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Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations

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Abstract: In this paper, we prove some existence results of solutions for a class of nonlocal initial value problem for nonlinear fractional hybrid implicit differential equations under generalized Hilfer fractional derivative. The result is based on a fixed point theorem on Banach algebras. Further, examples are provided to illustrate our results.

Keywords: Generalized Hilfer fractional derivative, initial value problem, nonlocal, existence, hybrid fractional differential equations, implicit differential equations, fixed point

MSC: 34A08, 26A33

1 Introduction

Fractional calculus is a branch of classical mathematics, which is concerned with the generalization of the integer order differentiation and integration of a function to non-integer order, its is a solid and growing field both in theory and in its applications [2–4, 15, 32]. In the last few decades, fractional differentiation and fractional integration have found many applications in various fields of science and engineering. There are numerous kinds of fractional derivatives, such as, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hilfer fractional derivative, Hadamard fractional derivative, Erdélyi-Kober, Katugampola and others; see [1, 5–8, 10, 11, 14, 21, 26–29] and the references therein. For some recent applications, see [22, 23, 25].

Another interesting class of problems involves hybrid fractional differential equations and has received attention of several researchers [9, 12, 19, 31].

In [13], the authors discussed the following terminal value problem for fractional differential equations with generalized Hilfer fractional derivative :

$$\begin{cases} \left({}^{\alpha}\mathbb{D}_{a^{+}}^{g,r}x \right)(t) = f\left(t, x(t), \left({}^{\alpha}\mathbb{D}_{a^{+}}^{g,r}x \right)(t)\right), & t \in I := [a, T], \quad a > 0, \\ x(T) = c \in \mathbb{R}. \end{cases}$$

Their reasoning is mainly based upon different types of classical fixed point theory such as the Banach contraction principle and the Krasnoselskii fixed point theorem.

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Using Krasnoselskii, Schaefer and Schauder fixed point theorems, Wang and Zhang [30] proved some existence results for the following nonlocal initial value problem for differential equations involving Hilfer fractional derivative :

$$\begin{cases} D_{a^+}^{\vartheta,r} u(t) = f(t, u(t)), & t \in (a, b], \\ \left(I_{a^+}^{1-\xi} u \right) (a^+) = \sum_{i=1}^m \lambda_i u(\tau_i). \end{cases}$$

Derbazi *et al.* [16] studied the existence and uniqueness of solutions of the following three-point boundary value problem for fractional hybrid differential equations with Caputo fractional derivative :

$$\begin{cases} {}^c\mathcal{D}_{0^+}^{\vartheta} \left(\frac{u(t)-f(t,u(t))}{g(t,u(t))} \right) = h(t, u(t)), & t \in J_1 := [0, T], \\ a_1 \left(\frac{u(0)-f(0,u(0))}{g(0,u(0))} \right) + b_1 \left(\frac{u(T)-f(T,u(T))}{g(T,u(T))} \right) = c_1, \\ a_2 {}^c\mathcal{D}_{0^+}^{\beta} \left(\frac{u(t)-f(t,u(t))}{g(t,u(t))} \right)_{t=\eta} + b_2 {}^c\mathcal{D}_{0^+}^{\beta} \left(\frac{u(t)-f(t,u(t))}{g(t,u(t))} \right)_{t=T} = c_2. \end{cases}$$

The proved results rely on a hybrid fixed point theorem for a sum of three operators due to Dhage.

Motivated by the works of the papers mentioned above, we establish in this paper, existence results to the nonlocal initial value problem (IVP for short) with nonlinear implicit hybrid generalized Hilfer type fractional differential equation :

$${}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) = \varphi \left(t, x(t), {}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) \right), \quad t \in (a, b], \quad (1)$$

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (a^+) = \sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right), \quad (2)$$

where ${}^{\alpha}\mathbb{D}_{a^+}^{\vartheta,r}$, ${}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}$ are the generalized Hilfer fractional derivative of order $\vartheta \in (0, 1)$ and type $r \in [0, 1]$ and generalized fractional integral of order $1 - \xi$, ($\xi = \vartheta + r - \vartheta r$) respectively, $c_i, i = 1, \dots, m$, are real numbers, $\epsilon_i, i = 1, \dots, m$, are pre-fixed points satisfying $a < \epsilon_1 \leq \dots \leq \epsilon_m < b$, $f \in C([a, b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\chi \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $\varphi \in C([a, b] \times \mathbb{R}^2, \mathbb{R})$. Further details and definitions are given in Section 2.

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about generalized Hilfer fractional derivative and auxiliary results. In Section 3, an existence result for the problem (1)-(2) is presented which is based on a fixed point theorem in Banach algebras [17, 18]. Finally, in the last section, we give an example to illustrate the applicability of our results.

2 Preliminaries

In this section, we introduce some preliminary facts, essential notations, definitions and results which are used throughout this paper. Let $0 < a < b$, $J = [a, b]$, and

$$C_{\xi,a}(J) = \left\{ x : (a, b] \rightarrow \mathbb{R} : \text{the function } t \rightarrow \Psi_{\xi}(t, a)x(t) \in C(J, \mathbb{R}) \right\}, \quad 0 \leq \xi < 1,$$

where

$$\Psi_{\xi}(t, a) = \alpha^{\xi-1} (t^{\alpha} - a^{\alpha})^{1-\xi},$$

and

$$\begin{aligned} C_{\xi,a}^n(J) &= \left\{ x \in C^{n-1}(J) : x^{(n)} \in C_{\xi,a}(J) \right\}, \quad n \in \mathbb{N}, \\ C_{\xi,a}^0(J) &= C_{\xi,a}(J). \end{aligned}$$

with the norm

$$\|x\|_{C_{\xi,a}} = \sup_{t \in J} |\Psi_{\xi}(t, a)x(t)|.$$

Consider the space $X_c^p(a, b)$, ($c \in \mathbb{R}$, $1 \leq p \leq \infty$) of those complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(a, b)$ coincides with the $L^p(a, b)$ space: $X_{\frac{1}{p}}^p(a, b) = L^p(a, b)$.

