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Multi-dimensional (ω, c) -almost periodic type functions and applications

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Abstract: In this paper, we analyze various classes of multi-dimensional (ω, c) -almost periodic type functions with values in complex Banach spaces. The main structural properties and characterizations for the introduced classes of functions are presented. We provide certain applications of our abstract theoretical results to the abstract Volterra integro-differential equations, as well.

Keywords: Multi-dimensional (ω, c) -almost periodic functions, multi-dimensional $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions, multi-dimensional c -almost periodic functions, (I', a, ω, c) -almost periodic type functions, abstract Volterra integro-differential equations

MSC: 42A75, 43A60, 47D99

1 Introduction and preliminaries

The notion of an almost periodic function was introduced by H. Bohr [7] around 1924-1926 and later generalized by many other mathematicians (for more details about the subject, we refer the reader to the research monographs [6], [9]-[11], [17]-[19], [21]-[23] and [25]). Let I be either $\mathbb{R}$ or $[0, \infty)$, and let $f : I \rightarrow X$ be a given continuous function, where X is a complex Banach space equipped with the norm $\|\cdot\|$. For any $\varepsilon > 0$, a number $\tau > 0$ is called a ε -period for $f(\cdot)$ if and only if $\|f(t + \tau) - f(t)\| \leq \varepsilon$, $t \in I$. The set consisting of all ε -periods for $f(\cdot)$ is denoted by $\mathcal{P}(f, \varepsilon)$. The function $f(\cdot)$ is said to be almost periodic if and only if for each $\varepsilon > 0$ the set $\mathcal{P}(f, \varepsilon)$ is relatively dense in $[0, \infty)$, i.e., there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\mathcal{P}(f, \varepsilon)$.

The notion of a periodic function has recently been reconsidered by E. Alvarez, A. Gómez and M. Pinto [2] in the following way: A continuous function $f : I \rightarrow X$ is said to be (ω, c) -periodic ($\omega > 0$, $c \in \mathbb{C} \setminus \{0\}$) if and only if $f(x + \omega) = cf(x)$ for all $x \in I$. We know that a continuous function $f : I \rightarrow X$ is (ω, c) -periodic if and only if the function $g(\cdot) \equiv c^{-1/\omega} f(\cdot)$ is periodic and $g(x + \omega) = g(x)$ for all $x \in I$; here, $c^{-1/\omega}$ denotes the principal branch of the exponential function (see also the research articles [3]-[4] by E. Alvarez, S. Castillo, M. Pinto for more details about the subject). In our joint paper [16] with M. T. Khalladi, A. Rahmani, M. Pinto and D. Velinov, we have recently extended the notion of (ω, c) -periodicity by examining various classes of (ω, c) -almost periodic type functions.

On the other hand, in our recent paper [15], we have recently introduced the class of c -almost periodic functions depending on one real variable. The multi-dimensional c -almost periodic type functions have recently been investigated in [16]. The main aim of this paper is to continue the above-mentioned research studies by introducing and analyzing various notions of (ω, c) -periodicity and (ω, c) -almost periodicity for vector-valued functions depending on several real variables; we provide certain applications to the abstract partial differential equations, as well. For the sake of simplicity and better exposition, we will consider the

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corresponding Stepanov classes of multi-dimensional (ω, c) -almost periodic type functions and the corresponding (Stepanov) classes of multi-dimensional semi- c -periodic functions somewhere else.

The organization and main ideas of the paper can be briefly described as follows. In Subsection 1.1, we recall the basic definitions and results about c -almost periodic functions in $\mathbb{R}^n$. The main aim of Section 2 is to introduce and analyze the classes of multi-dimensional (ω, c) -periodic functions and multi-dimensional $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions. The main structural results obtained in this section are Proposition 2.4-Proposition 2.5 and Proposition 2.8-Proposition 2.11. The corresponding classes of asymptotically (ω, c) -almost periodic type functions are introduced in Definition 2.15; Subsection 2.1 investigates $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic type functions. In Definition 2.16, we introduce the notion of $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodicity, $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniform recurrence and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -almost automorphy. The main structural propositions of function spaces introduced in Definition 2.16 are stated in Proposition 2.19; we also discuss the convolution invariance of function spaces introduced here before we move ourselves to the third section of paper, which is reserved for the study of $(I', \mathbf{a}, \omega, c)$ -uniform recurrence of type 1 (2) and the $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 1 and type 2. Here we continue our investigation from [16, Section 3] and prove two negative results, Theorem 3.3 and Theorem 3.5, saying that the introduction of Definition 3.1 is basically an unsatisfactory way to extend the notion of (ω, c) -almost periodicity. In the final section of paper, we provide certain application to the abstract Volterra integro-differential equations in Banach spaces.

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Notation and terminology. We assume henceforth that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are complex Banach spaces, $n \in \mathbb{N}$, $\mathcal{B}$ is a certain collection of subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. We will use the principal branch of the exponential function to take the powers of complex numbers.

If $\mathbf{t}_0 \in \mathbb{R}^n$ and $\epsilon > 0$, then we set $B(\mathbf{t}_0, \epsilon) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq \epsilon\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Set $\mathbb{N}_n := \{1, \dots, n\}$ and $I_M := \{\mathbf{t} \in I : |\mathbf{t}| \leq M\}$ ($I \subseteq \mathbb{R}^n$; $M > 0$).

1.1 c -Almost periodic functions

In [19], we have recently introduced the following notion:

Definition 1.1. Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $F : I \times X \rightarrow Y$ is a continuous function and $I + I' \subseteq I$. Then we say that:

- (i) $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, I', c)$ -almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\|F(\mathbf{t} + \tau; x) - cF(\mathbf{t}; x)\|_Y \leq \epsilon, \quad \mathbf{t} \in I, x \in B.$$

- (ii) $F(\cdot; \cdot)$ is $(\mathcal{B}, I', c)$ -uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I' such that $\lim_{k \rightarrow +\infty} |\tau_k| = +\infty$ and

$$\lim_{k \rightarrow +\infty} \sup_{\mathbf{t} \in I; x \in B} \|F(\mathbf{t} + \tau_k; x) - cF(\mathbf{t}; x)\|_Y = 0.$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot; \cdot)$ is Bohr (I', c) -almost periodic ((I', c) -uniformly recurrent).

The most important case is that in which $I' = I$, when we also say that the function $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, c)$ -almost periodic [$(\mathcal{B}, c)$ -uniformly recurrent]; if $X \in \mathcal{B}$, then it is also said that $F(\cdot; \cdot)$ is Bohr c -almost periodic (c -uniformly recurrent). The classes of Bohr $(\mathcal{B}, I')$ -almost periodic functions, $(\mathcal{B}, I')$ -uniformly recurrent functions, Bohr I' -almost periodic functions and I' -uniformly recurrent functions, obtained by plugging $c = 1$, are thoroughly investigated in [8]. If $c = -1$, then we also say that $F(\cdot; \cdot)$ is $(\mathcal{B}, I')$ -almost anti-periodic, $(\mathcal{B}, I')$ -uniformly anti-recurrent, I' -almost anti-periodic or I' -uniformly anti-recurrent.

We know the following:

- (i) Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' = I$ and $F : I \times X \rightarrow Y$ is Bohr $(\mathcal{B}, I', c)$ -almost periodic $((\mathcal{B}, I', c)$ -uniformly recurrent). If $F(\cdot; \cdot) \neq 0$, then $|c| = 1$.
- (ii) Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$ and $I + I' = I$. If the function $F : I \rightarrow \mathbb{R}$ is $(\mathcal{B}, I', c)$ -uniformly recurrent and $F \neq 0$, then $c = \pm 1$. Furthermore, if $F(\mathbf{t}) \geq 0$ for all $\mathbf{t} \in I$, then $c = 1$.
- (iii) Suppose that $l \in \mathbb{N}$, $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$ and $F : I \times X \rightarrow Y$ is Bohr $(\mathcal{B}, I', c)$ -almost periodic $((\mathcal{B}, I', c)$ -uniformly recurrent). Then $lI' \subseteq I$, $I + lI' \subseteq I$ and $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, lI', c^l)$ -almost periodic $((\mathcal{B}, lI', c^l)$ -uniformly recurrent).
- (iv) Suppose that

$$p \in \mathbb{Z} \setminus \{0\}, q \in \mathbb{N}, (p, q) = 1, |c| = 1 \text{ and } \arg(c) = \pi p/q. \quad (1.1)$$

Suppose, further, that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$ and $F : I \times X \rightarrow Y$ is Bohr $(\mathcal{B}, I', c)$ -almost periodic $((\mathcal{B}, I', c)$ -uniformly recurrent). Then the following holds:

- (a) If p is even, then $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, qI')$ -almost periodic $((\mathcal{B}, qI')$ -uniformly recurrent).
 - (b) If p is odd, then $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, qI')$ -almost anti-periodic $((\mathcal{B}, qI')$ -uniformly anti-recurrent).
- (v) Let $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, $lI' = I'$ for all $l \in \mathbb{N}$ and $F : I \times X \rightarrow Y$ is a bounded, Bohr $(\mathcal{B}, I', c)$ -almost periodic $((\mathcal{B}, I', c)$ -uniformly recurrent) function, then the function $F(\cdot; \cdot)$ is Bohr $(\mathcal{B}, I', c')$ -almost periodic $((\mathcal{B}, I', c')$ -uniformly recurrent) for all $c' \in S_1$.
- If $F : \mathbb{R}^n \rightarrow Y$ is an almost periodic function ($X = \{0\}$, $\mathcal{B} = \{X\}$, $I = I' = \mathbb{R}^n$), then the mean value

$$M(F) := \lim_{T \rightarrow +\infty} \frac{1}{(2T)^n} \int_{s+K_T} F(\mathbf{t}) d\mathbf{t}$$

exists and it does not depend on $s \in \mathbb{R}^n$, where $K_T := \{\mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n : |t_i| \leq T \text{ for } 1 \leq i \leq n\}$. The Bohr-Fourier coefficient $F_\lambda \in X$ is defined by

$$F_\lambda := M\left(e^{-i\langle \lambda, \cdot \rangle} F(\cdot)\right), \quad \lambda \in \mathbb{R}^n.$$

We know that the Bohr spectrum of $F(\cdot)$, defined by

$$\sigma(F) := \{\lambda \in \mathbb{R}^n : F_\lambda \neq 0\},$$

is at most a countable set. By $AP(\mathbb{R}^n : X)$ and $AP_\Lambda(\mathbb{R}^n : X)$ we denote respectively the Banach space consisting of all almost periodic functions $F : \mathbb{R}^n \rightarrow X$ and its subspace consisting of all almost periodic functions $F : \mathbb{R}^n \rightarrow X$ such that $\sigma(F) \subseteq \Lambda$.

