=0}^k (f_{w,c})_j^{(j)}$, where

$$\left((f_{w,c})_j \right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j \right)_{0 \leq j \leq k}.$$

Proof. See [12]. $\square$

Taking into account the notation (2.2), we have the following consequence of Theorem 4.

Corollary 1. *A distribution $T \in \mathcal{B}'_{w,c}$, if and only if, $T_{w,c} \in \mathcal{D}'_{L^\infty}$.*

Proof. From the equivalence (i) $\iff$ (ii) of Theorem 4 and Theorem 2, we have

$$T \in \mathcal{B}'_{w,c} \iff c^{\frac{t}{w}}(T_{w,c} \star \varphi) \in L^\infty_{w,c}, \forall \varphi \in \mathcal{D} \iff T_{w,c} \star \varphi \in L^\infty, \forall \varphi \in \mathcal{D} \iff T_{w,c} \in \mathcal{D}'_{L^\infty}.$$

□

The concept of (w, c) -almost periodicity of Schwartz distributions is given by the following

Definition 5. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c) -almost periodic, if and only if, $T_{w,c} \in \mathcal{B}'_{ap}$, i.e., the set $\{\tau_h T_{w,c}, h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of (w, c) -almost periodic distributions is denoted by $\mathcal{B}'_{AP_{w,c}}$.

Example 1. (i) The associated distribution of an (w, c) -almost periodic function (resp. Stepanov (p, w, c) -almost periodic function) is an (w, c) -almost periodic distribution, i.e.

$$AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}} \text{ (resp. } S^p AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}}).$$

(ii) When $c = 1$ it follows that $\mathcal{B}'_{AP_{w,c}} := \mathcal{B}'_{ap}$.

Characterizations of (w, c) -almost periodic distributions are given in the following result.

Theorem 5. Let $T \in \mathcal{B}'_{w,c}$, the following statements are equivalent:

- (i) $T \in \mathcal{B}'_{AP_{w,c}}$.
- (ii) $c^{\frac{t}{w}}(T_{w,c} \star \varphi) \in AP_{w,c}, \forall \varphi \in \mathcal{D}$.
- (iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{0 \leq j \leq k} \subset AP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^k (f_{w,c})_j^{(j)},$ where

$$\left((f_{w,c})_j \right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j \right)_{0 \leq j \leq k}.$$

Proof. (i) $\iff$ (ii) : By definition we have $T \in \mathcal{B}'_{AP_{w,c}} \iff T_{w,c} \in \mathcal{B}'_{ap}$, and from the characterization of Schwartz distributions (Theorem 3-(ii)), we obtain $T_{w,c} \star \varphi \in AP, \forall \varphi \in \mathcal{D}$, i.e. $c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (T_{w,c} \star \varphi) \right) \in AP, \forall \varphi \in \mathcal{D}$. According to definition of $AP_{w,c}$, this is equivalent to $c^{\frac{t}{w}} (T_{w,c} \star \varphi) \in AP_{w,c}, \forall \varphi \in \mathcal{D}$.

(i) $\iff$ (iii) : Using assertion (iii) of Theorem 3, we have

$$\begin{aligned} T \in \mathcal{B}'_{AP_{w,c}} &\iff T_{w,c} \in \mathcal{B}'_{ap} \iff \exists k \in \mathbb{Z}_+, \exists (g_j)_{0 \leq j \leq k} \subset AP : T_{w,c} = \sum_{j=0}^k g_j^{(j)} \\ &\iff \exists k \in \mathbb{Z}_+, \exists (g_j)_{0 \leq j \leq k} \subset AP : T = c^{\frac{t}{w}} \sum_{j=0}^k g_j^{(j)}. \end{aligned}$$

From the definition of (w, c) -almost periodic functions, $\exists (f_j)_{0 \leq j \leq k} \subset AP_{w,c}$ such that $(f_{w,c})_j = g_j, \forall j \in \{0, \dots, k\}$, where $\left((f_{w,c})_j \right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j \right)_{0 \leq j \leq k}$. Hence

$$\exists k \in \mathbb{Z}_+, \exists (f_j)_{0 \leq j \leq k} \subset AP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^k (f_{w,c})_j^{(j)}.$$

□

We recall that a distribution $T \in \mathcal{D}'$ is zero on an open subset V of $\mathbb{R}$ if

$$\langle T, \varphi \rangle = 0, \forall \varphi \in \mathcal{D}(V),$$

and that two distributions $T, S \in \mathcal{D}'$ coincide on V if $T - S = 0$ on V .