Definition 2.1. [20] Let $\vartheta \in \mathbb{R}_+$, $c \in \mathbb{R}$ and $h \in X_c^p(a, b)$. The generalized fractional integral of order ϑ is defined by

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} h \right) (t) = \int_a^t \tau^{\alpha-1} \bar{\Psi}_{\vartheta}(t, \tau) h(\tau) d\tau, \quad t > a, \alpha > 0,$$

where

$$\bar{\Psi}_{\vartheta}(t, \tau) = \frac{\alpha^{1-\vartheta}}{\Gamma(\vartheta)} (t^{\alpha} - \tau^{\alpha})^{\vartheta-1}.$$

If $h(t) = 1$, we obtain

$$\begin{aligned} \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} h \right) (t) &= \int_a^t \tau^{\alpha-1} \bar{\Psi}_{\vartheta}(t, \tau) d\tau \\ &= \frac{1}{\Gamma(\vartheta+1)} \left(\frac{t^{\alpha} - a^{\alpha}}{\alpha} \right)^{\vartheta} \\ &= \bar{\Psi}_{\vartheta+1}(t, a). \end{aligned}$$

Definition 2.2. [20] Let $\vartheta \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\alpha > 0$. The generalized fractional derivative ${}^{\alpha} \mathbb{D}_{a^+}^{\vartheta}$ of order ϑ is defined by

$$\begin{aligned} \left({}^{\alpha} \mathbb{D}_{a^+}^{\vartheta} h \right) (t) &= \delta_{\alpha}^n ({}^{\alpha} \mathbb{J}_{a^+}^{n-\vartheta} h)(t) \\ &= \left(t^{1-\alpha} \frac{d}{dt} \right)^n \int_a^t \tau^{\alpha-1} \bar{\Psi}_{n-\vartheta}(t, \tau) h(\tau) d\tau, \quad t > a, \alpha > 0, \end{aligned}$$

where $n = [\vartheta] + 1$ and $\delta_{\alpha}^n = \left(t^{1-\alpha} \frac{d}{dt} \right)^n$.

Lemma 2.3. [20, 24] Let $\vartheta > 0$, $0 \leq \xi < 1$. Then, ${}^{\alpha} \mathbb{J}_{a^+}^{\vartheta}$ is bounded from $C_{\xi, \alpha}(J)$ into $C_{\xi, \alpha}(J)$.

Lemma 2.4. [24] Let $\vartheta > 0$, $0 \leq \xi < 1$, and $h \in C_{\xi, \alpha}(J)$. Then,

$$\left({}^{\alpha} \mathbb{D}_{a^+}^{\vartheta} {}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} h \right) (t) = h(t).$$

Lemma 2.5. [24] Let $0 < \vartheta < 1$, $0 \leq \xi < 1$. If $h \in C_{\xi, \alpha}(J)$ and ${}^{\alpha} \mathbb{J}_{a^+}^{1-\vartheta} h \in C_{\xi, \alpha}^1(J)$, then

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} {}^{\alpha} \mathbb{D}_{a^+}^{\vartheta} h \right) (t) = h(t) - \bar{\Psi}_{\vartheta}(t, a) \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\vartheta} h \right) (a), \quad \text{for all } t \in (a, b].$$

Definition 2.6. [24] Let order ϑ and type r satisfy $n-1 < \vartheta < n$ and $0 \leq r \leq 1$, with $n \in \mathbb{N}$. The generalized Hilfer type fractional derivative with $\alpha > 0$ of a function $h \in C_{\xi, \alpha}(J)$, is defined by

$$\left({}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} h \right) (t) = \left({}^{\alpha} \mathbb{J}_{a^+}^{r(n-\vartheta)} \left(t^{\alpha-1} \frac{d}{dt} \right)^n {}^{\alpha} \mathbb{J}_{a^+}^{(1-r)(n-\vartheta)} h \right) (t)$$

$$= \left({}^{\alpha} \mathbb{J}_{a^+}^{r(n-\vartheta)} \delta_a^n {}^{\alpha} \mathbb{J}_{a^+}^{(1-r)(n-\vartheta)} h \right) (t).$$

Property 2.7. [24] The operator ${}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r}$ can be written as

$${}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} = {}^{\alpha} \mathbb{J}_{a^+}^{r(1-\vartheta)} \delta_a {}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} = {}^{\alpha} \mathbb{J}_{a^+}^{r(1-\vartheta)} {}^{\alpha} \mathbb{D}_{a^+}^{\xi},$$

where $0 \leq r \leq 1$ and $\xi = \vartheta + r - \vartheta r$.

Consider the following parameters ϑ, r, ξ satisfying

$$\xi = \vartheta + r - \vartheta r, \quad 0 < \vartheta, r, \xi < 1.$$

We define the spaces

$$C_{\xi, \alpha}^{\vartheta, r}(J) = \left\{ x \in C_{\xi, \alpha}(J), {}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} x \in C_{\xi, \alpha}(J) \right\},$$

and

$$C_{\xi, \alpha}^{\xi}(J) = \left\{ x \in C_{\xi, \alpha}(J), {}^{\alpha} \mathbb{D}_{a^+}^{\xi} x \in C_{\xi, \alpha}(J) \right\}.$$

Since ${}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} x = {}^{\alpha} \mathbb{J}_{a^+}^{r(1-\vartheta)} {}^{\alpha} \mathbb{D}_{a^+}^{\xi} x$, it follows from Lemma 2.3 that

$$C_{\xi, \alpha}^{\xi}(J) \subset C_{\xi, \alpha}^{\vartheta, r}(J) \subset C_{\xi, \alpha}(J).$$

Lemma 2.8. [24] If $x \in C_{\xi, \alpha}^{\xi}(J)$, then

$${}^{\alpha} \mathbb{J}_{a^+}^{\xi} {}^{\alpha} \mathbb{D}_{a^+}^{\xi} x = {}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} {}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} x,$$

and

$${}^{\alpha} \mathbb{D}_{a^+}^{\xi} {}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} x = {}^{\alpha} \mathbb{D}_{a^+}^{r(1-\vartheta)} x.$$

Lemma 2.9. [17] Let B be a closed, convex, bounded and nonempty subset of a Banach algebra $(X, \|\cdot\|)$, and let $\mathcal{P}, \mathcal{R} : X \rightarrow X$ and $\mathcal{Q} : B \rightarrow X$ be three operators such that

- 1) $\mathcal{P}$ and $\mathcal{R}$ are Lipschitzian with Lipschitz constants η_1 and η_2 , respectively,
- 2) $\mathcal{Q}$ is compact and continuous,
- 3) $x = \mathcal{P}x\mathcal{Q}y + \mathcal{R}x \Rightarrow x \in B$ for all $y \in B$,
- 4) $\eta_1\beta + \eta_2 < 1$, where $\beta = \|\mathcal{Q}(B)\| = \sup\{\|\mathcal{Q}(y)\| : y \in B\}$.