We also need the following definition from [8]:

Definition 1.2. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : I \times X \rightarrow Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{t \in \mathbb{D}, |t| \rightarrow +\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$. If $X = \{0\}$, then we abbreviate $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ to $C_{0,\mathbb{D},\mathcal{B}}(I : Y)$.

2 (ω, c) -Periodic functions and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions

Let us recall that a continuous function $F : I \rightarrow X$ is said to be Bloch $(\mathbf{p}, \mathbf{k})$ -periodic, or Bloch periodic with period $\mathbf{p}$ and Bloch wave vector or Floquet exponent $\mathbf{k}$, where $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{k} \in \mathbb{R}^n$, if and only if $F(\mathbf{t} + \mathbf{p}) = e^{i\langle \mathbf{k}, \mathbf{p} \rangle} F(\mathbf{t})$, $\mathbf{t} \in I$ (we assume here that $\mathbf{p} + I \subseteq I$). In [8, Example 2.15(viii)], we have observed that the Bloch $(\mathbf{p}, \mathbf{k})$ -periodicity of function $F(\cdot)$ implies the Bohr $(\mathcal{B}, I')$ -almost periodicity of function $e^{-i\langle \mathbf{k}, \cdot \rangle} F(\cdot)$ with I' being the intersection of I and the one-dimensional submanifold generated by the vector $\mathbf{p}$ as well as that the orthogonality of vectors $\mathbf{k}$ and $\mathbf{p}$ implies that the function $F(\cdot)$ is Bohr $(\mathcal{B}, I')$ -almost periodic. For more details about the Bloch periodic functions, we refer the reader to the research articles [13] by M. Hasler and [14] by M. Hasler, G. M. N'Guérékata.

Following the recent research analyses of E. Alvarez, A. Gómez, M. Pinto [2] and E. Alvarez, S. Castillo, M. Pinto [3]–[4], we generalize the notion of Bloch $(\mathbf{p}, \mathbf{k})$ -periodicity in the following way:

Definition 2.1. Let $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$ and $\omega + I \subseteq I$. A continuous function $F : I \rightarrow X$ is said to be (ω, c) -periodic if and only if $F(\mathbf{t} + \omega) = cF(\mathbf{t})$, $\mathbf{t} \in I$.

If $F : I \rightarrow X$ is a Bloch $(\mathbf{p}, \mathbf{k})$ -periodic function, then $F(\cdot)$ is $(\mathbf{p}, c)$ -periodic with $c = e^{i(\mathbf{k}, \mathbf{p})}$; conversely, if $|c| = 1$ and $F : I \rightarrow X$ is (ω, c) -periodic, then we can always find a point $\mathbf{k} \in \mathbb{R}^n$ such that the function $F(\cdot)$ is Bloch $(\mathbf{p}, \mathbf{k})$ -periodic. In the case that $|c| \neq 1$, we have the following: if $F : I \rightarrow X$ is (ω, c) -periodic, then $F(\mathbf{t} + m\omega) = c^m F(\mathbf{t})$, $\mathbf{t} \in I$, $m \in \mathbb{N}$, so that the existence of a point $\mathbf{t}_0 \in I$ such that $F(\mathbf{t}_0) \neq 0$ implies $\lim_{m \rightarrow \infty} \|F(\mathbf{t}_0 + m\omega)\| = +\infty$, provided that $|c| > 1$, and $\lim_{m \rightarrow \infty} \|F(\mathbf{t}_0 + m\omega)\| = 0$, provided that $|c| < 1$.

If $c = 1$, resp. $c = -1$, then we also say that the function $F(\cdot)$ is ω -periodic, resp. ω -anti-periodic. It is clear that, if $F(\cdot)$ is (ω, c) -periodic, $k \in \mathbb{N}$ and $c^k = 1$, resp. $c^k = -1$, then $F(\cdot)$ is $(k\omega)$ -periodic, resp. $(k\omega)$ -anti-periodic.

In [14, Definition 2.1], the authors have assumed that any Bloch $(\mathbf{p}, \mathbf{k})$ -periodic is bounded a priori, which is a slightly redundant condition as the following example shows:

Example 2.2. There exists a continuous, unbounded function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ which satisfies $F(\mathbf{t} + (1, 1, \dots, 1)) = F(\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^n$. We can simply construct such a function, with $n = 2$, as follows. Let $F_0 : \{(t_1, t_2) \in \mathbb{R}^2 : 0 \leq t_1 + t_2 \leq 2\}$ be any continuous function satisfying that:

$$F_0(t_1, t_2) = F(t_1 + 1, t_2 + 1), \text{ provided } (t_1, t_2) \in \mathbb{R}^2 \text{ and } t_1 + t_2 = 0, \quad (2.1)$$

$$\text{the set } \{(4k\sqrt{2}, -4k\sqrt{2}) : k \in \mathbb{N}\} \text{ is unbounded, and} \quad (2.2)$$

$$F_0((4k+2)\sqrt{2}, -(4k+2)\sqrt{2}) = 1, \quad k \in \mathbb{N}. \quad (2.3)$$

Due to condition (2.1), we can extend the function $F_0(\cdot)$ to a continuous function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ which satisfies $F(t_1 + 1, t_2 + 1) = F(t_1, t_2)$ for all $t_1, t_2 \in \mathbb{R}$. Clearly, this function is unbounded due to condition (2.2).

The following definition is also meaningful:

Definition 2.3. Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$). A continuous function $F : I \rightarrow X$ is said to be $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic if and only if $F(\mathbf{t} + \omega_j e_j) = c_j F(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$.

It is clear that, if $F : I \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, then $F(\mathbf{t} + m\omega_j e_j) = c_j^m F(\mathbf{t})$, $\mathbf{t} \in I$, $m \in \mathbb{N}$, $j \in \mathbb{N}_n$, so that the existence of a point $\mathbf{t}_0 \in I$ such that $F(\mathbf{t}_0) \neq 0$ implies $\lim_{m \rightarrow \infty} \|F(\mathbf{t}_0 + m\omega_j e_j)\| = +\infty$, provided that $|c_j| > 1$, and $\lim_{m \rightarrow \infty} \|F(\mathbf{t}_0 + m\omega_j e_j)\| = 0$, provided that $|c_j| < 1$, for some $j \in \mathbb{N}_n$.

If $c_j = 1$ for all $j \in \mathbb{N}_n$, resp. $c_j = -1$ for all $j \in \mathbb{N}_n$, then we also say that the function $F(\cdot)$ is $(\omega_j)_{j \in \mathbb{N}_n}$ -periodic, resp. $(\omega_j)_{j \in \mathbb{N}_n}$ -anti-periodic. It is clear that, if $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, $k \in \mathbb{N}$ and $c_j^k = 1$ for all $j \in \mathbb{N}_n$, resp. $c_j^k = -1$ for all $j \in \mathbb{N}_n$, then $F(\cdot)$ is $(k\omega_j)_{j \in \mathbb{N}_n}$ -periodic, resp. $(k\omega_j)_{j \in \mathbb{N}_n}$ -anti-periodic.

The classes of (ω, c) -periodic functions and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions are closed under the operation of the pointwise convergence of functions, as easily approved. In the scalar-valued case, the following holds: If the function $F : I \rightarrow \mathbb{C} \setminus \{0\}$ is (ω, c) -periodic, resp. $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, then the function $(1/F)(\cdot)$ is $(\omega, 1/c)$ -periodic, resp. $(\omega_j, 1/c_j)_{j \in \mathbb{N}_n}$ -periodic. It is also clear that we have the following:

Proposition 2.4. (i) Let $\omega, a \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$, $\omega + I \subseteq I$ and $a + I \subseteq I$. If the function $F : I \rightarrow X$ is (ω, c) -periodic, then $-\omega - I \subseteq -I$ and the function $\check{F} : -I \rightarrow X$, defined by $\check{F}(x) := F(-x)$, $x \in I$, is $(-\omega, c)$ -periodic. Moreover, $\|F(\cdot)\|$ is $(\omega, |c|)$ -periodic, the function $F_a : I \rightarrow X$ defined by $F_a(\mathbf{t}) := F(\mathbf{t} + a)$, $\mathbf{t} \in I$ is (ω, c) -periodic and the function $\alpha F(\cdot)$ is (ω, c) -periodic.

(ii) Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$, $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$) and $a + I \subseteq I$. If a continuous function $F : I \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, then $-\omega_j e_j - I \subseteq -I$ ($1 \leq j \leq n$) and the function $\check{F} : -I \rightarrow X$ is $(-\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic.