Let us also recall the following lemma.

Lemma 1. Let $f \in \mathcal{C}^\infty$ and $T \in \mathcal{D}'$. If $fT = 0$, then $T = 0$ on the set $G = \{x \in \mathbb{R} : f(x) \neq 0\}$.

Proof. Let $f \in \mathcal{C}^\infty$, $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ with $\text{supp } \varphi \in G$. Since $\frac{\varphi}{f} \in \mathcal{D}$, then $\langle T, f \frac{\varphi}{f} \rangle = \langle fT, \frac{\varphi}{f} \rangle = 0$. $\square$

Returning to the notation (2.2), we have the following proposition.

Proposition 4. Let $T \in \mathcal{D}'$. Then $T \in \mathcal{B}'_{AP_{w,c}}$, if and only if, there exists $S \in \mathcal{B}'_{ap}$, such that $T = c^{\frac{(\cdot)}{w}} S$ in $\mathcal{D}'$.

Proof. $(\Rightarrow)$: If $T \in \mathcal{B}'_{AP_{w,c}}$, then $T_{w,c} = c^{-\frac{(\cdot)}{w}} T \in \mathcal{B}'_{ap}$, so there exists $S \in \mathcal{B}'_{ap}$ such that $c^{-\frac{(\cdot)}{w}} T - S = 0$ in $\mathcal{D}'_{L^\infty}$, i.e. $c^{-\frac{(\cdot)}{w}} (T - c^{\frac{(\cdot)}{w}} S) = 0$ in $\mathcal{D}'_{L^\infty}$. By applying Lemma 1, it follows that $T = c^{\frac{(\cdot)}{w}} S$ in $\mathcal{D}'$.

$(\Leftarrow)$: Suppose that $T \in \mathcal{D}'$ and there exists $S \in \mathcal{B}'_{ap}$, such that $T = c^{\frac{(\cdot)}{w}} S$ in $\mathcal{D}'$, then $c^{-\frac{(\cdot)}{w}} T = S \in \mathcal{B}'_{ap}$, hence $T \in \mathcal{B}'_{AP_{w,c}}$. $\square$

We recall also the following space of smooth (w, c) -almost periodic functions

$$\mathcal{B}_{AP_{w,c}} := \left\{ \varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi_{w,c} \in \mathcal{B}_{ap} \right\},$$

where

$$\mathcal{B}_{ap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP, \forall j \in \mathbb{Z}_+ \right\},$$

is the space of smooth almost periodic functions introduced by L. Schwartz. We endow $\mathcal{B}_{AP_{w,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{w,c}}$. The main properties of $\mathcal{B}_{AP_{w,c}}$ and $\mathcal{B}'_{AP_{w,c}}$ are summarized in the following

Proposition 5. (i) $\mathcal{B}_{AP_{w,c}} = AP_{w,c} \cap \mathcal{D}_{L^\infty_{w,c}}$.

(ii) $\mathcal{B}_{AP_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$.

(iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $c^{\frac{(\cdot)}{w}} (f_{w,c} \star \varphi_{w,c}) \in \mathcal{B}_{AP_{w,c}}$.

(iv) If $f \in \mathcal{D}_{L^\infty_{w,c}}$ and $c^{\frac{(\cdot)}{w}} (f_{w,c} \star \varphi_{w,c}) \in AP_{w,c}$, $\forall \varphi \in \mathcal{D}$, then $f \in \mathcal{B}_{AP_{w,c}}$.

(v) If $T \in \mathcal{B}'_{AP_{w,c}}$, then $c^{\frac{(\cdot)}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{AP_{w,c}}$, $\forall j \in \mathbb{Z}_+$.

(vi) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c} T \in \mathcal{B}'_{AP_{w,c}}$.

(vii) If $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$, then $c^{\frac{(\cdot)}{w}} (T_{w,c} \star S_{w,c}) \in \mathcal{B}'_{AP_{w,c}}$.

(viii) $\mathcal{B}_{AP_{w,c}}$ is dense in $\mathcal{B}'_{AP_{w,c}}$.

Proof. See [12]. $\square$

We recall that a sequence $(T_j)_j$ converges to zero in $\mathcal{D}'_{L^\infty}$ if $(\tau_{-h} T_j)_j$ converges to 0 in $\mathcal{D}'$ uniformly in $h \in \mathbb{R}$.

