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The mutual singularity of the relative multifractal measures

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Abstract: M. Das proved that the relative multifractal measures are mutually singular for the self-similar measures satisfying the significantly weaker open set condition. The aim of this paper is to show that these measures are mutually singular in a more general framework. As examples, we apply our main results to quasi-Bernoulli measures.

Keywords: Multifractal analysis, multifractal formalism, singularity

MSC: 28A78, 28A80

1 Introduction and statements of results

In [4], Billingsley applies methods from ergodic theory to calculate the size of a level sets of the local dimension of μ with respect to another measure ν . Cajar [5] also has studies these sets in the code space. Anyone familiar with multifractal analysis will recognize this as a form of multifractal analysis. In several recent papers on multifractal analysis this type of multifractal analysis has re-emerged as mathematicians and physicists have begun to discuss the idea of performing multifractal analysis with respect to an arbitrary reference measure. Cole [6] has formalised these ideas by introducing a relative formalism for the multifractal analysis of one measure with respect to another. This formalism is based on the ideas of the "multifractal formalism" as clarified by Olsen [15]. Later, in [1, 3, 10, 21], Selmi et al. justified the relative multifractal formalism under less restrictive hypotheses. Other studies have been developed in the same direction such as [2, 9, 19, 20, 22, 24]. The purpose of this paper is to show that the relative multifractal Hausdorff and packing measures are mutually singular. The purpose of this paper is to study the multifractal structure of measures using the formalism introduced in [6].

We first, let us recall the multifractal formalism introduced by Cole in [6]. Let μ and ν be two probability measures on a metric space $\mathbb{R}^n$. For $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, write

$$\overline{\mathcal{P}}_{\mu, \nu, \delta}^{q, t}(E) = \sup \left\{ \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \right\}, \quad E \neq \emptyset,$$

where the supremum is taken over all centered δ -packing of E . Moreover we can set $\overline{\mathcal{P}}_{\mu, \nu, \delta}^{q, t}(\emptyset) = 0$. Also, we define

$$\overline{\mathcal{H}}_{\mu, \nu, \delta}^{q, t}(E) = \inf \left\{ \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^t \right\}, \quad E \neq \emptyset,$$

where the infimum is taken over all centered δ -covering of E . Moreover we can set $\overline{\mathcal{H}}_{\mu, \nu, \delta}^{q, t}(\emptyset) = 0$. Especially, we have the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

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The packing and Hausdorff pre-measures are defined respectively by

$$\overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E) = \inf_{\delta>0} \overline{\mathcal{P}}_{\mu,\nu,\delta}^{q,t}(E) \quad \text{and} \quad \overline{\mathcal{H}}_{\mu,\nu}^{q,t}(E) = \sup_{\delta>0} \overline{\mathcal{H}}_{\mu,\nu,\delta}^{q,t}(E).$$

The function $\overline{\mathcal{P}}_{\mu,\nu}^{q,t}$ is not necessarily countably subadditive, also the set function $\overline{\mathcal{H}}_{\mu,\nu}^{q,t}$ is not necessarily monotone. For these reasons, Cole introduced the packing and Hausdorff measures denoted respectively by $\mathcal{P}_{\mu,\nu}^{q,t}$ and $\mathcal{H}_{\mu,\nu}^{q,t}$ are defined as following

$$\mathcal{P}_{\mu,\nu}^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E_i) \quad \text{and} \quad \mathcal{H}_{\mu,\nu}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu,\nu}^{q,t}(F).$$