Then the operator equation $\mathcal{P}x\mathcal{Q}x + \mathcal{R}x = x$ has a solution in B .

3 Existence of Solutions

Let $v : J \rightarrow \mathbb{R}$ be a function such that $v(\cdot) \in C_{\xi, \alpha}(J)$, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and the function $\chi \in C(J \times \mathbb{R}, \mathbb{R})$. We consider the following simpler fractional differential equation related to (1)–(2) with $\varphi \equiv v$.

$${}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) = v(t), \quad t \in (a, b], \quad (3)$$

where $0 < \vartheta < 1$, $0 \leq r \leq 1$, $\alpha > 0$, with the nonlocal condition

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (a^+) = \sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right), \quad (4)$$

where $\xi = \vartheta + r - \vartheta r$, $c_i, i = 1, \dots, m$, are real numbers, $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\chi \in C(J \times \mathbb{R}, \mathbb{R})$, $\epsilon_i, i = 1, \dots, m$, are pre-fixed points satisfying $a < \epsilon_1 \leq \dots \leq \epsilon_m < b$, $v \in C_{\xi, a}(J)$ and $\sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a) \neq 1$. The following theorem shows that the problem (3)-(4) have a solution given by

$$x(t) = f(t, x(t)) \left[\frac{\sum_{i=1}^m c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\epsilon_i)}{1 - \sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a)} + \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (t) \right] + \chi(t, x(t)). \quad (5)$$

Theorem 3.1. *The function x satisfies equations (3) and (4) if and only if it satisfies (5).*

Proof. Assume x satisfies the equations (3) and (4) and such that the function $\sigma : t \rightarrow \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) \in C_{\xi, a}^{\xi}(J)$. We prove that x is a solution to the equation (5). From the definition of the space $C_{\xi, a}^{\xi}(J)$ and by using Lemma 2.3 and Definition 2.2, we have

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (t) \in C_{\xi, a}(J),$$

and

$${}^{\alpha} \mathbb{D}_{a^+}^{\xi} \sigma(t) = \left(\delta_a {}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (t) \in C_{\xi, a}(J).$$

We have

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (t) \in C_{\xi, a}^1(J).$$

Hence, Lemma 2.5 implies that

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{\xi} {}^{\alpha} \mathbb{D}_{a^+}^{\xi} \sigma(\tau) \right) (t) = \sigma(t) - \bar{\Psi}_{\xi}(t, a) \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a), \text{ for all } t \in (a, b].$$

Using Lemma 2.8 we have

$$\begin{aligned} \left({}^{\alpha} \mathbb{J}_{a^+}^{\xi} {}^{\alpha} \mathbb{D}_{a^+}^{\xi} \sigma(\tau) \right) (t) &= \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} {}^{\alpha} \mathbb{D}_{a^+}^{\vartheta, r} \sigma(\tau) \right) (t) \\ &= \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (t). \end{aligned}$$

Then,

$$\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} = \bar{\Psi}_{\xi}(t, a) \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a) + \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (t),$$

which implies that

$$x(t) = f(t, x(t)) \left[\bar{\Psi}_{\xi}(t, a) \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (a) + \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (t) \right] + \chi(t, x(t)). \quad (6)$$

Next, we substitute $t = \epsilon_i$ into (6), then we multiply c_i to both sides, we obtain

$$c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) = c_i \bar{\Psi}_{\xi}(\epsilon_i, a) \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a) + c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\epsilon_i).$$

Then by using condition (4), we have

$$\begin{aligned} \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a^+) &= \sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) \\ &= \left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a) \sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a) + \sum_{i=1}^m c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\epsilon_i), \end{aligned}$$

which implies

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right) (a^+) = \frac{\sum_{i=1}^m c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\epsilon_i)}{1 - \sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a)}. \quad (7)$$

Substituting (7) into (6), we obtain (5).

Reciprocally, assume x satisfies the equation (5) such that the function $\sigma : t \rightarrow \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) \in C_{\xi, \alpha}^{\xi}(J)$.

Applying operator ${}^{\alpha}\mathbb{D}_{a^+}^{\xi}$ on both sides of (5), and since $f(t, x(t)) \neq 0$ for all $t \in J$, then, from Lemma 2.8 we obtain

$$({}^{\alpha}\mathbb{D}_{a^+}^{\xi} \sigma(\tau))(t) = \left({}^{\alpha}\mathbb{D}_{a^+}^{r(1-\vartheta)} v(\tau) \right)(t). \quad (8)$$

Since $\sigma \in C_{\xi, \alpha}^{\xi}(J)$ we have ${}^{\alpha}\mathbb{D}_{a^+}^{\xi} \sigma \in C_{\xi, \alpha}(J)$, then (8) implies that

$$({}^{\alpha}\mathbb{D}_{a^+}^{\xi} \sigma(\tau))(t) = \left(\delta_{\alpha} {}^{\alpha}\mathbb{J}_{a^+}^{1-r(1-\vartheta)} v(\tau) \right)(t) = \left({}^{\alpha}\mathbb{D}_{a^+}^{r(1-\vartheta)} v(\tau) \right)(t) \in C_{\xi, \alpha}(J). \quad (9)$$

As $v(\cdot) \in C_{\xi, \alpha}(J)$ and from Lemma 2.3, follows

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-r(1-\vartheta)} v \right) \in C_{\xi, \alpha}(J). \quad (10)$$

From (9), (10) and by the definition of the space $C_{\xi, \alpha}^n(J)$, we obtain

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-r(1-\vartheta)} v \right) \in C_{\xi, \alpha}^1(J).$$