We have the following properties of $\mathcal{B}_{AP_{w,c}}$.

Theorem 6. Let $T \in \mathcal{B}'_{w,c}$. Then

(i) $T \in \mathcal{B}_{AP_{w,c}}$ if and only if for every sequence $(c_j)_j \subset \mathbb{R}$ there exists a subsequence $(b_j)_j$ of $(c_j)_j$ such that $(\tau_{-b_j} T_{w,c})_j$ converges in $\mathcal{D}'_{L^\infty}$ when $j \rightarrow +\infty$.

(ii) If $T \in \mathcal{B}'_{AP_{w,c}}$ is such that $T \neq 0$ and $\lim_{j \rightarrow +\infty} \tau_{-b_j} T_{w,c} = S$ in $\mathcal{D}'_{L^\infty}$ for a sequence $(b_j)_j \subset \mathbb{R}$, then $S \neq 0$ and $\lim_{j \rightarrow +\infty} \tau_{b_j} S = T_{w,c}$ in $\mathcal{D}'_{L^\infty}$, as well.

Proof. (i) It follows from the relation

$$\langle \tau_{-h} T_{w,c}, \varphi \rangle = \langle T_{w,c} \star \check{\varphi} \rangle(h), \text{ where } \check{\varphi}(\cdot) = \varphi(-\cdot),$$

and the equivalence (i) $\iff$ (ii) of Theorem 5.

(ii) Suppose that $S = 0$. Then

$$\lim_{j \rightarrow +\infty} \tau_{-b_j} (T_{w,c} \star \varphi)(\cdot) = 0 \text{ in } \mathcal{C}_b, \forall \varphi \in \mathcal{D}.$$

Therefore, $\forall \varepsilon > 0$, there exists $j_\varepsilon \in \mathbb{Z}_+$ such that

$$\sup_{x \in \mathbb{R}} \left| \tau_{-b_j} (T_{w,c} \star \varphi) (x + \cdot) \right| < \varepsilon, \quad \forall j \geq j_\varepsilon,$$

i.e. $\langle T_{w,c}, \varphi \rangle = 0$, $\forall \varphi \in \mathcal{D}$, thus by Lemma 1, we obtain $T = 0$, which is in contradiction with the hypothesis. The last part of (ii), follows straightforward from inequality

$$\sup_{x \in \mathbb{R}} \left| \tau_{-b_j} (T_{w,c} \star \varphi) (x + \cdot) - (S \star \varphi) (x) \right| < \varepsilon, \quad \forall j \geq j_\varepsilon, \quad \forall \varphi \in \mathcal{D},$$

and

$$\sup_{x \in \mathbb{R}} \left| (T_{w,c} \star \varphi) (x + \cdot) - \tau_{b_j} (S \star \varphi) (x) \right| < \varepsilon, \quad \forall j \geq j_\varepsilon, \quad \forall \varphi \in \mathcal{D}.$$

□

Now, we recall the space $\mathcal{B}'_{PAP}$ of pseudo almost periodic distributions. The space of bounded distributions with mean value vanishing at infinity is denoted and defined by

$$\mathcal{B}'_0 := \left\{ T \in \mathcal{D}'_{L^\infty} : \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(T \star \varphi)(x)| dx = 0, \quad \forall \varphi \in \mathcal{D} \right\}.$$

Definition 6. A distribution $T \in \mathcal{D}'_{L^\infty}$ is said to be a pseudo almost periodic, if there are $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_0$ such that $T = R + S$.

The decomposition of a pseudo almost periodic distribution is unique. In addition, we have the following characterization of pseudo almost periodic distributions.

Theorem 7. Let $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent :

- (i) $T \in \mathcal{B}'_{PAP}$.
- (ii) $T \star \varphi \in PAP, \forall \varphi \in \mathcal{D}$.
- (iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{j \leq k} \subset PAP : T = \sum_{j \leq k} f_j^{(j)}$.