The functions $\mathcal{H}_{\mu,\nu}^{q,t}$ and $\mathcal{P}_{\mu,\nu}^{q,t}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. It is easy to see that $\mathcal{P}_{\mu,\nu}^{q,t} \leq \overline{\mathcal{P}}_{\mu,\nu}^{q,t}$. Moreover, by using Besicovitch's theorem, there exists an integer $\xi \in \mathbb{N}$, such that $\mathcal{H}_{\mu,\nu}^{q,t} \leq \xi \overline{\mathcal{P}}_{\mu,\nu}^{q,t}$ (see [15]). The measure $\mathcal{H}_{\mu,\nu}^{q,t}$ is a multifractal generalization of the centered Hausdorff measure, whereas $\mathcal{P}_{\mu,\nu}^{q,t}$ is a multifractal generalization of the packing measure. In fact, in the case when $t \geq 0$ and ν is the Lebesgue measure $\mathcal{L}^n$ on $\mathbb{R}^n$, $\mathcal{H}_{\mu,\nu}^{0,t} = \mathcal{H}^t$ and $\mathcal{P}_{\mu,\nu}^{0,t} = \mathcal{P}^t$, where $\mathcal{H}^t$ denotes the t -dimensional centered Hausdorff measure and $\mathcal{P}^t$ denotes the t -dimensional packing measure.

The measures $\mathcal{H}_{\mu,\nu}^{q,t}$ and $\mathcal{P}_{\mu,\nu}^{q,t}$ and the pre-measure $\overline{\mathcal{P}}_{\mu,\nu}^{q,t}$ assign in the usual way a multifractal dimension to each subset E of $\mathbb{R}^n$. They are respectively denoted by $b_{\mu,\nu}^q(E)$, $B_{\mu,\nu}^q(E)$ and $\Lambda_{\mu,\nu}^q(E)$ and satisfy

$$b_{\mu,\nu}^q(E) = \inf \left\{ t \in \mathbb{R}; \quad \mathcal{H}_{\mu,\nu}^{q,t}(E) = 0 \right\}, \quad B_{\mu,\nu}^q(E) = \inf \left\{ t \in \mathbb{R}; \quad \mathcal{P}_{\mu,\nu}^{q,t}(E) = 0 \right\},$$

$$\Lambda_{\mu,\nu}^q(E) = \inf \left\{ t \in \mathbb{R}; \quad \overline{\mathcal{P}}_{\mu,\nu}^{q,t}(E) = 0 \right\}.$$

The number $b_{\mu,\nu}^q(E)$ and $B_{\mu,\nu}^q(E)$ are obvious multifractal analogues of the ν -Hausdorff dimension $\dim_\nu(E)$ and the ν -packing dimension $\text{Dim}_\nu(E)$ of E respectively. In fact, it follows immediately from the definitions that

$$\dim_\nu(E) = b_{\mu,\nu}^0(E) \quad \text{and} \quad \text{Dim}_\nu(E) = B_{\mu,\nu}^0(E).$$

Next, for $q \in \mathbb{R}$, we define the dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$ by

$$b_{\mu,\nu}(q) = b_{\mu,\nu}^q(\text{supp } \mu \cap \text{supp } \nu), \quad B_{\mu,\nu}(q) = B_{\mu,\nu}^q(\text{supp } \mu \cap \text{supp } \nu) \quad \text{and} \quad \Lambda_{\mu,\nu}(q) = \Lambda_{\mu,\nu}^q(\text{supp } \mu \cap \text{supp } \nu).$$

It is well known that the functions $b_{\mu,\nu}$, $B_{\mu,\nu}$ and $\Lambda_{\mu,\nu}$ are decreasing and $B_{\mu,\nu}$, $\Lambda_{\mu,\nu}$ are convex and satisfying

$$b_{\mu,\nu} \leq B_{\mu,\nu} \leq \Lambda_{\mu,\nu}.$$

Relative multifractal analysis is a natural framework to finely describe geometrically the heterogeneity in the distribution at small scales of the elements of compactly supported Borel positive and finite measures on $\mathbb{R}^n$. Specifically, this heterogeneity can be described via the lower and upper local dimensions of a measure μ with respect to an arbitrary probability measure ν , namely

$$\underline{\alpha}_{\mu,\nu}(x) = \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} \quad \text{and} \quad \overline{\alpha}_{\mu,\nu}(x) = \limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))}.$$