Applying operator ${}^{\alpha}\mathbb{J}_{a^+}^{r(1-\vartheta)}$ on both sides of (9) and using Lemma 2.5 and Property 2.7, we have

$$\begin{aligned} \left({}^{\alpha}\mathbb{D}_{a^+}^{\vartheta, r} \sigma(\tau) \right)(t) &= {}^{\alpha}\mathbb{J}_{a^+}^{r(1-\vartheta)} \left({}^{\alpha}\mathbb{D}_{a^+}^{\xi} \sigma(\tau) \right)(t) \\ &= v(t) - \tilde{\Psi}_{r(1-\vartheta)}(t, a) \left({}^{\alpha}\mathbb{J}_{a^+}^{1-r(1-\vartheta)} v(\tau) \right)(a) \\ &= v(t), \end{aligned}$$

that is, (3) holds. Now, applying ${}^{\alpha}\mathbb{J}_{a^+}^{1-\xi}$ on both sides of (5) we get

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi} \sigma(\tau) \right)(t) = \frac{\sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^m c_i \tilde{\Psi}_{\xi}(\epsilon_i, a)} + \left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi+\vartheta} v(\tau) \right)(t). \quad (11)$$

Taking the limit $t \rightarrow a^+$ of (11) we obtain

$$\left({}^{\alpha}\mathbb{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right)(a^+) = \frac{\sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^m c_i \tilde{\Psi}_{\xi}(\epsilon_i, a)}. \quad (12)$$

Substituting $t = \epsilon_i$ into (5), we have

$$\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} = \tilde{\Psi}_{\xi}(\epsilon_i, a) \frac{\sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^m c_i \tilde{\Psi}_{\xi}(\epsilon_i, a)} + \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i).$$

Then, we have

$$\sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) = \sum_{i=1}^m c_i \tilde{\Psi}_{\xi}(\epsilon_i, a) \frac{\sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i)}{1 - \sum_{i=1}^m c_i \tilde{\Psi}_{\xi}(\epsilon_i, a)} + \sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right)(\epsilon_i),$$

thus,

$$\sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right) = \frac{\sum_{i=1}^m c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\theta} v(\tau) \right) (\epsilon_i)}{1 - \sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a)}. \quad (13)$$

From (12) and (13), we find that

$$\left({}^{\alpha} \mathbb{J}_{a^+}^{1-\xi} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (a^+) = \sum_{i=1}^m c_i \left(\frac{x(\epsilon_i) - \chi(\epsilon_i, x(\epsilon_i))}{f(\epsilon_i, x(\epsilon_i))} \right),$$

which shows that the initial condition (4) is satisfied.

Conditions on φ so that $\varphi(\cdot, x(\cdot), y(\cdot)) \in C_{\xi, \alpha}(J)$, for any $x, y \in C_{\xi, \alpha}(J)$ will be given below.

Lemma 3.2. Let $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\chi \in C(J \times \mathbb{R}, \mathbb{R})$. If the function $t \rightarrow \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) \in C_{\xi, \alpha}^{\xi}(J)$, then x satisfies the problem (1) – (2) if and only if x is the fixed point of the operator $\mathfrak{S} : C_{\xi, \alpha}(J) \rightarrow C_{\xi, \alpha}(J)$ defined by

$$\mathfrak{S}x(t) = f(t, x(t)) \left[K \bar{\Psi}_{\xi}(t, a) \sum_{i=1}^m c_i \left({}^{\alpha} \mathbb{J}_{a^+}^{\theta} v(\tau) \right) (\epsilon_i) + \left({}^{\alpha} \mathbb{J}_{a^+}^{\theta} v(\tau) \right) (t) \right] + \chi(t, x(t)), \quad (14)$$

where $K = \left[1 - \sum_{i=1}^m c_i \bar{\Psi}_{\xi}(\epsilon_i, a) \right]^{-1}$ and $v : J \rightarrow \mathbb{R}$ be a function satisfying the functional equation

$$v(t) = \varphi(t, x(t), v(t)).$$

Since the functions f and χ are continuous and $\varphi(\cdot, x(\cdot), y(\cdot)) \in C_{\xi, \alpha}(J)$, then, by Lemma 2.3, we have $\mathfrak{S}x \in C_{\xi, \alpha}(J)$.

(Ax1) $\varphi : J \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous on J and

$$\varphi(\cdot, x(\cdot), y(\cdot)) \in C_{\xi, \alpha}^{r(1-\theta)}(J), \text{ for any } x, y \in C_{\xi, \alpha}(J).$$

(Ax2) The functions $f : J \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$ and $\chi : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and there exist two functions $p, q \in C(J, [0, \infty))$ such that

$$|f(t, x) - f(t, \bar{x})| \leq p(t) \Psi_{\xi}(t, a) |x - \bar{x}|$$

and

$$|\chi(t, x) - \chi(t, \bar{x})| \leq q(t) |x - \bar{x}|$$

for any $x, \bar{x} \in \mathbb{R}$ and $t \in (a, b]$.

(Ax3) There exists functions $\lambda_1, \lambda_2, \lambda_3 \in C(J, [0, \infty))$ such that

$$|\varphi(t, x, y)| \leq \lambda_1(t) + \lambda_2(t) |x| + \lambda_3(t) |y| \text{ for } t \in (a, b], \text{ and } x, y \in \mathbb{R}.$$

(Ax4) There exists a number $\ell > 0$ such that

$$\ell \geq \frac{f^* M + \chi^*}{1 - p^* M - q^*}, \quad (15)$$

where

$$p^* = \sup_{t \in J} p(t), \quad q^* = \sup_{t \in J} q(t),$$

$$\lambda_i^* = \sup_{t \in J} \lambda_i(t), \quad i = 1, 2, \quad \lambda_3^* = \sup_{t \in J} \lambda_3(t) < 1,$$

$$f^* = \sup_{t \in J} |f(t, 0)|, \quad \chi^* = \sup_{t \in J} |\chi(t, 0)|, \quad \Lambda := \frac{\Psi_{\xi}(b, a) \lambda_1^* + \lambda_2^* \ell}{1 - \lambda_3^*}$$

and

$$M = \frac{\Lambda \alpha^{-\vartheta}}{\Gamma(\vartheta + \xi)} \left[|K| \alpha^{1-\xi} \sum_{i=1}^m |c_i| (\epsilon_i^\alpha - a^\alpha)^{\vartheta+\xi-1} + \Gamma(\xi) (b^\alpha - a^\alpha)^\vartheta \right].$$

Theorem 3.3. Assume (Ax1)–(Ax4) hold. If

$$\max\{p^* M, p^* \Psi_\xi(b, a)M\} + q^* < 1, \quad (16)$$

then the problem (1)–(2) has at least one solution in $C_{\xi, \alpha}(J)$.