Proof. It follows by using the same arguments as in ([7], Theorem 1) and its proof by replacing the space $\mathcal{B}'_0$ of bounded distributions with mean value vanishing at infinity instead of the space $\mathcal{B}'_0$ of bounded distributions tending to zero at infinity. □

It was shown in ([12], Theorem 3.4) that, if $T \in \mathcal{B}'_{AP_{w,c}}, T_{w,c} \star \varphi \in AP, \forall \varphi \in \mathcal{D}$, thus $\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(T_{w,c} \star \varphi)(x)| dx$ exists. This allows us to extend the notion of c -ergodicity to $\mathcal{B}'_{w,c}$ space. Set

$$\mathcal{B}'_{0;w,c} := \left\{ T \in \mathcal{B}'_{w,c} : \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(T_{w,c} \star \varphi)(x)| dx = 0, \quad \forall \varphi \in \mathcal{D} \right\}.$$

Some properties, easy to prove, of $\mathcal{B}'_{0;w,c}$ distributions are given in the following.

Proposition 6. (i) $T \in \mathcal{B}'_{0;w,c}$ if and only if $T_{w,c} \in \mathcal{B}'_0$.

(ii) If $T \in \mathcal{B}'_{0;w,c}$, then $c^{\frac{t}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{0;w,c}, \forall j \in \mathbb{Z}_+$.

(iii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{0;w,c}$, then $\varphi_{w,c} T \in \mathcal{B}'_{0;w,c}$.

Proof. (i) Obvious.

(ii) Let $T \in \mathcal{B}'_{0;w,c}$, by definition we have $T_{w,c} \in \mathcal{D}'_{L^\infty}$. Since $\mathcal{D}'_{L^\infty}$ is invariant under derivation it follows that $(T_{w,c})^{(j)} \in \mathcal{D}'_{L^\infty}, \forall j \in \mathbb{Z}_+$, so $c^{\frac{t}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{w,c}, \forall j \in \mathbb{Z}_+$. Furthermore, for all $\varphi \in \mathcal{D}$ we have

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(T_{w,c} \star \varphi)(x)| dx = 0, \quad \forall \varphi \in \mathcal{D}.$$

Thus

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t \left| \left(c^{\frac{t}{w}} (T_{w,c})^{(j)} \right)_{w,c} \star \varphi \right| (x) dx = \lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t \left| (T_{w,c} \star \varphi^{(j)}) (x) \right| dx = 0, \quad \forall \varphi \in \mathcal{D},$$

which shows that $c^{\frac{t}{w}} (T_{w,c})^{(j)} \in \mathcal{B}'_{0;w,c}$.

(iii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{0;w,c}$, then $\varphi_{w,c} \in \mathcal{B}_{ap}$, $T_{w,c} \in \mathcal{B}'_0$ and $\varphi_{w,c} T_{w,c} \in \mathcal{B}'_0$, because of the inclusion $\mathcal{B}_{ap} \times \mathcal{B}'_0 \subset \mathcal{B}'_0$. Moreover, $\left(c^{\frac{t}{w}} (\varphi_{w,c} T_{w,c}) \right)_{w,c} = c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} (\varphi_{w,c} T_{w,c}) \right) \in \mathcal{B}'_0$ and from (i) we get $c^{\frac{t}{w}} (\varphi_{w,c} T_{w,c}) \in \mathcal{B}'_{0;w,c}$, i.e. $\varphi_{w,c} T \in \mathcal{B}'_{0;w,c}$. $\square$

Now we can introduce the notion of (w, c) -pseudo almost periodicity in the setting of Schwartz distributions.

Definition 7. A distribution $T \in \mathcal{B}'_{w,c}$ is said to be (w, c) -pseudo almost periodic if it can be written as

$$T = R + S \quad (3.1)$$

where $R \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{B}'_{0;w,c}$.

Denote by $\mathcal{B}_{PAP_{w,c}}$ the space of (w, c) -pseudo almost periodic distributions.

It is clear that $\mathcal{B}'_{AP_{w,c}} \subset \mathcal{B}'_{PAP_{w,c}} \subset \mathcal{B}'_{w,c}$ and when $c = 1$, we get $\mathcal{B}'_{PAP_{w,c}} := \mathcal{B}'_{PAP}$ the space of pseudo almost periodic distributions.

Proposition 7. The decomposition (3.1) is unique.