Moreover, $\|F(\cdot)\|$ is $(\omega_j, |c_j|)_{j \in \mathbb{N}_n}$ -periodic, the function $F_a : I \rightarrow X$ defined above is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic and the function $\alpha F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic.

Proposition 2.5. Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$). If a continuous function $F : I \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, then $\omega + I \subseteq I$, where $\omega := \sum_{j=1}^n \omega_j e_j$, and the function $F(\cdot)$ is (ω, c) -periodic with $c := \prod_{j=1}^n c_j$.

The converse statement is not true in general case $n > 1$, as the following simple counterexample shows:

Example 2.6. Consider the function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ from Example 2.2. Then there do not exist numbers $\omega_1, \omega_2 \in \mathbb{R} \setminus \{0\}$ and numbers $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ such that the function $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_2}$ -periodic. If we assume the contrary, then we would have $F(t_1 + \omega_1, 0) = c_1 F(t_1, 0)$ for all $t_1 \in \mathbb{R}$. If $|c_1| \leq 1$, then the contradiction is obvious since (2.2) implies the unboundedness of function $F(\cdot, 0)$, because $F(8k, 0) = F(4k\sqrt{2}, -4k\sqrt{2})$ for all $k \in \mathbb{N}$. If $|c_1| > 1$, then the contradiction is obvious due to condition (2.3), which implies that the function $F(\cdot, 0)$ cannot tend to plus infinity as the time variable tends to plus infinity (see also [15, Remark 2.4]).

Sometimes the converse in Proposition 2.5 is possible to be made:

Example 2.7. (cf. also [19, Example 2.8]) Suppose that $I := \{(x, y) \in \mathbb{R}^2 : x + y \geq 0\}$ and $I' := \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$. Set $F(x, y) := 2^{-x-y}$, $(x, y) \in I$. Then, for every $(a, b) \in I'$, we have $F((x, y) + (a, b)) = 2^{-1} F(x, y)$, $(x, y) \in I$, so that $F(\cdot, \cdot)$ is $(\omega, 2^{-1})$ -periodic provided that $\omega \in I'$. Similarly, we have that $F(\cdot, \cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_2}$ -periodic for every $\omega_1 > 0$ and $\omega_2 > 0$, with $c_1 = 2^{-\omega_1}$ and $c_2 = 2^{-\omega_2}$.

Concerning the boundedness of $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions, we will state only one result:

Proposition 2.8. Suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $M > 0$, $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$), the set I is closed, the function $F : I \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, $|c_j| \leq 1$ for all $j \in \mathbb{N}_n$ and, for every $\mathbf{t} = (t_1, t_2, \dots, t_n) \in I$, there exist a point $\boldsymbol{\eta} = (\eta_1, \eta_2, \dots, \eta_n) \in I_M$ and integers $k_j \in \mathbb{N}$ ($1 \leq j \leq n$) such that $t_j = k_j \omega_j + \eta_j$ ($1 \leq j \leq n$). Then the function $F(\cdot)$ is bounded.

Proof. Let a point $\mathbf{t} = (t_1, t_2, \dots, t_n) \in I$ be fixed, and let $\boldsymbol{\eta} \in I$ and integers $k_j \in \mathbb{N}$ ($1 \leq j \leq n$) satisfy the above requirements. Then we have $\mathbf{t} = \boldsymbol{\eta} + \sum_{j=1}^n k_j \omega_j e_j$ so that $F(\mathbf{t}) = \prod_{j=1}^n c_j^{k_j} F(\boldsymbol{\eta})$. Since I is closed, I_M is compact and there exist a finite constant $M_1 > 0$ such that $\|F(\mathbf{x})\| \leq M_1$ for all $\mathbf{x} \in I_M$. Then $\|F(\mathbf{t})\| \leq M_1$ since $|c_j| \leq 1$ for all $j \in \mathbb{N}_n$. $\square$

In the following extension of [2, Proposition 2], we profile the class of (ω, c) -periodic functions in the following way:

Proposition 2.9. Let $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{R}^n \setminus \{0\}$, $\omega + I \subseteq I$, $c \in \mathbb{C} \setminus \{0\}$ and $S := \{i \in \mathbb{N}_n : \omega_i \neq 0\}$. Denote by A the collection of all tuples $\mathbf{a} = (a_1, a_2, \dots, a_{|S|}) \in \mathbb{R}^{|S|}$ such that $\sum_{i \in S} a_i = 1$. Then a continuous function $F : I \rightarrow X$ is (ω, c) -periodic if and only if, for every (some) $\mathbf{a} \in A$, the function $G_{\mathbf{a}} : I \rightarrow X$, defined by

$$G_{\mathbf{a}}(t_1, t_2, \dots, t_n) := c^{-\sum_{i \in S} \frac{a_i t_i}{\omega_i}} F(t_1, t_2, \dots, t_n), \quad \mathbf{t} = (t_1, t_2, \dots, t_n) \in I, \quad (2.4)$$

is $(\omega, 1)$ -periodic.

Proof. Let a point $\mathbf{t} = (t_1, t_2, \dots, t_n) \in I$ be fixed. Then it is clear that $G_{\mathbf{a}}(\mathbf{t} + \omega) = G_{\mathbf{a}}(\mathbf{t})$ if and only if

$$c^{-\sum_{i \in S} \frac{a_i(t_i + \omega_i)}{\omega_i}} F(t_1 + \omega_1, t_2 + \omega_2, \dots, t_n + \omega_n) = c^{-\sum_{i \in S} \frac{a_i t_i}{\omega_i}} F(t_1, t_2, \dots, t_n)$$

if and only if $F(\mathbf{t} + \omega) = cF(\mathbf{t})$. $\square$

We illustrate Proposition 2.9 with the following example:

Example 2.10. (see also [14, pp. 22-23]) Suppose that $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{R}^n \setminus \{0\}$, $\mathbf{k} \in \mathbb{R}^n \setminus \{0\}$, $\mathbf{a} \in \mathbf{A}$, (b_l) is any sequence of complex numbers such that $|b_l| = O(l^{-2})$, $\langle \mathbf{k}, \omega \rangle = 2\pi/3$ and

$$F(t_1, t_2, \dots, t_n) = c^{\sum_{i \in S} \frac{a_i t_i}{\omega_i}} \sum_{l \in 1+3\mathbb{N}} b_l e^{il \langle \mathbf{t}, \mathbf{k} \rangle}, \quad \mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathbb{R}^n.$$

Then $F(\cdot)$ is $(3\omega, c)$ -almost periodic.

Similarly, we can prove the following:

Proposition 2.11. Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$) and the function $F : I \rightarrow X$ is continuous. For each $j \in \mathbb{N}_n$, we define the function $G_j : I \rightarrow X$ by

$$G_j(t_1, t_2, \dots, t_n) := c_j^{-\frac{t_j}{\omega_j}} F(t_1, t_2, \dots, t_n), \quad \mathbf{t} = (t_1, t_2, \dots, t_n) \in I. \quad (2.5)$$

Then $F(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic if and only if, for every $\mathbf{t} = (t_1, t_2, \dots, t_n) \in I$ and $j \in \mathbb{N}_n$, we have

$$G_j(t_1, t_2, \dots, t_j + \omega_j, \dots, t_n) = G_j(t_1, t_2, \dots, t_j, \dots, t_n).$$

Therefore, we have the following:

Example 2.12. Let $c_j \in \mathbb{C} \setminus \{0\}$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^n c_j^{\frac{t_j}{2\pi}} \sin t_j, \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$$

is $(2\pi, c_j)_{j \in \mathbb{N}_n}$ -periodic.

If $\omega \in \mathbb{R}^n \setminus \{0\}$, $c_i \in \mathbb{C} \setminus \{0\}$ for $i = 1, 2$, $\omega + I \subseteq I$, the function $G : I \rightarrow \mathbb{C}$ is (ω, c_1) -periodic and the function $H : I \rightarrow X$ is (ω, c_2) -periodic, then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(\omega, c_1 c_2)$ -periodic. For the class of $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions, we can clarify the following result (cf. also [16, Proposition 2.2]):

Proposition 2.13. Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_{j,i} \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$, $1 \leq i \leq 2$). Suppose that the function $G : I \rightarrow \mathbb{C}$ is $(\omega_j, c_{j,1})_{j \in \mathbb{N}_n}$ -periodic and the function $H : I \rightarrow X$ is $(\omega_j, c_{j,2})_{j \in \mathbb{N}_n}$ -periodic. Set $c_j := c_{j,1} c_{j,2}$, $1 \leq j \leq n$. Then the function $F(\cdot) := G(\cdot)H(\cdot)$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic.

Concerning the convolution invariance of spaces introduced in Definition 2.1 and Definition 2.3, we will state and prove the following result:

Proposition 2.14. Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $S = \{i \in \mathbb{N}_n : \omega_i \neq 0\}$, $\mathbf{a} = (a_1, a_2, \dots, a_{|S|}) \in \mathbb{R}^{|S|}$ and $\sum_{i \in S} a_i = 1$, resp. $\omega_j \in \mathbb{R} \setminus \{0\}$ and $c_j \in \mathbb{C} \setminus \{0\}$ ($1 \leq j \leq n$). Suppose, further, that $F : \mathbb{R}^n \rightarrow X$ is (ω, c) -periodic and the function $G_{\mathbf{a}}(\cdot)$, defined through (2.4) is bounded, resp. $F : \mathbb{R}^n \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic and for each $j \in \mathbb{N}_n$ the function $G_j(\cdot)$, defined through (2.5) is bounded. If the function $c^{-\sum_{i \in S} a_i t_i / \omega_i} h(t_1, \dots, t_n)$ belongs to the space $L^1(\mathbb{R}^n)$, resp. for each $j \in \mathbb{N}_n$ the function $c^{-t_j / \omega_j} h(t_1, \dots, t_n)$ belongs to the space $L^1(\mathbb{R}^n)$, then the function

$$(h \star F)(\mathbf{t}) := \int_{\mathbb{R}^n} h(y) F(\mathbf{t} - y) dy, \quad \mathbf{t} \in \mathbb{R}^n$$

is (ω, c) -periodic, resp. $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic.