Proof. We define a subset Ω of $C_{\xi, \alpha}(J)$ by

$$\Omega = \{x \in C_{\xi, \alpha}(J) : \|x\|_{\xi, \alpha} \leq \ell\}.$$

We consider the operator $\mathfrak{S}$ defined in (14), and define three operators $\mathfrak{S}, \mathfrak{N} : C_{\xi, \alpha}(J) \rightarrow C_{\xi, \alpha}(J)$, $\mathfrak{T} : \Omega \rightarrow C_{\xi, \alpha}(J)$ by

$$(\mathfrak{S}x)(t) = f(t, x(t)), \quad t \in (a, b], \quad (17)$$

$$(\mathfrak{N}x)(t) = \chi(t, x(t)), \quad t \in (a, b], \quad (18)$$

and

$$(\mathfrak{T}x)(t) = K \bar{\Psi}_\xi(t, a) \sum_{i=1}^m c_i \left({}^\alpha \mathbb{J}_{a^+}^\vartheta v(\tau) \right) (\epsilon_i) + \left({}^\alpha \mathbb{J}_{a^+}^\vartheta v(\tau) \right) (t), \quad t \in (a, b]. \quad (19)$$

Then we get $\mathfrak{S}x = \mathfrak{S}x\mathfrak{T}x + \mathfrak{N}x$.

Step 1: The operators $\mathfrak{S}$ and $\mathfrak{N}$ are Lipschitzian on $C_{\xi, \alpha}(J)$.

Let $x, y \in C_{\xi, \alpha}(J)$ and $t \in (a, b]$. Then by (Ax2) we have

$$\begin{aligned} |((\mathfrak{S}x)(t) - (\mathfrak{S}y)(t)) \Psi_\xi(t, a)| &\leq \Psi_\xi(t, a) |f(t, x) - f(t, y)|, \\ &\leq p(t) \Psi_\xi(t, a) \|x - y\|_{\xi, \alpha}, \\ &\leq p^* \Psi_\xi(b, a) \|x - y\|_{\xi, \alpha}, \end{aligned}$$

then for each $t \in (a, b]$ we obtain

$$\|\mathfrak{S}x - \mathfrak{S}y\|_{\xi, \alpha} \leq p^* \Psi_\xi(b, a) \|x - y\|_{\xi, \alpha}.$$

Also, for each $t \in (a, b]$ we have

$$\begin{aligned} |((\mathfrak{N}x)(t) - (\mathfrak{N}y)(t)) \Psi_\xi(t, a)| &\leq \Psi_\xi(t, a) |\chi(t, x(t)) - \chi(t, y(t))|, \\ &\leq q(t) \|x - y\|_{\xi, \alpha}, \\ &\leq q^* \|x - y\|_{\xi, \alpha}, \end{aligned}$$

then,

$$\|\mathfrak{N}x - \mathfrak{N}y\|_{\xi, \alpha} \leq q^* \|x - y\|_{\xi, \alpha}.$$

Step 2: The operator $\mathfrak{T}$ is completely continuous on Ω .

We firstly show that the operator $\mathfrak{T}$ is continuous on Ω . Let $\{x_n\}$ be sequence in Ω such that $x_n \rightarrow x$ in Ω .

Let $x, y \in C_{\xi, \alpha}(J)$.

Then for each $t \in (a, b]$, we have

$$|(\mathfrak{T}x_n)(t) - (\mathfrak{T}x)(t)| \Psi_\xi(t, a) \leq \frac{|K|}{\Gamma(\xi)} \sum_{i=1}^m |c_i| \left({}^\alpha \mathbb{J}_{a^+}^\vartheta |v_n(\tau) - v(\tau)| \right) (\epsilon_i)$$

$$+ \Psi_{\xi}(t, a) \left({}^a\mathbb{J}_{a^+}^{\vartheta} |v_n(\tau) - v(\tau)| \right) (t),$$

where $v_n, v \in C_{\xi, a}(J)$ such that

$$\begin{aligned} v_n(t) &= \varphi(t, x_n(t), v_n(t)), \\ v(t) &= \varphi(t, x(t), v(t)). \end{aligned}$$

Since $x_n \rightarrow x$ and φ is a continuous function on J then we get $v_n(t) \rightarrow v(t)$ as $n \rightarrow \infty$ for each $t \in (a, b]$, so by Lebesgue dominated convergence theorem, we have

$$\|\mathcal{T}x_n - \mathcal{T}x\|_{C_{\xi, a}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then $\mathcal{T}$ is continuous.

Next we prove that $\mathcal{T}(\Omega)$ is uniformly bounded on $C_{\xi, a}(J)$. Let any $x \in \Omega$. By (Ax3), we have for each $t \in (a, b]$

$$\begin{aligned} |\Psi_{\xi}(t, a)v(t)| &= |\Psi_{\xi}(t, a)\varphi(t, x(t), v(t))| \\ &\leq \Psi_{\xi}(t, a)(\lambda_1(t) + \lambda_2(t)|x(t)| + \lambda_3(t)|v(t)|) \\ &\leq \Psi_{\xi}(b, a)\lambda_1^* + \lambda_2^*\ell + \lambda_3^*|\Psi_{\xi}(t, a)v(t)|. \end{aligned}$$

Hence

$$|\Psi_{\xi}(t, a)v(t)| \leq \frac{\Psi_{\xi}(b, a)\lambda_1^* + \lambda_2^*\ell}{1 - \lambda_3^*}.$$

Then, we have

$$\sup_{t \in (a, b]} |\Psi_{\xi}(t, a)v(t)| \leq \frac{\Psi_{\xi}(b, a)\lambda_1^* + \lambda_2^*\ell}{1 - \lambda_3^*} := \Lambda.$$