If $\underline{\alpha}_{\mu,\nu}(x) = \overline{\alpha}_{\mu,\nu}(x)$, we refer to the common value as the local dimension of μ with respect to ν at x , and we denote it by $\alpha_{\mu,\nu}(x)$. For $\alpha \geq 0$, let us introduce the fractal sets which are also very natural, and the most studied in the literature,

$$E_\alpha = \left\{ x \in \text{supp } \mu \cap \text{supp } \nu; \quad \underline{\alpha}_{\mu,\nu}(x) \geq \alpha \right\},$$

$$\overline{E}^\alpha = \left\{ x \in \text{supp } \mu \cap \text{supp } \nu; \quad \overline{\alpha}_{\mu,\nu}(x) \leq \alpha \right\}$$

and

$$E(\alpha) = E_\alpha \cap \overline{E}^\alpha.$$

Inspired by the observations made by physicists of turbulence and statistical mechanics, mathematicians derived, and in many situations justified the heuristic claiming that for a measure possessing a self-conformal like property, its Hausdorff spectrum should be obtained as the Legendre transform of a kind of free energy function called L^q -spectrum. This gave birth to the abundant literature on the so-called relative multifractal formalisms, which aim at linking the asymptotic statistical properties of a given measure with its fine geometric properties. One of the main importance of the relative multifractal measures $\mathcal{H}_{\mu,\nu}^{q,t}$ and $\mathcal{P}_{\mu,\nu}^{q,t}$, and the corresponding dimension functions $b_{\mu,\nu}$, $B_{\mu,\nu}$, and $\Lambda_{\mu,\nu}$ is due to the fact that the ν -multifractal spectra functions $\dim_\nu$ and Dim_ν are bounded above by the Legendre transforms of $b_{\mu,\nu}$ and $B_{\mu,\nu}$, respectively, i.e.,

$$\dim_\nu(E(\alpha)) \leq b_{\mu,\nu}^*(\alpha) \quad \text{and} \quad \text{Dim}_\nu(E(\alpha)) \leq B_{\mu,\nu}^*(\alpha) \quad \text{for all } \alpha \geq 0.$$

These inequalities may be viewed as rigorous versions of the multifractal formalism. Furthermore, for many natural families of measures we have

$$\dim_\nu(E(\alpha)) = \text{Dim}_\nu(E(\alpha)) = b_{\mu,\nu}^*(\alpha) = B_{\mu,\nu}^*(\alpha), \quad \text{where } \alpha = -B_{\mu,\nu}'(q).$$

The interest of mathematicians in singularly continuous measures and probability distributions were fairly weak, which can be explained, on the one hand, by the absence of adequate analytic apparatus for specification and investigation of these measures, and, on the other hand, by a widespread opinion about the absence of applications of these measures. Due to the fractal explosion and a deep connection between the theory of fractals and singular measures, the situation has radically changed in the last years. It was proved that singular distributions of probabilities are dominant for many classes of random variables. Possible applications in the spectral theory of self-adjoint operators serve as an additional stimulus for a further investigation of singularly continuous measures. The authors in [11, 12, 15, 23] provided some examples of the mutual singularity of multifractal Hausdorff and packing measures for graph directed self-similar measures in $\mathbb{R}^n$ with totally disconnected support, cookie-cutter measures [15], for some homogeneous Moran measures [11, 12], in the spacial case where ν is the Lebesgue measure $\mathcal{L}^n$. Also, in [8, 9], M. Das proved that the relative multifractal Hausdorff and packing measures are mutually singular for the self-similar measures satisfying the significantly weaker open set condition. The aim of this article is to show that the relative multifractal Hausdorff and packing measures are mutually singular in a more general setting. The results in this paper generalize many known results and in particular provides a positive answer to Olsen's questions. Our main results apply to quasi-Bernoulli measures.

These more general results are stated as follows:

Theorem 1.