Proof. We will consider only (ω, c) -periodicity. By Proposition 2.9, it suffices to show that

$$c^{-\sum_{i \in S} \frac{a_i(t_i + \omega_i)}{\omega_i}} (h \star F)(\mathbf{t} + \omega) = c^{-\sum_{i \in S} \frac{a_i t_i}{\omega_i}} (h \star F)(\mathbf{t}) \quad (2.6)$$

for every fixed point $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$. Note first that the value $(h \star F)(\mathbf{t})$ is well defined, since we have assumed that the function $c^{-\sum_{i \in S} a_i t_i / \omega_i} h(t_1, \dots, t_n)$ belongs to the space $L^1(\mathbb{R}^n)$, as well as that the function $G_{\mathbf{a}}(\cdot)$, defined through (2.4), is bounded and

$$\begin{aligned} & c^{-\sum_{i \in S} a_i t_i / \omega_i} (h \star F)(\mathbf{t}) \\ &= \int_{\mathbb{R}^n} \left[c^{-\sum_{i \in S} a_i y_i / \omega_i} h(y_1, \dots, y_n) \right] \cdot \left[c^{-\sum_{i \in S} a_i (t_i - y_i) / \omega_i} F(\mathbf{t} - \mathbf{y}) \right] dy. \end{aligned} \quad (2.7)$$

Keeping in mind (2.7) and the dominated convergence theorem, we get that the function $(h \star F)(\cdot)$ is continuous. Similarly, by plugging $\mathbf{t} + \omega$ in place of $\mathbf{t}$ in (2.7), we get that (2.6) holds because $\sum_{i \in S} a_i = 1$. $\square$

Concerning asymptotically (ω, c) -periodic type functions, we will use the following definition, only:

Definition 2.15. Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $\omega + I \subseteq I$, $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$, $1 \leq i \leq 2$) and $F : I \rightarrow X$. Then we say that the function $F(\cdot)$ is $\mathbb{D}$ -asymptotically (ω, c) -periodic, resp. $\mathbb{D}$ -asymptotically $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, if and only if there exists a (ω, c) -periodic, resp. $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, function $F_0 : I \rightarrow X$ and a function $Q \in C_{0, \mathbb{D}, \mathbb{B}}(I : X)$ such that $F(\mathbf{t}) = F_0(\mathbf{t}) + Q(\mathbf{t})$, $\mathbf{t} \in I$.

2.1 $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -Almost periodic type functions

Following our idea from [16, Definition 2.1], we can introduce and analyze several various generalizations of the class of multi-dimensional $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions with the help of Proposition 2.11. For example, suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$ and $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$); if a function $F : I \rightarrow X$ is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic, then for each $j \in \mathbb{N}_n$ and for every $k \in \mathbb{N}$ we have $F(\mathbf{t} + k\omega_j e_j) = c_j^k F(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$ and $G_j(\mathbf{t} + k\omega_j e_j) = G_j(\mathbf{t})$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$. Set $W_+ := \{j \in \mathbb{N}_n : \omega_j > 0\}$ and $W_- := \{j \in \mathbb{N}_n : \omega_j < 0\}$, as well as $I_{j, \mathbf{t}} := \{x \geq 0 : \mathbf{t} + x e_j \in I\}$ if $j \in W_+$, resp. $I_{j, \mathbf{t}} := \{x \geq 0 : \mathbf{t} - x e_j \in I\}$ if $j \in W_-$ ($\mathbf{t} \in I$), and $G_{j, \mathbf{t}}(x) := G_j(\mathbf{t} + x e_j)$, $x \in I_{j, \mathbf{t}}$ if $j \in W_+$, resp. $G_{j, \mathbf{t}}(x) := G_j(\mathbf{t} - x e_j)$, $x \in I_{j, \mathbf{t}}$ if $j \in W_-$ ($\mathbf{t} \in I$). Then we can generalize the class of $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic functions as follows:

Definition 2.16. Suppose that $\omega_j \in \mathbb{R} \setminus \{0\}$, $c_j \in \mathbb{C} \setminus \{0\}$, $\omega_j e_j + I \subseteq I$ ($1 \leq j \leq n$) and $F : I \rightarrow X$ is a continuous function. Let $r_j \in \mathbb{C} \setminus \{0\}$ for $1 \leq j \leq n$, and let $\emptyset \neq \mathbb{I}'_{j, \mathbf{t}} \subseteq I_{j, \mathbf{t}} \subseteq \mathbb{R}$, $I_{j, \mathbf{t}} + \mathbb{I}'_{j, \mathbf{t}} \subseteq I_{j, \mathbf{t}}$ for $1 \leq j \leq n$, $\mathbf{t} \in I$. Set $\mathbb{I}'_j := \{\mathbb{I}'_{j, \mathbf{t}} : \mathbf{t} \in I\}$. Then we say that the function $F(\cdot)$ is:

- (i) $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic if and only if, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j, \mathbf{t}}(\cdot)$ defined above is $(\mathbb{I}'_{j, \mathbf{t}}, r_j)$ -almost periodic;
- (ii) $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent if and only if, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j, \mathbf{t}}(\cdot)$ is $(\mathbb{I}'_{j, \mathbf{t}}, r_j)$ -uniformly recurrent.

Suppose that $I = \mathbb{R}^n$. Then we say that the function $F(\cdot)$ is:

- (iii) $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -almost automorphic if and only if, for every $j \in \mathbb{N}_n$, for every $\mathbf{t} \in \mathbb{R}^n$ and for every real sequence (b_k) , there exist a subsequence (a_k) of (b_k) and a function $F_{j, \mathbf{t}}^* : \mathbb{R} \rightarrow X$ such that

$$\lim_{k \rightarrow +\infty} G_j(\mathbf{t} + (x + a_k) e_j) = F_{j, \mathbf{t}}^*(x) \text{ and } \lim_{k \rightarrow +\infty} F_{j, \mathbf{t}}^*(x - a_k) = G_j(\mathbf{t} + x e_j), \quad (2.8)$$

pointwise for $x \in \mathbb{R}$; if, moreover, the convergence in (2.8) is uniform in the variable x on compact subsets of $\mathbb{R}$, then we say that the function $F(\cdot)$ is compactly $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -almost automorphic.

Remark 2.17. (i) It is clear that $I = \mathbb{R}^n$ is equivalent to saying that $I + \eta e_j \subseteq I$ for all $\eta \in \mathbb{R} \setminus \{0\}$ and $j \in \mathbb{N}_n$.
(ii) It is clear that (i) implies (ii) and that the almost periodicity of the function $\widetilde{G}_{j, \mathbf{t}}(x) := G_j(\mathbf{t} + x e_j)$, $x \in \mathbb{R}$ for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$ implies (iii), which is equivalent to saying that the function $\widetilde{G}_{j, \mathbf{t}}(\cdot)$ defined above is almost automorphic for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$.

Now we will provide an illustrative example in which we have $I = \mathbb{R}^n$, $\omega_j = c_j = 1$ and $I_{j,\mathbf{t}} = I'_{j,\mathbf{t}} = [0, \infty)$ for all $j \in \mathbb{N}_n$ and $\mathbf{t} \in \mathbb{R}^n$:

Example 2.18. (i) Suppose that $r_j = 1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^n [\sin t_j + \sin(\sqrt{2}t_j)], \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$$

is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic but not $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodic.

(ii) ([12], [15], [19]) Suppose that $r_j = -1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^n \left[(\sin t_j) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\left(\frac{t_j}{3^n}\right) \right], \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$$

is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent but not $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic.

(iii) Suppose that $r_j = 1$ for all $j \in \mathbb{N}_n$. Then the function

$$F(t_1, \dots, t_n) := \prod_{j=1}^n \sin\left(\frac{1}{2 + \sin t_j + \sin(\sqrt{2}t_j)}\right), \quad \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$$

is $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -almost automorphic but not $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic.