For $t \in (a, b]$, by (19) we have

$$\begin{aligned} &|\Psi_{\xi}(t, a)(\mathcal{T}x)(t)| \\ &\leq \frac{|K|}{\Gamma(\xi)} \sum_{i=1}^m |c_i| \left({}^a\mathbb{J}_{a^+}^{\vartheta} |v(\tau)| \right) (\epsilon_i) + \Psi_{\xi}(t, a) \left({}^a\mathbb{J}_{a^+}^{\vartheta} |v(\tau)| \right) (t) \\ &\leq \Lambda |K| \sum_{i=1}^m |c_i| \left({}^a\mathbb{J}_{a^+}^{\vartheta} \bar{\Psi}_{\xi}(\tau, a) \right) (\epsilon_i) + \Lambda \Psi_{\xi}(t, a) \Gamma(\xi) \left({}^a\mathbb{J}_{a^+}^{\vartheta} \bar{\Psi}_{\xi}(\tau, a) \right) (t) \\ &\leq \Lambda |K| \sum_{i=1}^m |c_i| \bar{\Psi}_{\vartheta+\xi}(\epsilon_i, a) + \Lambda \Psi_{\xi}(t, a) \Gamma(\xi) \bar{\Psi}_{\vartheta+\xi}(t, a) \\ &\leq \frac{\Lambda |K|}{\Gamma(\vartheta + \xi)} \sum_{i=1}^m |c_i| \left(\frac{\epsilon_i^{\alpha} - a^{\alpha}}{\alpha} \right)^{\vartheta+\xi-1} + \frac{\Lambda \Gamma(\xi)}{\Gamma(\vartheta + \xi)} \left(\frac{b^{\alpha} - a^{\alpha}}{\alpha} \right)^{\vartheta}. \end{aligned}$$

Then we obtain

$$\|\mathcal{T}x\|_{C_{\xi, a}} \leq M.$$

This prove that the operator $\mathcal{T}$ is uniformly bounded on Ω . Next we prove that the operator $\mathcal{T}\Omega$ is equicontinuous. We take $x \in \Omega$ and $a < \varepsilon_1 < \varepsilon_2 \leq b$. Then,

$$\begin{aligned} &\left| \Psi_{\xi}(\varepsilon_1, a)(\mathcal{T}x)(\varepsilon_1) - \Psi_{\xi}(\varepsilon_2, a)(\mathcal{T}x)(\varepsilon_2) \right| \\ &\leq \left| \Psi_{\xi}(\varepsilon_1, a) \left({}^a\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\varepsilon_1) - \Psi_{\xi}(\varepsilon_2, a) \left({}^a\mathbb{J}_{a^+}^{\vartheta} v(\tau) \right) (\varepsilon_2) \right| \\ &\leq \int_a^{\varepsilon_1} |\Psi_{\xi}(\varepsilon_1, a) \bar{\Psi}_{\vartheta}(\varepsilon_1, \tau) - \Psi_{\xi}(\varepsilon_2, a) \bar{\Psi}_{\vartheta}(\varepsilon_2, \tau)| \tau^{\alpha-1} v(\tau) d\tau \\ &+ \Psi_{\xi}(\varepsilon_2, a) \left({}^a\mathbb{J}_{\varepsilon_1^+}^{\vartheta} |v(\tau)| \right) (\varepsilon_2). \end{aligned}$$

Then we have for each $t \in (a, b]$

$$\begin{aligned} & |\Psi_{\xi}(\varepsilon_1, a)(\mathcal{T}x)(\varepsilon_1) - \Psi_{\xi}(\varepsilon_2, a)(\mathcal{T}x)(\varepsilon_2)| \\ & \leq \Lambda\Gamma(\xi) \int_a^{\varepsilon_1} \tau^{\alpha-1} |\Psi_{\xi}(\varepsilon_1, a)\bar{\Psi}_{\theta}(\varepsilon_1, \tau) - \Psi_{\xi}(\varepsilon_2, a)\bar{\Psi}_{\theta}(\varepsilon_2, \tau)| \bar{\Psi}_{\xi}(\tau, a) d\tau, \\ & \quad + \Lambda\Gamma(\xi) \Psi_{\xi}(\varepsilon_2, a) \bar{\Psi}_{\theta+\xi}(\varepsilon_2, \varepsilon_1). \end{aligned}$$

Note that

$$|\Psi_{\xi}(\varepsilon_1, a)(\mathcal{T}x)(\varepsilon_1) - \Psi_{\xi}(\varepsilon_2, a)(\mathcal{T}x)(\varepsilon_2)| \rightarrow 0 \quad \text{as } \varepsilon_1 \rightarrow \varepsilon_2.$$

This proves that $\mathcal{T}\Omega$ is equicontinuous on J . Therefore by the Arzelà-Ascoli theorem, $\mathcal{T}$ is completely continuous on Ω .

Step 3: The third hypothesis of Lemma 2.9 is satisfied.

Let $x \in C_{\xi, \alpha}(J)$ and $y \in \Omega$ be arbitrary such that $x = \mathcal{S}x\mathcal{T}y + \mathcal{N}x$ and $\tilde{v} \in C_{\xi, \alpha}(J)$ with

$$\tilde{v}(t) = \varphi(t, y(t), \tilde{v}(t)).$$

Then, for $t \in (a, b]$ we have

$$\begin{aligned} & |\Psi_{\xi}(t, a)x(t)| \\ & = |\Psi_{\xi}(t, a)(\mathcal{S}x\mathcal{T}y)(t) + \Psi_{\xi}(t, a)(\mathcal{N}x)(t)| \\ & \leq \Psi_{\xi}(t, a) |(\mathcal{S}x)(t)| |(\mathcal{T}y)(t)| + |\Psi_{\xi}(t, a)(\mathcal{N}x)(t)| \\ & \leq |f(t, x(t))| \left[\frac{|K|}{\Gamma(\xi)} \sum_{i=1}^m |c_i| \left({}^{\alpha}\mathbb{J}_{a^+}^{\theta} |\tilde{v}(\tau)| \right) (\varepsilon_i) + \Psi_{\xi}(t, a) \left({}^{\alpha}\mathbb{J}_{a^+}^{\theta} |\tilde{v}(\tau)| \right) (t) \right] \\ & \quad + \Psi_{\xi}(t, a) |\chi(t, x(t))| \\ & \leq M (|f(t, x(t)) - f(t, 0)| + |f(t, 0)|) + \Psi_{\xi}(t, a) (|\chi(t, x(t)) - \chi(t, 0)| + |\chi(t, 0)|) \\ & \leq M (p^* \|x\|_{C_{\xi, \alpha}} + f^*) + q^* \|x\|_{C_{\xi, \alpha}} + \chi^*, \end{aligned}$$

then,

$$\|x\|_{C_{\xi, \alpha}} = \frac{f^*M + \chi^*}{1 - p^*M - q^*} \leq \ell.$$

Then $x \in \Omega$, thus the third hypothesis of Lemma 2.9 is satisfied.