1. Assume that $b_{\mu,\nu} = B_{\mu,\nu}$ and $B_{\mu,\nu}$ is differentiable at p and q with $B_{\mu,\nu}'(p) \neq B_{\mu,\nu}'(q)$. Then

$$\mathcal{H}_{\mu,\nu}^{p,b_{\mu,\nu}(p)} \perp \mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)} \quad \text{on } \text{supp } \mu \cap \text{supp } \nu.$$

2. Assume that $B_{\mu,\nu}$ is differentiable at p and q and there exists a Gibbs measure ν_q for (μ, ν) at a state $(q, B_{\mu,\nu}(q))$, i.e., the existence of a measure ν_q on $\text{supp } \mu \cap \text{supp } \nu$ and constants $\underline{C}, \bar{C} > 0$ with $\bar{C} = \underline{C}^{-1}$ and $\delta > 0$ such that for every $x \in \text{supp } \mu \cap \text{supp } \nu$ and every $0 < r < \delta$,

$$\underline{C} \mu(B(x, r))^q \nu(B(x, r))^{B_{\mu,\nu}(q)} \leq \nu_q(B(x, r)) \leq \bar{C} \mu(B(x, r))^q \nu(B(x, r))^{B_{\mu,\nu}(q)}.$$

Then, for all $p, q \in \mathbb{R}$ with $B_{\mu,\nu}'(p) \neq B_{\mu,\nu}'(q)$ we have

$$\mathcal{H}_{\mu,\nu}^{p,b_{\mu,\nu}(p)} \perp \mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)} \quad \text{and} \quad \mathcal{P}_{\mu,\nu}^{p,B_{\mu,\nu}(p)} \perp \mathcal{P}_{\mu,\nu}^{q,B_{\mu,\nu}(q)} \quad \text{on } \text{supp } \mu \cap \text{supp } \nu.$$

3. The previous assertions hold if we replace the multifractal function $B_{\mu,\nu}$ by the function $\Lambda_{\mu,\nu}$.

2 Proof of the main results

1. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and $x \in \mathbb{R}$, we denote the left and right derivative of f at x (if they exist) by $D_-f(x)$ and $D_+f(x)$. Let $K := \text{supp } \mu \cap \text{supp } \nu$. The proof of Assertion (1) it follows from the following lemma.

Lemma 1. *Let $q \in \mathbb{R}$. If $b_{\mu,\nu}$ is convex, then we have for $\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}$ -a.a.x*

$$-D_+b_{\mu,\nu}(q) \leq \bar{\alpha}_{\mu,\nu}(x) \quad \text{and} \quad -D_-b_{\mu,\nu}(q) \geq \underline{\alpha}_{\mu,\nu}(x).$$

Proof. Let us prove the first inequality. The proof of the second statement is identical to the proof of the statement in the first inequality and is therefore omitted. Write $a = D_+b_{\mu,\nu}(q)$. Fix $\varepsilon, \eta > 0$ and let

$$E = \left\{ x \in K : \liminf_{r \rightarrow 0} \frac{\mu(B(x, r))}{\nu(B(x, r))^{-a-\varepsilon}} > \eta \right\}.$$

Then it is sufficient to prove that $\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) = 0$. It follows from the convexity of $b_{\mu,\nu}(q)$ that there exists $h > 0$ such that

$$\frac{b_{\mu,\nu}(q+h) - b_{\mu,\nu}(q)}{h} < a + \varepsilon$$

and thus

$$b_{\mu,\nu}(q+h) < b_{\mu,\nu}(q) + h(a + \varepsilon).$$

It results that

$$\mathcal{H}_{\mu,\nu}^{q+h,b_{\mu,\nu}(q)+h(a+\varepsilon)}(K) = 0. \quad (2.1)$$

Also observe that for each $x \in E$, there exists $r_x > 0$ such that

$$\mu(B(x, r))^q \nu(B(x, r))^{b_{\mu,\nu}(q)} \leq \eta^{-h} \mu(B(x, r))^{q+h} \nu(B(x, r))^{b_{\mu,\nu}(q)+h(a+\varepsilon)}, \quad (2.2)$$

for all $0 < r < r_x$. It follows easily from (2.1) and (2.2) that

$$\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E) \leq \eta^{-h} \mathcal{H}_{\mu,\nu}^{q+h,b_{\mu,\nu}(q)+h(a+\varepsilon)}(E) = 0.$$