The function spaces introduced in Definition 2.16 are translation invariant and closed under the pointwise multiplications with complex scalars. Furthermore, if the function $F(\cdot)$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic, then it can be easily proved that the function $\|F(\cdot)\|$ is $(\omega_j, |c_j|; |r_j|, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic. Suppose now that the function $F : I \rightarrow \mathbb{C} \setminus \{0\}$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic, $|F(\mathbf{t})| \geq m > 0$ for all $\mathbf{t} \in I$, and $|c_j| = 1$ for all $j \in \mathbb{N}_n$. Then the function $(1/F)(\cdot)$ is $(\omega_j, 1/c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic, which can be simply proved as follows (cf. also [16, Proposition 2.5]). Let $j \in \mathbb{N}_n$, $\mathbf{t} \in I$ and $\epsilon > 0$ be fixed; without loss of generality, we may assume that $j \in W_+$. Let $\tau \in I'_{j,\mathbf{t}}$ and $|G_{j,\mathbf{t}}(x + \tau) - r_j G_{j,\mathbf{t}}(x)| < \epsilon$, $x \geq 0$. After multiplication with $c_j^{t_j/\omega_j}$, we get

$$\left| c_j^{-\frac{x+\tau}{\omega_j}} F(t_1, t_2, \dots, t_j + x, \dots, t_n) - r_j F(t_1, t_2, \dots, t_j + (x + \tau), \dots, t_n) \right| \leq \epsilon, \quad x \geq 0. \quad (2.9)$$

Hence, for every $x \geq 0$, we have:

$$\begin{aligned} & \left| \frac{c_j^{\frac{t_j+x+\tau}{\omega_j}}}{F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n)} - r_j^{-1} \frac{c_j^{\frac{t_j+x}{\omega_j}}}{F(t_1, t_2, \dots, t_j + x, \dots, t_n)} \right| \\ &= \left| \frac{c_j^{\frac{\tau}{\omega_j}} F(t_1, t_2, \dots, t_j + x, \dots, t_n) - r_j^{-1} F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n)}{F(t_1, t_2, \dots, t_j + x + \tau, \dots, t_n) \cdot |F(t_1, t_2, \dots, t_j + x, \dots, t_n)|} \right| \\ & \leq m^{-2} |r_j|^{-1} \epsilon, \end{aligned}$$

where we have employed (2.9) in the last estimate. This simply implies the required.

By a careful examination of the notion introduced in Definition 2.16 and the paragraph preceding it, we may deduce that the $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodicity of the function $F : I \rightarrow X$ implies the $(-\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodicity of the function $\check{F}(\cdot)$. We leave all details concerning the proof of this fact to the interested readers.

Using the statements (i)-(v) from Subsection 1.1 and corresponding definitions, we may deduce the following proposition:

Proposition 2.19. (i) Suppose that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, we have $I_{j,\mathbf{t}} + I'_{j,\mathbf{t}} = I_{j,\mathbf{t}}$ and the function $F : I \rightarrow X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent. Then, for every $j \in \mathbb{N}_n$, we have $r_j = \pm 1$; if, additionally, $F(\mathbf{t}) \geq 0$ for all $\mathbf{t} \in I$, then, for every $j \in \mathbb{N}_n$, we have $r_j = 1$.

- (iii) Suppose that $l \in \mathbb{N}$, and $F : I \rightarrow X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent). Then, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, we have $\mathbb{I}'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}}, I_{j,\mathbf{t}} + \mathbb{I}'_{j,\mathbf{t}} \subseteq I_{j,\mathbf{t}}$ and $F(\cdot)$ is $(\omega_j, c_j; r_j^l, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; r_j^l, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent), where $\mathbb{I}'_j := \{\mathbb{I}'_{j,\mathbf{t}} : \mathbf{t} \in I\}$ for all $j \in \mathbb{N}_n$.
- (iv) Suppose that (1.1) holds and $F : I \rightarrow X$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent). Then the following holds:
- If p is even, then $F(\cdot)$ is $(\omega_j, c_j; 1, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; 1, q\mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent), where $q\mathbb{I}'_j := \{q\mathbb{I}'_{j,\mathbf{t}} : \mathbf{t} \in I\}$ for all $j \in \mathbb{N}_n$.
 - If p is odd, then $F(\cdot)$ is $(\omega_j, c_j; -1, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; -1, q\mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent).
- (v) Let $|c| = 1$ and $\arg(c)/\pi \notin \mathbb{Q}$. If, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, $\mathbb{I}'_{j,\mathbf{t}} = I'_{j,\mathbf{t}}$ for all $l \in \mathbb{N}$ and $F : I \rightarrow X$ is a bounded, $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent) function, then the function $F(\cdot)$ is $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -almost periodic ($(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniformly recurrent) for all $(r'_1, \dots, r'_n) \in (S_1)^n$.

Concerning the convolution invariance of spaces introduced in Definition 2.16, the following important fact should be said: we have introduced the notion of $(I'_{j,\mathbf{t}}, r_j)$ -almost periodicity, for example, by requiring that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in I$, the function $G_{j,\mathbf{t}}(\cdot)$ is $(I'_{j,\mathbf{t}}, r_j)$ -almost periodic. Unfortunately, sometimes we need to assume that, for every $j \in \mathbb{N}_n$, the function $G_{j,\mathbf{t}}(\cdot)$ is $(I'_{j,\mathbf{t}}, r_j)$ -almost periodic uniformly in the variable $\mathbf{t} \in I$, in a certain sense. For simplicity, let us assume that $I = \mathbb{R}^n$, which immediately implies that, for every $j \in \mathbb{N}_n$ and $\mathbf{t} \in \mathbb{R}^n$, we have $I_{j,\mathbf{t}} = [0, \infty)$. Assume, further, that for each $j \in \mathbb{N}_n$ there exists a set $A_j \subseteq [0, \infty)$ such that $A_j = I'_{j,\mathbf{t}}$ for every $\mathbf{t} \in \mathbb{R}^n$, as well as that for each $\epsilon > 0$ there exists $l > 0$ such that for each $x_0 \in A_j$ we have the existence of a number $x \in B(x_0, l) \cap A_j$ such that

$$\|G_{j,\mathbf{t}}(x + \tau) - r_j G_{j,\mathbf{t}}(x)\| \leq \epsilon, \quad x \geq 0, \mathbf{t} \in \mathbb{R}^n,$$

i.e.,

$$\left\| c_j^{-\frac{t_j+x+\tau}{\omega_j}} F(\mathbf{t} + (x + \tau)e_j) - r_j c_j^{-\frac{t_j+x}{\omega_j}} F(\mathbf{t} + xe_j) \right\| \leq \epsilon, \quad x \geq 0, \mathbf{t} \in \mathbb{R}^n. \quad (2.10)$$

If $h \in L^1(\mathbb{R}^n)$ and $F(\cdot)$ is a bounded, continuous function, then the function $(h \star F)(\cdot)$ is well defined on $\mathbb{R}^n$, bounded and continuous. If we assume, in addition to the all above, that $|c_j| = 1$ for all $j \in \mathbb{N}_n$, then the estimate (2.10) will be invariant under the action of convolution $h \star \cdot$, since

$$\begin{aligned} & \left\| c_j^{-\frac{t_j+x+\tau}{\omega_j}} (h \star F)(\mathbf{t} + (x + \tau)e_j) - r_j c_j^{-\frac{t_j+x}{\omega_j}} (h \star F)(\mathbf{t} + xe_j) \right\| \\ & \leq \int_{\mathbb{R}^n} |h(y)| \left\| c_j^{-\frac{t_j+x+\tau}{\omega_j}} F(\mathbf{t} + (x + \tau)e_j - y) - r_j c_j^{-\frac{t_j+x}{\omega_j}} F(\mathbf{t} + xe_j - y) \right\| dy \\ & = \int_{\mathbb{R}^n} |h(y)| \left\| c_j^{-\frac{t_j-y_j+x+\tau}{\omega_j}} F(\mathbf{t} + (x + \tau)e_j - y) - r_j c_j^{-\frac{t_j-y_j+x}{\omega_j}} F(\mathbf{t} + xe_j - y) \right\| dy \\ & \leq \epsilon, \quad x \geq 0, \mathbf{t} \in \mathbb{R}^n, \end{aligned}$$

where we have employed (2.10) in the last estimate. We close this section with the observation that a similar result can be established for the $(\omega_j, c_j; r_j, \mathbb{I}'_j)_{j \in \mathbb{N}_n}$ -uniform recurrence and the $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -almost automorphy.

3 Further generalizations of (ω, c) -periodicity and $(\omega_j, c_j)_{j \in \mathbb{N}_n}$ -periodicity

Unless stated otherwise, in this section we will assume that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, $\omega \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$. Define $S := \{i \in \mathbb{N}_n : \omega_i \neq 0\}$ and A to be the collection of all tuples $\mathbf{a} = (a_1, a_2, \dots, a_{|S|}) \in \mathbb{R}^{|S|}$ such that $\sum_{i \in S} a_i = 1$. Let $\mathbf{a} \in A$.

Following our analyses from [16, Section 3], we introduce the next notion:

Definition 3.1. We say that a continuous function $F : I \rightarrow X$ is:

- (i) $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2, if and only if there exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I' such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and

$$\lim_{k \rightarrow +\infty} \sup_{\mathbf{t} \in I} \left\| F(\mathbf{t} + \alpha_k) - c^{\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}) \right\| = 0, \quad (3.1)$$

resp.

$$\lim_{k \rightarrow +\infty} \sup_{\mathbf{t} \in I} \left\| c^{-\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t} + \alpha_k) - F(\mathbf{t}) \right\| = 0; \quad (3.2)$$

- (ii) $(I', \mathbf{a}, \omega, c)$ -almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, c)$ -almost periodic of type 2, if and only if for each $\epsilon > 0$ and $\mathbf{t}_0 \in I'$ there exist a finite number $l > 0$ and a point $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that

$$\sup_{\mathbf{t} \in I} \left\| F(\mathbf{t} + \tau) - c^{\sum_{i \in S} \frac{a_i \tau_i}{\omega_i}} F(\mathbf{t}) \right\| < \epsilon,$$

resp.

$$\sup_{\mathbf{t} \in I} \left\| c^{-\sum_{i \in S} \frac{a_i \tau_i}{\omega_i}} F(\mathbf{t} + \tau) - F(\mathbf{t}) \right\| < \epsilon. \quad (3.3)$$

If $|c| = 1$, then the concept $(I', \mathbf{a}, \omega, c)$ -uniform recurrence of type 1 and the concept $(I', \mathbf{a}, \omega, c)$ -uniform recurrence of type 2 coincide, as easily approved by multiplying (3.1) with $c^{-\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}}$; this also holds for the concepts $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 1 and $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 2, but then we can say a little bit more. Speaking-matter-of-factly, we can multiply the both sides of (3.3) with $c^{-\sum_{i \in S} \frac{a_i \tau_i}{\omega_i}}$ in order to see that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -almost periodic of type 2 (1) if and only if the function $G_{\mathbf{a}}(\cdot)$, defined through (2.4), is I' -almost periodic; in the usually considered case $I = I' = \mathbb{R}^n$, this is equivalent to saying that the function $F(\cdot)$ is almost periodic.