Step 4: The fourth hypothesis of Lemma 2.9 is satisfied.

We show that $p^*\Psi_{\xi}(b, a)L + q^* < 1$, where

$$L = \|\mathcal{T}(\Omega)\|_{C_{\xi, \alpha}} = \sup\{\|\mathcal{T}y\|_{C_{\xi, \alpha}} : y \in \Omega\}.$$

Since $L \leq M$, we have

$$p^*\Psi_{\xi}(b, a)L + q^* \leq p^*\Psi_{\xi}(b, a)M + q^* < 1.$$

That is, the last hypothesis of Lemma 2.9 is satisfied. Thus, the operator equation $\mathfrak{S}x = \mathcal{S}x\mathcal{T}x + \mathcal{N}x = x$ has at least one solution $x^* \in C_{\xi, \alpha}$, which is a fixed point for the operator $\mathfrak{S}$.

Step 5: We prove that for such fixed point $x^* \in C_{\xi, \alpha}(J)$, the function $\sigma : t \rightarrow \frac{x^*(t) - \chi(t, x^*(t))}{f(t, x^*(t))}$ is in $C_{\xi, \alpha}^{\xi}(J)$.

Since x^* is a fixed point of operator $\mathfrak{S}$ in $C_{\xi, \alpha}(J)$, then for each $t \in (a, b]$, we have

$$\mathfrak{S}x^*(t) = f(t, x^*(t)) \left[K\bar{\Psi}_{\xi}(t, a) \sum_{i=1}^m c_i \left({}^{\alpha}\mathbb{J}_{a^+}^{\theta} v(\tau) \right) (\varepsilon_i) + \left({}^{\alpha}\mathbb{J}_{a^+}^{\theta} v(\tau) \right) (t) \right] + \chi(t, x^*(t)). \quad (20)$$

where $v \in C_{\xi, \alpha}(J)$ such that

$$v(t) = \varphi(t, x^*(t), v(t)).$$

Applying ${}^{\alpha}\mathbb{D}_{a^{+}}^{\xi}$ to both sides of 20, and by Lemma 2.8, we have

$$\begin{aligned} {}^{\alpha}\mathbb{D}_{a^{+}}^{\xi} \left(\frac{x^{*}(t) - \chi(t, x^{*}(t))}{f(t, x^{*}(t))} \right) &= \left({}^{\alpha}\mathbb{D}_{a^{+}}^{\xi} {}^{\alpha}\mathbb{J}_{a^{+}}^{\vartheta} v(\tau) \right)(t) \\ &= \left({}^{\alpha}\mathbb{D}_{a^{+}}^{r(1-\vartheta)} v(\tau) \right)(t). \end{aligned}$$

Since $\xi \geq \vartheta$, by (Ax1), the right hand side is in $C_{\xi, \alpha}(J)$ and thus ${}^{\alpha}\mathbb{D}_{a^{+}}^{\xi} \sigma \in C_{\xi, \alpha}(J)$. It is clear that $\sigma \in C_{\xi, \alpha}(J)$, since $f \in C(J \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\})$ and $\chi \in C(J \times \mathbb{R} \rightarrow \mathbb{R})$, then $\sigma \in C_{\xi, \alpha}^{\xi}(J)$. As a consequence of Steps 1–5 with Lemma 3.2, we can conclude that the problem (1) – (2) has at least a solution in $C_{\xi, \alpha}(J)$. This completes the proof of Theorem 3.3.

4 Examples

Example 4.1. Consider the problem

$${}^1\mathbb{D}_{1^{+}}^{\frac{1}{2}, 0} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) = \frac{\sqrt{t-1} \left(x(t) + {}^1\mathbb{D}_{1^{+}}^{\frac{1}{2}, 0} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) + 1 \right)}{111e^{-t+2}(1 + \sqrt{t-1}|x(t)|)}, \text{ for each } t \in (1, 2], \quad (21)$$

$$\left({}^1\mathbb{J}_{1^{+}}^{\frac{1}{2}} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (1^{+}) = 2 \left(\frac{x(\frac{3}{2}) - \chi(\frac{3}{2}, x(\frac{3}{2}))}{f(\frac{3}{2}, x(\frac{3}{2}))} \right), \quad (22)$$

where $I = [1, 2]$, $a = 1$, $b = 2$ and

$$f(t, x(t)) = \frac{(t-1)(|x(t)| + 1)}{41e^{-t+4}}, \quad t \in I, \quad x \in C_{\frac{1}{2}, 1}(I),$$

and

$$\chi(t, x(t)) = \frac{\sqrt{t-1} \ln(2)x(t)}{33e^3\sqrt{6-t}} + \frac{1}{55e^{-t+2}}, \quad t \in I, \quad x \in C_{\frac{1}{2}, 1}(I).$$

Set

$$\varphi(t, x, y) = \frac{\sqrt{t-1}(x+y+1)}{111e^{-t+2}(1 + |x|\sqrt{t-1})}, \quad t \in I, \quad x, y \in \mathbb{R}.$$

We have

$$C_{\xi, \alpha}^{r(1-\vartheta)}(I) = C_{\frac{1}{2}, 1}^0(I) = \left\{ v : (1, 2] \rightarrow \mathbb{R} : t \rightarrow (\sqrt{t-1})v(t) \in C(I, \mathbb{R}) \right\},$$

with $\xi = \vartheta = \frac{1}{2}$, $\alpha = 1$, $r = 0$. Clearly, the continuous function $\varphi \in C_{\frac{1}{2}, 1}^0(I)$. Hence the condition (Ax1) is satisfied. For each $x, \bar{x} \in \mathbb{R}$ and $t \in I$, we have

$$|f(t, x) - f(t, \bar{x})| \leq \frac{t-1}{41e^{-t+4}} |x - \bar{x}|,$$

and

$$|\chi(t, x) - \chi(t, \bar{x})| \leq \frac{\sqrt{t-1} \ln(2)}{33e^3\sqrt{6-t}} |x - \bar{x}|.$$