□

Let us return to the proof of Assertion (1). Since $b_{\mu,\nu}(q) = B_{\mu,\nu}(q)$ for all $q \in \mathbb{R}$, it follows from Lemma 1 that

$$\lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} = -B'_{\mu,\nu}(q), \quad \mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)} \text{ - a.e.}$$

It holds that

$$\mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)}(E(-B'_{\mu,\nu}(q))) = 1.$$

Consequently, if $p, q \in \mathbb{R}$ with $B'_{\mu,\nu}(p) \neq B'_{\mu,\nu}(q)$, then

$$\mathcal{H}_{\mu,\nu}^{p,b_{\mu,\nu}(p)} \perp \mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)} \quad \text{on } K.$$

2. We present some tools, as well as lemmas, which will be used in the proof of our main result.

Lemma 2. *For any $q \in \mathbb{R}$, we have*

$$\underline{C} \nu_q \leq \mathcal{H}_{\mu,\nu}^{q,b_{\mu,\nu}(q)} \quad \text{on } K.$$

Proof. Fix $\delta > 0$ and let $(B(x_i, r_i))_{i \in \mathbb{N}}$ be a centered δ -covering of K . One gets

$$\begin{aligned} \nu_q(K) &\leq \sum_i \nu_q(B(x_i, r_i)) \\ &\leq \bar{C} \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^{b_{\mu,\nu}(q)} \\ &= \bar{C} \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^{b_{\mu,\nu}(q)}. \end{aligned}$$

Therefore

$$\underline{C} \nu_q(K) \leq \overline{\mathcal{H}}_{\mu, \nu, \delta}^{q, b_{\mu, \nu}(q)}(K) \leq \overline{\mathcal{H}}_{\mu, \nu}^{q, b_{\mu, \nu}(q)}(K) \leq \mathcal{H}_{\mu, \nu}^{q, b_{\mu, \nu}(q)}(K).$$

which achieves the proof of Lemma 2.

Lemma 3. For any $q \in \mathbb{R}$, we have

$$\mathcal{D}_{\mu, \nu}^{q, B_{\mu, \nu}(q)} \leq \overline{C} \nu_q \quad \text{on } K.$$

Proof. Let F be a closed subset of K . For $\delta > 0$ write

$$B(F, \delta) = \{x \in K; \quad \text{dist}(x, F) \leq \delta\}.$$

When δ tend to 0, the set $B(F, \delta)$ decreases to F . Then for all $\varepsilon > 0$, we can choose $\delta_0 > 0$ satisfying

$$\nu_q(B(F, \delta)) \leq \nu_q(F) + \varepsilon, \quad \forall 0 < \delta < \delta_0.$$

Fix $\delta > 0$ and let $(B(x_i, r_i))_i$ be a centered δ -packing of F . Then, one has

$$\begin{aligned} \underline{C} \sum_i \mu(B(x_i, r_i))^q \nu(B(x_i, r_i))^{B_{\mu, \nu}(q)} &\leq \sum_i \nu_q(B(x_i, r_i)) \\ &\leq \nu_q(B(F, \delta)) \leq \nu_q(F) + \varepsilon \\ &\leq \nu_q(K) + \varepsilon. \end{aligned}$$

Which leads to

$$\overline{\mathcal{D}}_{\mu, \nu}^{q, B_{\mu, \nu}(q)}(F) \leq \overline{C} (\nu_q(K) + \varepsilon).$$

As ε tend to 0, we can conclude that

$$\mathcal{D}_{\mu, \nu}^{q, B_{\mu, \nu}(q)}(K) \leq \overline{\mathcal{D}}_{\mu, \nu}^{q, B_{\mu, \nu}(q)}(K) \leq \overline{C} \nu_q(K)$$

which proves the desired result.