Proof. Suppose that $T = R + S = P + Q$ in $\mathcal{B}'_{w,c}$, with $R, P \in \mathcal{B}'_{AP_{w,c}}$ and $S, Q \in \mathcal{B}'_{0;w,c}$. Then $R - P \in \mathcal{B}'_{AP_{w,c}} \cap \mathcal{B}'_{0;w,c}$. From Theorem 5-(ii), it follows that $c^{\frac{t}{w}} ((R_{w,c} - P_{w,c}) \star \varphi) \in AP_{w,c} \cap PAP_{0;w,c}$, $\forall \varphi \in \mathcal{D}$, which implies that $(R_{w,c} - P_{w,c}) \star \varphi = 0$, $\forall \varphi \in \mathcal{D}$. Hence $R_{w,c} - P_{w,c} = 0$, i.e. $R - P = 0$. $\square$

It is easy to prove that every (w, c) -pseudo almost periodic function is a (w, c) -pseudo almost periodic distribution, and that

$$PAP_{w,c} \hookrightarrow \mathcal{B}'_{PAP_{w,c}}.$$

Proposition 8. A distribution $T \in \mathcal{B}'_{PAP_{w,c}}$, if and only if, $T_{w,c} \in \mathcal{B}'_{PAP}$.

Proof. It follows from the definition of (w, c) -almost periodic distributions and Proposition 6-(i). $\square$

The following result gives a characterization of (w, c) -pseudo almost periodic distributions.

Theorem 8. Let $T \in \mathcal{B}'_{w,c}$, the following statements are equivalent :

- (i) $T \in \mathcal{B}'_{PAP_{w,c}}$.
- (ii) $c^{\frac{t}{w}} (T_{w,c} \star \varphi) \in PAP_{w,c}$, $\forall \varphi \in \mathcal{D}$.
- (iii) $\exists k \in \mathbb{Z}_+$, $\exists (f_j)_{j \leq k} \subset PAP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^k (f_{w,c})_j^{(j)}$, where $\left((f_{w,c})_j \right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j \right)_{0 \leq j \leq k}$.

Proof. According to Proposition 8, a distribution $T \in \mathcal{B}'_{PAP_{w,c}}$, if and only if, $T_{w,c} \in \mathcal{B}'_{PAP}$, thus the result follows immediately from Theorem 7. $\square$

Another characterization of (w, c) -pseudo almost periodic distributions is given by the following result.

Theorem 9. Let $T \in \mathcal{B}'_{w,c}$. Then $T \in \mathcal{B}'_{PAP_{w,c}}$, if and only if there exists $S \in \mathcal{B}'_{PAP}$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{B}'_{w,c}$.

Proof. $(\Rightarrow)$: If $T \in \mathcal{B}'_{PAP_{w,c}}$, then by Proposition 8, $T_{w,c} = c^{-\frac{t}{w}} T \in \mathcal{B}'_{PAP}$, so there exists $S \in \mathcal{B}'_{PAP}$ such that $c^{-\frac{t}{w}} T - S = 0$ in $\mathcal{D}'_{L^\infty}$, i.e. $c^{-\frac{t}{w}} \left(T - c^{\frac{t}{w}} S \right) = 0$ in $\mathcal{D}'_{L^\infty}$. By applying Lemma 1, it follows that $T = c^{\frac{t}{w}} S$ in $\mathcal{B}'_{w,c}$.

($\Leftarrow$): Let $T \in \mathcal{B}'_{w,c}$ and suppose that there exists $S \in \mathcal{B}'_{PAP}$, such that $T = c^{\frac{(j)}{w}} S$ in $\mathcal{B}'_{w,c}$, then $c^{-\frac{(j)}{w}} T = S \in \mathcal{B}_{PAP}$, hence $T \in \mathcal{B}_{PAP_{w,c}}$. $\square$

Proposition 9. Let $T \in \mathcal{B}'_{PAP}$ such that $T = R + S$, where $R \in \mathcal{B}'_{ap}$ and $S \in \mathcal{B}'_0$. Suppose that for $(b_j)_j \subset \mathbb{R}$, $\lim_{j \rightarrow +\infty} b_j = +\infty$, the sequence $(\tau_{-b_j} R)_j$ converges in $\mathcal{D}'_{L^\infty}$ to R_0 , then

(i) If $\varphi \in \mathcal{B}_{ap}$ and $(\tau_{-b_j} \varphi)_j$ converges in $\mathcal{D}'_{L^\infty}$ to φ_0 , then $\varphi T \in \mathcal{B}'_{PAP}$ and $(\tau_{-b_j} (\varphi T))_j$ converges in $\mathcal{D}'$ to $\varphi_0 R_0$, when $j \rightarrow +\infty$.