The function spaces introduced in Definition 2.16 are translation invariant and closed under the pointwise multiplications with complex scalars; if $I = \mathbb{R}^n$ and $F(\cdot)$ is a bounded, continuous function which belongs to any of the above introduced function spaces, then for each $h \in L^1(\mathbb{R}^n)$ the function $(h * F)(\cdot)$ is also bounded and belongs to the same space. Furthermore, if the function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, c)$ -almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, c)$ -almost periodic of type 2], then $F(\cdot)$ is $(I', \mathbf{a}, \omega, |c|)$ -uniformly recurrent of type 1, resp. $(I', \mathbf{a}, \omega, |c|)$ -uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, |c|)$ -almost periodic of type 1, resp. $(I', \mathbf{a}, \omega, |c|)$ -almost periodic of type 2]. Concerning the invariance of function spaces under the operation of uniform convergence, we will only state that the assumptions $a_j \omega_j > 0$ for all $j \in S$, $|c| \leq 1$, $I' \subseteq [0, \infty)^n$ and the sequence (F_k) of $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent functions of type 1 [$(I', \mathbf{a}, \omega, c)$ -almost periodic functions of type 1] uniformly converges to a function $F : I \rightarrow X$ imply that the function $F(\cdot)$ is likewise $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1 [$(I', \mathbf{a}, \omega, c)$ -almost periodic of type 1].

For the sequel, we need the following extension of [16, Lemma 3.4]:

Lemma 3.2. Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, $I = -I$, $I + I' = I$ and the function $F : I \rightarrow X$ is continuous. Then $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1 [$(I', \mathbf{a}, \omega, c)$ -almost periodic of type 1] if and only if $\check{F}(\cdot)$ is $(I', \mathbf{a}, \omega, 1/c)$ -uniformly recurrent of type 2 [$(I', \mathbf{a}, \omega, 1/c)$ -almost periodic of type 2].

Proof. We will present the main details of the proof provided that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1. Then there exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I' such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and (3.1) holds. Since we have assumed $I = -I$ and $I + I' = I$, the proof simply follows from the next computation ($k \in \mathbb{N}$):

$$\begin{aligned} & \sup_{\mathbf{t} \in I} \left\| (1/c)^{-\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} \check{F}(\mathbf{t} + \alpha_k) - \check{F}(\mathbf{t}) \right\| \\ &= \sup_{\mathbf{t} \in I} \left\| (1/c)^{-\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(-\mathbf{t} - \alpha_k) - F(-\mathbf{t}) \right\| \\ &= \sup_{\mathbf{t} \in -(I+I')} \left\| c^{\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t} + \alpha_k) \right\| \\ &= \sup_{\mathbf{t} \in -I} \left\| c^{\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t} + \alpha_k) \right\| \\ &= \sup_{\mathbf{t} \in I} \left\| c^{\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}) - F(\mathbf{t} + \alpha_k) \right\|. \end{aligned}$$

□

Concerning [16, Theorem 3.2(i)] and its proof, we will first state and prove the following result:

Theorem 3.3. Suppose that $\emptyset \neq I' \subseteq I \subseteq \mathbb{R}^n$, I' is unbounded, $F : I \rightarrow X$ is continuous, $I + I' = I$, $\omega \in \mathbb{R}^n \setminus \{0\}$, $|c| > 1$, $S = \mathbb{N}_n$ and any component of a tuple $\mathbf{a} \in A$ is positive. Suppose further that, for every $\mathbf{t} \in I$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geq 0$. Then the following assertions are equivalent:

- (i) The function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1.
- (ii) The function $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2.
- (iii) There exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I' such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and the function $G_{\mathbf{a}} : I \rightarrow X$, defined through (2.4), satisfies $G_{\mathbf{a}}(\mathbf{t} + \alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$.
- (iv) There exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I' such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and

$$F(\mathbf{t} + \alpha_k) = c^{\sum_{i \in S} \frac{a_i \alpha_{k,i}}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I, \quad k \in \mathbb{N}. \quad (3.4)$$

- (v) There exists a point $\omega \in I' \setminus \{0\}$ such that

$$F(\mathbf{t} + \omega) = c^{\sum_{i \in S} \frac{a_i \omega_i}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I. \quad (3.5)$$

Proof. If $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1, then our assumptions $a_j > 0$ and $\alpha_{k,j}/\omega_j > 0$ ($k \in \mathbb{N}$, $j \in \mathbb{N}_n$) imply that $|c^{-\sum_{j=1}^n \frac{a_j \alpha_{k,j}}{\omega_j}}| \leq 1$, so that (3.1) implies (3.2) after multiplication with $c^{-\sum_{j=1}^n \frac{a_j \alpha_{k,j}}{\omega_j}}$; hence, (i) implies (ii). Suppose now that $F(\cdot)$ is $(I', \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2 and the sequence (α_k) in I' satisfies (3.2). Let $k \in \mathbb{N}$ be fixed. Then (3.2) implies the existence of a finite real number $M \geq 1$ such that

$$\sup_{\mathbf{t} \in I} \left\| G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t}) \right\| \leq M |c|^{-\sum_{j=1}^n a_j t_j / \omega_j}.$$

Since we have assumed that $a_j > 0$ and $t_j/\omega_j > 0$ for all $j \in \mathbb{N}_n$, the above estimate yields

$$\left\| G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t}) \right\| \leq M |c|^{-\min\{a_j; j \in \mathbb{N}_n\} \left[\max\{\omega_j; j \in \mathbb{N}_n\} \right]^{-1} \left[|t_1| + \dots + |t_n| \right]},$$

for all $\mathbf{t} \in I$, which implies that $\lim_{|t| \rightarrow \infty} \|G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t})\| = 0$. On the other hand, (3.2) implies

$$\lim_{m \rightarrow +\infty} \left\| G_{\mathbf{a}}(\mathbf{t} + \alpha_m + \alpha_k) - G_{\mathbf{a}}(\mathbf{t} + \alpha_m) \right\| = 0.$$

Therefore, the function $\mathbf{t} \mapsto G_{\mathbf{a}}(\mathbf{t} + \alpha_k) - G_{\mathbf{a}}(\mathbf{t})$, $\mathbf{t} \in I$ is I' -uniformly recurrent and tends to zero as $|\mathbf{t}| \rightarrow +\infty$. Since we have assumed that $I + I' = I$, we may apply [19, Corollary 2.11] in order to see that $G_{\mathbf{a}}(\mathbf{t} + \alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$, which implies (iii). The implications (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (v) are trivial. To complete the proof, it suffices to show that (v) implies (iv). This follows by plugging $\alpha_k := k\omega$ for all $k \in \mathbb{N}$ since (3.5) implies inductively

$$F(\mathbf{t} + k\omega) = c^{\sum_{i \in S} \frac{a_i k \omega_i}{\omega_i}} F(\mathbf{t}), \quad \mathbf{t} \in I, \quad k \in \mathbb{N}.$$

□

- Remark 3.4.** (i) Since I' is unbounded, it is clear that the $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 1 implies the $(I', \mathbf{a}, \omega, c)$ -uniform recurrence of type 1 for $F(\cdot)$ as well as that the $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 1 implies the $(I', \mathbf{a}, \omega, c)$ -almost periodicity of type 2 for $F(\cdot)$, which further implies the $(I', \mathbf{a}, \omega, c)$ -uniform recurrence of type 2 for $F(\cdot)$.
- (ii) Let (α_k) be a sequence from (iv). Then it is clear that (iv) implies that for each number $k \in \mathbb{N}$ the function $F(\cdot)$ is $(I'_k, \mathbf{a}, \omega, c)$ -almost periodic of type 1, where $I'_k := \{m\alpha_k : m \in \mathbb{N}\}$. Keeping Theorem 3.3 and this observation in mind, we have extended so the first part of [16, Theorem 3.2(i)], where we have assumed that $I = [0, \infty)$.