Now, let us prove Assertion (2). By using Lemmas 2 and 3, we have

$$\underline{C} \nu_q \leq \mathcal{H}_{\mu, \nu}^{q, b_{\mu, \nu}(q)} \leq \xi \mathcal{D}_{\mu, \nu}^{q, B_{\mu, \nu}(q)} \leq \xi \overline{C} \nu_q \quad \text{on } K.$$

It results that

$$\frac{1}{\xi \overline{C}} \mathcal{H}_{\mu, \nu}^{q, b_{\mu, \nu}(q)} \leq \nu_q \leq \frac{1}{\underline{C}} \mathcal{H}_{\mu, \nu}^{q, b_{\mu, \nu}(q)} \quad \text{on } K$$

and

$$\frac{1}{\overline{C}} \mathcal{D}_{\mu, \nu}^{q, B_{\mu, \nu}(q)} \leq \nu_q \leq \frac{\xi}{\underline{C}} \mathcal{D}_{\mu, \nu}^{q, B_{\mu, \nu}(q)} \quad \text{on } K.$$

Since ν_q is the Gibbs measure for (μ, ν) at $(q, B_{\mu, \nu}(q))$ and if we assume that $B_{\mu, \nu}$ is differentiable, by similar technics in [15, 16, 21], we have

$$\lim_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log \nu(B(x, r))} = -B'_{\mu, \nu}(q), \quad \nu_q - \text{a.e.}$$

This implies that $\nu_q(E(-B'_{\mu, \nu}(q))) = 1$. We therefore infer that if $p, q \in \mathbb{R}$ with $B'_{\mu, \nu}(p) \neq B'_{\mu, \nu}(q)$, then

$$\nu_p \perp \nu_q.$$

This completes the proof of Assertion (2).

3. The proof of Assertion (3) is identical to the proof of the statement in the second assertion and is therefore omitted.

3 An example

Let $\mathcal{F} = \bigcup_{n \geq 1} \mathcal{F}_n$ such that $\mathcal{F}_n$ stands for a sequence of the 5-adic intervals. If x belongs to $[0, 1[$, $I_n(x)$ stands for the interval $\mathcal{F}_n$ which contains x . Now, considering $I = I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}$ and $J = I_{\varepsilon'_1 \varepsilon'_2 \dots \varepsilon'_p}$, we set

$$IJ = I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \varepsilon'_1 \varepsilon'_2 \dots \varepsilon'_p}.$$

A probability measure on $[0, 1[$ is said to be quasi-Bernoulli if there exists $C > 0$ such that, for any $I, J \in \mathcal{F}$, one has

$$\frac{1}{C} \mu(I)\mu(J) \leq \mu(IJ) \leq C \mu(I)\mu(J).$$

We say that the quasi-Bernoulli measure μ has a strong separation condition if

$$\mu \in \mathcal{D} := \left\{ \begin{array}{ll} \mu(I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}) = 0, & \text{if } \exists i, \varepsilon_i \notin \{1, 3\} \\ \mu(I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}) \neq 0, & \text{if } \forall i, \varepsilon_i \in \{1, 3\} \end{array} \right\}.$$

Throughout this section, we assume that both μ and ν are two quasi-Bernoulli measures that have the above strong separation condition. For any $q, t \in \mathbb{R}$, one defines

$$\mathcal{H}_{\mu, \nu}(q, t) = \limsup_{\delta \rightarrow 0} \sup \left\{ \sum_j^* \mu(I_j)^q \nu(I_j)^t; I_j \in \mathcal{F}, |I_j| \leq \delta, I_i \cap I_j = \emptyset, \forall i \neq j \right\},$$

where the star means that the terms for which $\mu(I_j) = 0$ and $\nu(I_j) = 0$ are removed, and let

$$\tau_{\mu, \nu}(q) = \sup \left\{ t \in \mathbb{R}; \mathcal{H}_{\mu, \nu}(q, t) = +\infty \right\}.$$

In the next lemma we investigate the relationship between the multifractal functions $\tau_{\mu, \nu}(q)$ and $\Lambda_{\mu, \nu}(q)$.