(ii) $T^{(i)} \in \mathcal{B}'_{PAP}$ and $(\tau_{-b_j} (T^{(i)}))_j, i \in \mathbb{Z}_+$, converges in $\mathcal{D}'$ to $R_0^{(i)}$, when $j \rightarrow +\infty$.

Proof. (i) The fact that $\varphi T \in \mathcal{B}'_{PAP}$ is trivial. The convergence of $(\tau_{-b_j} (\varphi T))_j$ in $\mathcal{D}'$ to $\varphi_0 R_0$, when $j \rightarrow +\infty$, follows from the relation

$$\langle \tau_{-b_j} (\varphi T), \psi \rangle = \langle \tau_{-b_j} T, \tau_{-b_j} \varphi \psi \rangle, \psi \in \mathcal{D},$$

and the fact that $\{\tau_{-b_j} \varphi \psi, b_j \in \mathbb{R}, j \in \mathbb{Z}_+\}$ is a bounded set in $\mathcal{D}$.

(ii) It is clear that $T^{(i)} \in \mathcal{B}'_{PAP}$. We have

$$\langle \tau_{-b_j} (T^{(i)}), \psi \rangle = (-1)^i \langle T, (\tau_{b_j} \psi)^{(i)} \rangle = (-1)^i \langle \tau_{-b_j} T, \psi^{(i)} \rangle, \psi \in \mathcal{D},$$

which gives

$$\lim_{j \rightarrow +\infty} \tau_{-b_j} (T^{(i)}) = R_0^{(i)}.$$

$\square$

4 Application to linear differential systems

In this section we will apply the above theoretical results to study the distributional (w, c) -pseudo almost periodic solutions of some linear differential equations. Let us first consider the linear system of differential equations with constant coefficients

$$X' = AX + W, \quad (4.1)$$

where $A = (a_{ij})_{1 \leq i, j \leq k}$, $k \in \mathbb{N}$, is a given square matrix of complex numbers, $W = (W_i)_{1 \leq i \leq k} \in (\mathcal{D}')^k$ is a vector distribution and $X = (X_i)_{1 \leq i \leq k}$ is the unknown vector distribution.

Let us consider the system (4.1) with $W \in (\mathcal{B}'_{AP_{w,c}})^k$ and recall the following main result obtained in [12].

Theorem 10. Let $W \in (\mathcal{B}'_{AP_{w,c}})^k$. If the matrix $A - \frac{\log c}{w} I_k$, where I_k be the unit matrix of order k , has no eigenvalues with real part zero, then the system (4.1) admits a unique solution $R \in (\mathcal{B}'_{w,c})^k$ which is a (w, c) -almost periodic vector distribution.

Proof. See [12]. $\square$

The following result gives the (w, c) -pseudo almost periodicity of the (w, c) -bounded solution of system (4.1).

Theorem 11. Assume the matrix A is such that $A - \frac{\log c}{w} I_k$, has no eigenvalues with real part zero. If $W = Y + Z$, where $Y \in (\mathcal{B}'_{AP_{w,c}})^k$ and $Z \in (\mathcal{B}'_{0,w,c})^k$, then the distribution $T = R + S \in (\mathcal{B}'_{w,c})^k$, where $R, S \in (\mathcal{B}'_{w,c})^k$,

$$R' = AR + Y,$$

and

$$S' = AS + Z,$$

is a (w, c) -pseudo almost periodic vector solution of (4.1).

Proof. Define $T = R + S$, then

$$T' = AT + W,$$

which shows that T is a (w, c) -bounded solution of system (4.1). It remains to prove the (w, c) -pseudo almost periodicity of T . Since $R \in (\mathcal{B}'_{w,c})^k$ is a (w, c) -bounded solution of system $R' = AR + Y$, with $Y \in (\mathcal{B}'_{AP_{w,c}})^k$, then by Theorem 10, it follows that $R \in (\mathcal{B}'_{AP_{w,c}})^k$. To have $S \in (\mathcal{B}'_{0;w,c})^k$, it suffices to show that

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(S_{w,c} \star \varphi)(x)| dx = 0, \quad \forall \varphi \in \mathcal{D},$$

where

$$S_{w,c} \star \varphi = ((S_{w,c})_i \star \varphi)_{1 \leq i \leq k} = \left((c^{-\frac{t}{w}} S_i) \star \varphi \right)_{1 \leq i \leq k}.$$