Concerning the statement of [16, Theorem 3.2(i)] with the interval $I = \mathbb{R}$, we would like to note that it can be extended to the higher dimensions as follows. Suppose that $I = I_0 \cup I_1$, where $\emptyset \neq I'_0 \subseteq I_0 \subseteq \mathbb{R}^n$, $I_0 + I'_0 = I_0$ and the function $F : I \rightarrow X$ is $(I_0, \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2, where $|c| > 1$, $S = \mathbb{N}_n$, any component of a tuple $\mathbf{a} \in A$ is positive and, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geq 0$. Then the restriction of function $F(\cdot)$ to the interval I_0 is $(I'_0, \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2, as well, so that we can apply Theorem 3.3 in order to conclude that (3.4) holds for every $\mathbf{t} \in I_0$ and $k \in \mathbb{N}$. To show the validity of this condition for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$, we may assume additionally that:

- (a) For every $\mathbf{t} \in I_1$, there exists $m_0 \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$ with $m \geq m_0$, we have $\mathbf{t} + \alpha_m \in I_0$. Applying (3.4) twice, with $\mathbf{t} + \alpha_m$ and $\mathbf{t}$ the first time, and with $\mathbf{t} + \alpha_k + \alpha_m$ and $\mathbf{t} + \alpha_k$ the second time, we easily get that (3.4) holds for every $\mathbf{t} \in I$. Therefore, we have proved the following:

Theorem 3.5. Suppose that $\emptyset \neq I'_0 \subseteq I_0 \subseteq \mathbb{R}^n$, I'_0 is unbounded, $I_0 + I'_0 = I_0$, $I = I_0 \cup I_1$, condition (a) holds and $F : I \rightarrow X$ is continuous. Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $|c| > 1$, $S = \mathbb{N}_n$ and any component of a tuple $\mathbf{a} \in A$ is positive. Suppose further that, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geq 0$. Then the following assertions are equivalent:

- (i) The function $F(\cdot)$ is $(I'_0, \mathbf{a}, \omega, c)$ -uniformly recurrent of type 1.
(ii) The function $F(\cdot)$ is $(I'_0, \mathbf{a}, \omega, c)$ -uniformly recurrent of type 2.
(iii) There exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I'_0 such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and the function $G_{\mathbf{a}} : I \rightarrow X$, defined through (2.4), satisfies $G_{\mathbf{a}}(\mathbf{t} + \alpha_k) = G_{\mathbf{a}}(\mathbf{t})$ for all $\mathbf{t} \in I$ and $k \in \mathbb{N}$.
(iv) There exists a sequence $(\alpha_k = (\alpha_{k,1}, \dots, \alpha_{k,n}))$ in I'_0 such that $\lim_{k \rightarrow +\infty} |\alpha_k| = +\infty$ and (3.4) holds.
(v) There exists a point $\omega \in I' \setminus \{0\}$ such that (3.5) holds.

Suppose now that $|c| < 1$, $S := \mathbb{N}_n$, any component of a tuple $\mathbf{a} \in A$ is positive and, for every $\mathbf{t} \in I_0$ and $j \in \mathbb{N}_n$, we have $\omega_j t_j \geq 0$. Applying Lemma 3.2, we can simply extend the statement of [16, Theorem 3.2(ii)] to the higher dimensions, provided that condition (a) holds with $I_1 = -I_0$. Details can be left to the interested readers.

In the case that $a_j \omega_j > 0$ for all $j \in S = \mathbb{N}_n$, $|c| < 1$, $I = I' = [0, \infty)^n$, then it can be simply proved (cf. [16, Proposition 3.6, Corollary 3.8]) that the function $F : I \rightarrow X$ is $(I'_0, \mathbf{a}, \omega, c)$ -almost periodic of type 1 if and only if there exists a finite constant $M \geq 1$ such that

$$\|F(\mathbf{t})\| \leq M|c|^{\sum_{i \in S} a_i t_i / \omega_i}, \quad \mathbf{t} \in I;$$

the statement of [16, Proposition 3.11] can be also extended to the higher dimensions provided that the function $F(\cdot)$ is bounded, $a_j \omega_j > 0$ for all $j \in S = \mathbb{N}_n$ and $|c| < 1$. Without any essential changes of the proof of [16, Proposition 3.12], we may deduce the following:

Proposition 3.6. *Suppose that $a_j \omega_j > 0$ for all $j \in S = \mathbb{N}_n$, $|c| < 1$ and $I = I' = [0, \infty)^n$. Then a continuous function $F : I \rightarrow X$ is $(I'_0, \mathbf{a}, \omega, c)$ -almost periodic of type 2 if and only if the function $\mathbf{t} \mapsto G(\mathbf{t}) \equiv c^{\sum_{i \in S} -a_i t_i / \omega_i} F(\mathbf{t})$, $\mathbf{t} \in I$ is bounded, continuous and satisfies that for each $\epsilon > 0$, $\mathbf{t}_0 \in I$ and $N > 0$ there exist a finite number $l > 0$ and a point $\tau \in B(\mathbf{t}_0, l) \cap I$ such that*

$$\|G(\mathbf{t} + \tau) - G(\mathbf{t})\| \leq \epsilon, \quad \mathbf{t} \in I_N.$$

In connection with Proposition 3.6, we want to note that the notion of a complex-valued Levitan N -almost periodic function was introduced by B. M. Levitan in 1937 (see [21]–[22] and references cited therein) and later studied on topological groups by B. Ya. Levin [20] in 1949. Let us recall that a continuous function $f : [0, \infty) \rightarrow X$ is said to be Levitan N -almost periodic if and only if for each $\epsilon > 0$ and $N > 0$ the set of all positive reals $\tau > 0$ such that $\|f(t + \tau) - f(t)\| \leq \epsilon$, $t \in [0, N]$ is relatively dense in $[0, \infty)$. The study of vector-valued Levitan N -almost periodic functions on topological (semi-)groups and multi-dimensional vector-valued Levitan N -almost periodic functions will be carried out somewhere else.

In our previous research studies of the multi-dimensional almost periodicity, we have also analyzed the invariance of almost periodicity under the actions of the finite convolution products and the infinite convolution products. In the one-dimensional case, this theme is crucially important for giving the most intriguing applications in the qualitative analysis of almost periodic type solutions for various classes of the abstract Volterra integro-differential equations. In the multi-dimensional case, the results obtained so far are not so easily applicable and, because of that, we will skip all related details with regards to this question.

4 Applications

In this section, we will present several illustrative examples and applications of our results to the abstract Volterra integro-differential equations in Banach spaces. We feel it is our duty to say that the the points [1,3,4,5] have been also considered in [8] for multi-dimensional almost periodic functions.

1. We start with the observation that all established results on the convolution invariance of introduced function spaces can be applied to the Gaussian semigroup

$$(G(t)F)(x) := (4\pi t)^{-(n/2)} \int_{\mathbb{R}^n} F(x-y) e^{-\frac{|y|^2}{4t}} dy, \quad t > 0, f \in Y, x \in \mathbb{R}^n;$$

see [5, Example 3.7.6] for more details. Suppose, for example, that $I = \mathbb{R}^n$, $|c| = 1$ and $F(\cdot)$ is a bounded, (ω, c) -periodic function. Then, due to Proposition 2.14, we have that for each $t_0 > 0$ the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is likewise bounded and (ω, c) -periodic. A similar result can be given for the Poisson semigroup; see [5, Example 3.7.9] and [8] for more details. Concerning the strongly continuous semigroups, we would like to note that our recent consideration from [19, Example 1.1] can be used to justify the introduction of analyzed function spaces, as well.

2. In [19, Example 1.2], we have recently observed an interesting feature of the famous d'Alembert formula, which has been used by S. Zaidman [25, Example 5] for almost periodic functions of one real variable, a long time ago (see also [14, Example 2.3]). Suppose that $a > 0$. Then it is well known that the regular solution of the wave equation $u_{tt} = a^2 u_{xx}$ in domain $\{(x, t) : x \in \mathbb{R}, t > 0\}$, equipped with the initial conditions $u(x, 0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x, 0) = g(x) \in C^1(\mathbb{R})$, is given by the d'Alembert formula

$$u(x, t) = \frac{1}{2} [f(x-at) + f(x+at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds, \quad x \in \mathbb{R}, t > 0.$$

In the above-mentioned example, if the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is c -almost periodic, where $g^{[1]}(\cdot) \equiv \int_0^\cdot g(s) ds$; then the conclusion is: the solution $u(x, t)$ can be extended to the whole real line in the time variable and this solution is c -almost periodic in $(x, t) \in \mathbb{R}^2$.

2.1. We assume here that there exist numbers $\omega \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{C} \setminus \{0\}$ such that the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is (ω, c) -periodic. Then it is clear that the solution $u(x, t)$ can be extended to the whole real line in the time variable and now we will prove that

$$u(x + \omega, t) = cu(x, t), \quad x, t \in \mathbb{R},$$

i.e., the function $u(\cdot; \cdot)$ is $((\omega, 0), c)$ -periodic. But, the last equality simply follows from the next calculation:

$$\begin{aligned} u(x + \omega, t) &= \frac{1}{2} [f(x - at + \omega) + f(x + at + \omega)] \\ &\quad + \frac{1}{2a} [g^{[1]}(x + at + \omega) - g^{[1]}(x - at + \omega)] \\ &= \frac{1}{2} [cf(x - at) + cf(x + at)] \\ &\quad + \frac{c}{2a} [g^{[1]}(x + at) - g^{[1]}(x - at)] = cu(x, t), \quad x, t \in \mathbb{R}. \end{aligned}$$

2.2. We assume here that there exist numbers $\omega \in \mathbb{R} \setminus \{0\}$, $k \in \mathbb{N}$ and $c \in \mathbb{C} \setminus \{0\}$ such that $c^{k-1} = 1$ and the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is (ω, c) -periodic. Set

$$\omega_1 := \frac{1+k}{2}\omega \text{ and } \omega_2 := \frac{k-1}{2a}\omega.$$

Then $(\omega_1, \omega_2) \neq (0, 0)$, $\omega_1 - a\omega_2 = \omega$, $\omega_1 + a\omega_2 = k\omega$, $c^k = c$, $f(x + \omega) = cf(x) = c^kf(x) = f(x + k\omega)$, $g^{[1]}(x + \omega) = cg^{[1]}(x) = c^kg^{[1]}(x) = g^{[1]}(x + k\omega)$ for all $x \in \mathbb{R}$, and we can simply show as above that

$$u(x + \omega_1, t + \omega_2) = cu(x, t), \quad x, t \in \mathbb{R},$$

i.e., the function $u(\cdot; \cdot)$ is $((\omega_1, \omega_2), c)$ -periodic.