Lemma 4. We have $\Lambda_{\mu, \nu}(q) = \tau_{\mu, \nu}(q)$.

Proof. It is easy to see that if $\mu, \nu \in \mathcal{D}$, then $\text{supp } \mu = \text{supp } \nu$ and

$$K = \bigcap_{n \geq 1} \bigcup_{\substack{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \\ \varepsilon_i \in \{1, 3\}}} I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}.$$

Let $\delta > 0$, $t > \tau_{\mu, \nu}(q)$ and $(B(x_j, r_j))_j$ be a δ -packing of K . Fix j , since $x_j \in K$, there exists $n_j \in \mathbb{N}^*$ such that

$$\frac{1}{5^{n_j}} \leq r_j < \frac{1}{5^{n_j-1}},$$

which implies that

$$I_{n_j}(x_j) \subseteq B(x_j, r_j).$$

Also, each $B(x_j, r_j)$ is covered by at most three 5-adic intervals $I_{j_1}, I_{n_j-1}(x_j), I_{j_2}$. Moreover, the strong separation condition ensures that

$$B(x_j, r_j) \subseteq I_{n_j-1}(x_j).$$

From the construction of measures μ and ν that, there exists $C, C' > 0$ such that

$$\mu(B(x_j, r_j)) \leq C \mu(I_{n_j}(x_j))$$

and

$$\nu(B(x_j, r_j)) \leq C' \nu(I_{n_j}(x_j)).$$

It results that, there exists a δ -packing $(I_{n_j})_j$ of $[0, 1[$ such that

$$\mu(I_{n_j}) \leq \mu(B(x_j, r_j)) \leq C \mu(I_{n_j}) \quad \text{and} \quad \nu(I_{n_j}) \leq \nu(B(x_j, r_j)) \leq C' \nu(I_{n_j}). \quad (3.1)$$

Therefore, one gets

$$\begin{aligned} \sum_j \mu(B(x_j, r_j))^q v(B(x_j, r_j))^t &\leq \gamma_{q,t} \sum_j \mu(I_{n_j})^q v(I_{n_j})^t \\ &\leq \gamma_{q,t} \sup_j \sum_j \mu(I_{n_j})^q v(I_{n_j})^t, \end{aligned}$$

where $\gamma_{q,t}$ is a constant depends only on q and t . It results that

$$\overline{\mathcal{P}}_{\mu,v}^{q,t}(K) \leq \gamma_{q,t} \mathcal{H}_{\mu,v}(q, t) < +\infty.$$

It follows from this that

$$\Lambda_{\mu,v}(q) \leq t,$$

and we thus deduce that

$$\Lambda_{\mu,v}(q) \leq \tau_{\mu,v}(q).$$

Now we will prove the other inequality. For $\delta > 0$, let $(I_j)_j$ be a δ -packing of $[0, 1[$ and $t > \Lambda_{\mu,v}(q)$. For any j there exists $n_j \in \mathbb{N}^*$ such that $I_j \in \mathcal{F}_{n_j}$, $x_j \in I_j \cap K$,

$$I_j \subseteq B(x_j, 5^{-n_j}), \quad \mu(I_j) = \mu(B(x_j, 5^{-n_j})) \quad \text{and} \quad v(I_j) = v(B(x_j, 5^{-n_j})).$$

The strong separation condition implies that $(B(x_j, 5^{-n_j}))_j$ is a δ -packing of K . Then, one has

$$\begin{aligned} \sum_j \mu(I_j)^q v(I_j)^t &\leq \sum_j \mu(B(x_j, 5^{-n_j}))^q v(B(x_j, 5^{-n_j}))^t \\ &\leq \overline{\mathcal{P}}_{\mu,v,\delta}^{q,t}(K), \end{aligned}$$

and we therefore deduce that

$$\sup_j \sum_j \mu(I_j)^q v(I_j)^t \leq \overline{\mathcal{P}}_{\mu,v,\delta}^{q,t}(K) \quad \text{and} \quad \mathcal{H}_{\mu,v}(q, t) \leq \overline{\mathcal{P}}_{\mu,v}^{q,t}(K) < +\infty.$$