Indeed, since $S \in (\mathcal{B}'_{w,c})^k$ satisfies equation

$$S' = AS + Z,$$

then

$$c^{-\frac{t}{w}} S' \star \varphi = A c^{-\frac{t}{w}} S \star \varphi + c^{-\frac{t}{w}} Z \star \varphi,$$

and due to the fact that

$$c^{-\frac{t}{w}} S' \star \varphi = \left(c^{-\frac{t}{w}} S \star \varphi \right)' + \frac{\log c}{w} c^{-\frac{t}{w}} S \star \varphi,$$

we get

$$(S_{w,c} \star \varphi)' = \left(A - \frac{\log c}{w} I_k \right) (S_{w,c} \star \varphi) + Z_{w,c} \star \varphi. \quad (4.2)$$

It is clear that $Z_{w,c} \star \varphi = ((Z_{w,c})_i \star \varphi)_{1 \leq i \leq k} = \left((c^{-\frac{t}{w}} Z_i) \star \varphi \right)_{1 \leq i \leq k} \in (PAP_0)^k$. We use the same arguments as in the proof of Theorem 10 (see [12], Theorem 4.2), it follows that $S_{w,c} \star \varphi \in (PAP_0)^k$ is a unique bounded solution of (4.2), i.e.

$$\lim_{t \rightarrow +\infty} \frac{1}{2t} \int_{-t}^t |(S_{w,c})_i \star \varphi(x)| dx = 0, \quad 1 \leq i \leq k,$$

thus $c^{\frac{t}{w}} (S_{w,c} \star \varphi) \in (PAP_{0;w,c})^k$. Due to the following equivalence

$$S \in (\mathcal{B}'_{0;w,c})^k \text{ if and only if } c^{\frac{t}{w}} (S_{w,c} \star \varphi) \in (PAP_{0;w,c})^k,$$

we conclude that $S \in (\mathcal{B}'_{0;w,c})^k$. □

Now, consider the following linear differential equation

$$X' = fX + W, \quad (4.3)$$

where $f \in \mathcal{B}_{AP_{w,c}}$, $f \neq 0$, $W \in \mathcal{D}'$ is a given distribution and X is the unknown distribution. We have the following result.

Theorem 12. Let $W \in \mathcal{B}'_{PAP_{w,c}}$ such that $W = Y + Z$, where $Y \in \mathcal{B}'_{AP_{w,c}}$ and $Z \in \mathcal{B}'_{0;w,c}$. Then, the equation (4.3) has a solution $T \in \mathcal{B}'_{PAP_{w,c}}$ if and only if there exists $P \in \mathcal{B}'_{AP_{w,c}}$ and $Q \in \mathcal{B}'_{0;w,c}$ such that

$$P' = fP + Y, \quad (4.4)$$

and

$$Q' = fQ + Z. \quad (4.5)$$

Proof. Since the sufficient condition is trivial, it suffices to prove that the given condition is necessary. Let $(c_j)_j \subset \mathbb{R}$ such that $\lim_{j \rightarrow +\infty} c_j = +\infty$. Then there exists a subsequence $(b_j)_j$ of $(c_j)_j$, $g \in \mathcal{D}_{L^\infty}$ and $U, V \in \mathcal{D}'_{L^\infty}$ such that

$$\lim_{j \rightarrow +\infty} \tau_{-b_j} f_{w,c} = g \text{ in } \mathcal{D}_{L^\infty}, \quad \lim_{j \rightarrow +\infty} \tau_{-b_j} P_{w,c} = U \text{ and } \lim_{j \rightarrow +\infty} \tau_{-b_j} Y_{w,c} = V \text{ in } \mathcal{D}'_{L^\infty},$$

and

$$\lim_{j \rightarrow +\infty} \tau_{-b_j} Q_{w,c} = 0 \text{ and } \lim_{j \rightarrow +\infty} \tau_{-b_j} Z_{w,c} = 0 \text{ in } \mathcal{D}'.$$

By the Proposition 9, using the limit in $\mathcal{D}'$ when $j \rightarrow +\infty$ in the equation

$$\tau_{-b_j} T'_{w,c} = \tau_{-b_j} f_{w,c} \tau_{-b_j} T_{w,c} + \tau_{-b_j} W_{w,c},$$

we obtain

$$U' = gU + V,$$

and by Property (ii) of Theorem 6, we have

$$P' = fP + Y.$$

Thanks to the linearity of equation (4.3), it follows that

$$Q' = fQ + Z. \quad \square$$

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