Hence condition (Ax2) is satisfied with

$$p(t) = \frac{\sqrt{t-1}}{41e^{-t+4}}, \quad \text{and} \quad q(t) = \frac{\sqrt{t-1} \ln(2)}{33e^3\sqrt{6-t}},$$

so we have

$$p^{*} = \frac{1}{41e^2}, \quad \text{and} \quad q^{*} = \frac{\ln(2)}{66e^3}.$$

Let $x, y \in \mathbb{R}$. Then we have

$$|\varphi(t, x, y)| \leq \frac{\sqrt{t-1}}{111e^{-t+2}} (|x| + |y| + 1), \quad t \in I,$$

and so the condition (Ax3) is satisfied with

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \frac{\sqrt{t-1}}{111e^{-t+2}},$$

and

$$\lambda_1^* = \lambda_2^* = \lambda_3^* = \frac{1}{111}.$$

Setting

$$f^* = \frac{1}{41e^2}, \quad \chi^* = \frac{1}{55}, \quad K = \frac{\sqrt{\pi}}{\sqrt{\pi-2}\sqrt{2}}$$

and

$$\Lambda = \frac{1+\ell}{110}.$$

Using these values, it follows by (15) and (16) that the constant ℓ satisfies the inequality

$$0.01835103855 \leq \ell < \frac{4510e^2(66e^3 - \ln(2))(2\sqrt{2} - \sqrt{\pi})}{66e^3(2\sqrt{\pi} + 2\sqrt{2\pi} - \pi)} - 1 \approx 6492.31995291809.$$

Thus, all the conditions of Theorem 3.3 are satisfied. Hence, our problem (21)–(22) has at least in $C_{\frac{1}{2},1}(I)$.

Example 4.2. Consider the problem

$${}^1\mathbb{D}_{1^+}^{\frac{1}{2},0} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) = \frac{\sqrt{t-1} \left(x(t) + {}^1\mathbb{D}_{1^+}^{\frac{1}{2},0} \left(\frac{x(t) - \chi(t, x(t))}{f(t, x(t))} \right) + 1 \right)}{111e^{-t+2}(1 + \sqrt{t-1}|x(t)|)}, \text{ for each } t \in (1, 2], \quad (23)$$

$$\left({}^1\mathbb{J}_{1^+}^{\frac{1}{2}} \left(\frac{x(\tau) - \chi(\tau, x(\tau))}{f(\tau, x(\tau))} \right) \right) (1^+) = 3 \left(\frac{x(\frac{5}{4}) - \chi(\frac{5}{4}, x(\frac{5}{4}))}{f(\frac{5}{4}, x(\frac{5}{4}))} \right) + 2 \left(\frac{x(\frac{4}{3}) - \chi(\frac{4}{3}, x(\frac{4}{3}))}{f(\frac{4}{3}, x(\frac{4}{3}))} \right), \quad (24)$$

where $I = [1, 2]$, $a = 1$, $b = 2$ and

$$f(t, x(t)) = \frac{(t-1)(|x(t)| + 1)}{41e^{-t+4}}, \quad t \in I, \quad x \in C_{\frac{1}{2},1}(I),$$

and

$$\chi(t, x(t)) = \frac{\sqrt{t-1} \ln(2)x(t)}{33e^3\sqrt{6-t}} + \frac{1}{55e^{-t+2}}, \quad t \in I, \quad x \in C_{\frac{1}{2},1}(I).$$

Set

$$\varphi(t, x, y) = \frac{\sqrt{t-1}(x+y+1)}{111e^{-t+2}(1 + |x|\sqrt{t-1})}, \quad t \in I, \quad x, y \in \mathbb{R}.$$

We have

$$C_{\xi, \alpha}^{r(1-\vartheta)}(I) = C_{\frac{1}{2},1}^0(I) = \left\{ v : (1, 2] \rightarrow \mathbb{R} : t \rightarrow (\sqrt{t-1})v(t) \in C(I, \mathbb{R}) \right\},$$

with $\xi = \vartheta = \frac{1}{2}$, $\alpha = 1$, $r = 0$. Clearly, the continuous function $\varphi \in C_{\frac{1}{2},1}^0(I)$. Hence the condition (Ax1) is satisfied. For each $x, \bar{x} \in \mathbb{R}$ and $t \in I$, we have

$$|f(t, x) - f(t, \bar{x})| \leq \frac{t-1}{41e^{-t+4}} |x - \bar{x}|,$$

and

$$|\chi(t, x) - \chi(t, \bar{x})| \leq \frac{\sqrt{t-1} \ln(2)}{33e^3\sqrt{6-t}} |x - \bar{x}|.$$

Hence condition (Ax2) is satisfied with

$$p(t) = \frac{\sqrt{t-1}}{41e^{-t+4}}, \quad \text{and} \quad q(t) = \frac{\sqrt{t-1} \ln(2)}{33e^3\sqrt{6-t}},$$

so we have

$$p^* = \frac{1}{41e^2}, \quad \text{and} \quad q^* = \frac{\ln(2)}{66e^3}.$$

Let $x, y \in \mathbb{R}$. Then we have

$$|\varphi(t, x, y)| \leq \frac{\sqrt{t-1}}{111e^{-t+2}} (|x| + |y| + 1), \quad t \in I,$$

and so the condition (Ax3) is satisfied with

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \frac{\sqrt{t-1}}{111e^{-t+2}},$$

and

$$\lambda_1^* = \lambda_2^* = \lambda_3^* = \frac{1}{111}.$$

Setting

$$m = 2, \quad f^* = \frac{1}{41e^2}, \quad \chi^* = \frac{1}{55}, \quad K = \frac{\sqrt{\pi}}{\sqrt{\pi-2}\sqrt{3-6}},$$

and

$$\Lambda = \frac{1+\ell}{110}.$$

Same as the last example, it follows by (15) and (16) that the constant ℓ satisfies the inequality

$$0.0183510386 \leq \ell < \frac{4510e^2(66e^3 - \ln(2))(2\sqrt{3} - \sqrt{\pi} + 6)}{66e^3(11\sqrt{\pi} + 2\sqrt{3}\pi - \pi)} - 1 \approx 11387.$$

Then the problem (23)–(24) has at least one solution in $C_{\frac{1}{2},1}(I)$.

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