Which means that

$$\tau_{\mu,v}(q) \leq \Lambda_{\mu,v}(q).$$

□

Now, by similar technics in [17], we can prove that if $\mu, v \in \mathcal{D}$, then there exists a measure ν_q such that for all $I \in \left\{ I_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n} \in \mathcal{F}_n; \quad \forall i, \quad \varepsilon_i \in \{1, 3\} \right\}$,

$$\frac{1}{C} \mu(I)^q v(I)^{\tau_{\mu,v}(q)} \leq \nu_q(I) \leq C \mu(I)^q v(I)^{\tau_{\mu,v}(q)}, \quad C > 0.$$

From Lemma 4 and (3.1), there exist two constants $\underline{C} > 0$ and $\overline{C} > 0$ such that

$$\underline{C} \mu(B(x, r))^q v(B(x, r))^{\Lambda_{\mu,v}(q)} \leq \nu_q(B(x, r)) \leq \overline{C} \mu(B(x, r))^q v(B(x, r))^{\Lambda_{\mu,v}(q)}$$

for all $x \in K$ and $0 < r < 1$. Assume that the function $\Lambda_{\mu,v}$ is differentiable at $q \in \mathbb{R}$. Now, it follows from Assertion (3) of Theorem 1 that, for any $p, q \in \mathbb{R}$ with $\Lambda'_{\mu,v}(p) \neq \Lambda'_{\mu,v}(q)$,

$$\mathcal{H}_{\mu,v}^{p, b_{\mu,v}(p)} \perp \mathcal{H}_{\mu,v}^{q, b_{\mu,v}(q)} \quad \text{and} \quad \mathcal{P}_{\mu,v}^{p, B_{\mu,v}(p)} \perp \mathcal{P}_{\mu,v}^{q, B_{\mu,v}(q)} \quad \text{on } K.$$

Remarks 1.

1. We note that our results, due to the use of the relative multifractal Hausdorff and packing measures introduced in [6], appear as natural multifractal generalizations of some of the main results in [11, 12, 15, 23].

2. The interesting case is, of course, the case where the measure ν is different from the normalized Lebesgue measure $\mathcal{L}^n$ on an open and bounded set containing the support of μ . If ν is the normalized Lebesgue measure $\mathcal{L}^n$ then our main results follow immediately from the (substantially more general) theorems in [11, 12, 15, 23] (provided that certain conditions are satisfied).
3. All the above results hold if we replace the centered δ -coverings (δ -packings) by the centered ν - δ -coverings (ν - δ -packings) and we suppose that the measure ν satisfies the following condition

For any $0 < \lambda < 1$ given, there exists $\delta > 0$, such that if $\nu(B(x, r)) \leq \delta$
for every $x \in \text{supp } \nu$, then $r \leq \lambda$.

Note that this assumption is not restrictive, as it encompasses a fairly broad class of measures, namely: quasi-Bernoulli measures, inhomogeneous Bernoulli measures and homogeneous Moran measures, etc. The reader is referred to [14] for a systematic discussion of these measures.

4. Let (X, d) be a metric space and $\mathcal{B}$ stand for the set of balls of X and $\mathcal{F}$ for the set of maps from $\mathcal{B}$ to $[0, +\infty)$. The set of $\mu \in \mathcal{F}$ such that $\mu(B) = 0$ implies $\mu(B') = 0$ for all $B' \subseteq B$ will be denoted by $\mathcal{F}^*$. For such a μ , one defines its support $\text{supp } \mu$ to be the complement of the set $\bigcup\{B \in \mathcal{B} \mid \mu(B) = 0\}$. Then, all the above results hold for any $\mu \in \mathcal{F}^*$.
5. Our main results in Theorem 1 also hold for the vectorial multifractal measures introduced by Peyrière in [18], the φ -Mixed multifractal measures introduced in [13] and the relative multifractal Hausdorff measure and the multifractal packing measure in a probability space [7].

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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