2.3. Let the assumptions of the previous point hold. Assume, further, that the function $x \mapsto (f_0(x), g_0^{[1]}(x))$, $x \in \mathbb{R}$ satisfies $\lim_{x \rightarrow \pm\infty} f_0(x) = \lim_{x \rightarrow \pm\infty} g_0^{[1]}(x) = 0$. Set $B := \{(x, t) \in \mathbb{R}^2 : x = \pm at\}$. If $\mathbb{D}$ is any subset of $\mathbb{R}^2$ satisfying that

$$\lim_{|(x,t)| \rightarrow +\infty, (x,t) \in \mathbb{D}} \text{dist}((x, t); B) = +\infty,$$

then the solution given by the d'Alembert formula, with the functions $f(\cdot)$ and $g(\cdot)$ replaced therein with the functions $(f + f_0)(\cdot)$ and $(g + g_0)(\cdot)$, is $\mathbb{D}$ -asymptotically $((\omega_1, \omega_2), c)$ -periodic.

3. Let $\omega \in \mathbb{R} \setminus \{0\}$ and $|c| = 1$. Equipped with the sup-norm, the space $B_{\omega, c}(\mathbb{R}^n : X)$ consisting of all X -valued, bounded and (ω, c) -periodic functions becomes a Banach space. In a series of our previous research studies, we have analyzed the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$:

$$y(\mathbf{t}) = g(\mathbf{t}) + \int_{\mathbb{R}^n} k(\mathbf{t} - \mathbf{s})G(\mathbf{s}, y(\mathbf{s})) d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n. \quad (4.1)$$

Suppose now that $g : \mathbb{R}^n \rightarrow X$ is bounded and (ω, c) -periodic, $k \in L^1(\mathbb{R}^n)$, $G : \mathbb{R}^n \times X \rightarrow X$ is continuous and satisfies that for each bounded subset of X we have that the set $\{G(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n, x \in B\}$ is bounded as well as that $G(\mathbf{t} + \omega, x) = cG(\mathbf{t}, x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. If there exists a finite real constant $L \geq 1$ such that

$$\|G(\mathbf{t}, x) - G(\mathbf{t}, y)\| \leq L\|x - y\|, \quad \mathbf{t} \in \mathbb{R}^n; x, y \in X$$

and $L \int_{\mathbb{R}^n} |k(\mathbf{y})| d\mathbf{y} < 1$, then we can apply the Banach contraction principle and Proposition 2.14 in order to see that there exists a unique solution of the integral equation (4.1) which belongs to the space $B_{\omega, c}(\mathbb{R}^n : X)$.

4. Of concern is the system of abstract partial differential equations

$$u_s(s, t) = Au(s, t) + f_1(s, t), \quad u_t(s, t) = Bu(s, t) + f_2(s, t); \quad u(0, 0) = x, \quad s \geq 0, \quad t \geq 0. \quad (4.2)$$

Motivated by the recent results of S. M. A. Alsulami given in [1, Section 2.1], we have recently considered, in [8], the case in which A and B are two complex matrices of format $n \times n$, $AB = BA$, and A , resp. B , generate an exponentially decaying, strongly continuous semigroup $(T_1(s))_{s \geq 0}$, resp. $(T_2(t))_{t \geq 0}$. The following assumptions have been made there: the functions $f_1(s, t)$ and $f_2(s, t)$ are continuously differentiable, the compatibility condition $(f_2)_s - Af_2 = (f_1)_t - Bf_1$ holds ($s, t \geq 0$), $\mathbb{D} := \{(s, t) \in [0, \infty)^2 : c_1s \leq t \leq c_2s \text{ for some positive real numbers } c_1 \text{ and } c_2\}$, and

- (i) There is a finite real constant $M > 0$ such that $|f_1(v, 0)| + |f_2(0, \omega)| \leq M$, for $v, \omega \geq 0$;
(ii) The mappings $g_i : \mathbb{R}^2 \rightarrow \mathbb{C}^n$ are continuous, bounded ($i = 1, 2$) and satisfy that, for every $\epsilon > 0$, there exists $l > 0$ such that any subinterval I of $\mathbb{R}$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s, t \geq 0$, we have $|g_1(s + \tau, t) - g_1(s, t)| \leq \epsilon$ and $|g_2(s, t + \tau) - g_2(s, t)| \leq \epsilon$;
(iii) The function $q_i : [0, \infty)^2 \rightarrow \mathbb{C}^n$ is bounded, $q_i \in C_{0,\mathbb{D}}([0, \infty)^2 : \mathbb{C}^n)$ and $f_i(s, t) = g_i(s, t) + q_i(s, t)$ for $(s, t) \in [0, \infty)^2$ and $i = 1, 2$.

The conclusion is: there is a unique classical solution $u(s, t)$ of (4.2) (cf. also [1, Definition 2.13]), and moreover, there exist a continuous function $u_{ap}(s, t)$ on $[0, \infty)^2$ and a function $u_0 \in C_{0,\mathbb{D}}([0, \infty)^2 : \mathbb{C}^n)$ such that $u(s, t) = u_{ap}(s, t) + u_0(s, t)$ for all $(s, t) \in [0, \infty)^2$, as well as for every $\epsilon > 0$, there exists $l > 0$ such that any subinterval I of $[0, \infty)$ of length $l > 0$ contains a number $\tau \in I$ such that, for every $s, t \geq 0$, we have $|u_{ap}(s + \tau, t) - u_{ap}(s, t)| \leq \epsilon$ and $|u_{ap}(s, t + \tau) - u_{ap}(s, t)| \leq \epsilon$.

If we replace condition (ii) with condition:

- (ii)' The mappings $g_i : \mathbb{R}^2 \rightarrow \mathbb{C}^n$ are continuous, bounded ($i = 1, 2$) and satisfy that there exist positive real numbers $\omega_1 > 0$ and $\omega_2 > 0$ as well as complex numbers c_1 and c_2 such that $|c_1| = |c_2| = 1$ and, for every $s, t \in \mathbb{R}$, we have $g_1(s + \omega_1, t) = c_1 g_1(s, t)$ and $g_2(s, t + \omega_2) = c_2 g_2(s, t)$,

and accept all remaining assumptions, then we similarly may deduce that there exist a continuous function $u_h(s, t)$ on $[0, \infty)^2$ and a function $u_0 \in C_{0,\mathbb{D}}([0, \infty)^2 : \mathbb{C}^n)$ such that $u(s, t) = u_h(s, t) + u_0(s, t)$ for all $(s, t) \in [0, \infty)^2$, as well as that, for every $s, t \geq 0$, we have $u_h(s + \omega_1, t) = c_1 u_h(s, t)$ and $u_h(s, t + \omega_2) = c_2 u_h(s, t)$.

5. Finally, it is worth recalling that the existence and uniqueness of almost periodic solutions for a class of boundary value problems for hyperbolic equations have been investigated by B. I. Ptashnic and P. I. Shtabalyuk in [24]. In the region $D_p = (0, T) \times \mathbb{R}^p$ ($T > 0, p \in \mathbb{N}$), these authors have analyzed the almost periodic type solutions of the following initial value problem:

$$Lu \equiv \sum_{s=0}^n \sum_{|\alpha|=2s} a_\alpha \frac{\partial^{2n} u(t, x)}{\partial t^{2n-2s} \partial x_1^{\alpha_1} \dots \partial x_p^{\alpha_p}} = 0, \quad (4.3)$$

$$\left. \frac{\partial^{j-1} u}{\partial t^{j-1}} \right|_{t=0} = \varphi_j(x), \quad \left. \frac{\partial^{j-1} u}{\partial t^{j-1}} \right|_{t=T} = \varphi_{j+n}(x) \quad (1 \leq j \leq n). \quad (4.4)$$

Suppose that any of the functions $\varphi_1(x), \dots, \varphi_{2n}(x)$ is almost periodic in $\mathbb{R}^p$ and $M_p = \{\mu_k : k \in \mathbb{Z}^p\}$ is the union of all Bohr-Fourier spectrum of functions $\varphi_1(x), \dots, \varphi_{2n}(x)$. Under certain assumptions, the solutions $u(t, x)$ of problem (4.3)-(4.4) have been found in the form

$$u(t, x) = \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x \rangle}, \quad \mu_k \in M_p, \quad (4.5)$$

where the functions $u_k(t)$ have the form [24, (8), p. 670] (cf. [24, Theorem 1, Theorem 2] for more details concerning the existence and uniqueness of solutions to (4.3)-(4.4)).

Suppose now that $\omega \in \mathbb{R}^p \setminus \{0\}$ and $C \in \mathbb{R}$. We want to observe here that the assumption $\varphi_j \in AP_A(\mathbb{R}^n : \mathbb{C})$ for all $j \in \mathbb{N}_{2n}$, where

$$\Lambda := \{(x_1, \dots, x_p) \in \mathbb{R}^p ; x_1 \omega_1 + \dots + x_p \omega_p = C\},$$

implies that the solution $u(t, x)$ of problem (4.3)-(4.4) is (ω, e^{iC}) -periodic in the space variable x . This follows from the computation ($t \in (0, T), x \in \mathbb{R}^p$):

$$\begin{aligned} u(t, x + \omega) &= \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x + \omega \rangle} = \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x \rangle} e^{i\langle \mu_k, \omega \rangle} \\ &= e^{iC} \sum_{k \in \mathbb{Z}^p} u_k(t) e^{i\langle \mu_k, x \rangle} = cu(t, x). \end{aligned}$$

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