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Delay-Dependent Stability Conditions for Non-autonomous Functional Differential Equations with Several Delays in a Banach Space

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Abstract: Let $B_j(t)$ ($j = 1, \dots, m$) and $B(t, \tau)$ ($t \geq 0, 0 \leq \tau \leq 1$) be bounded variable operators in a Banach space. We consider the equation

$$u'(t) = \sum_{k=1}^m B_k(t)u(t - h_k(t)) + \int_0^1 B(t, \tau)u(t - h_0(\tau))d\tau \quad (t \geq 0),$$

where $h_k(t)$ ($t \geq 0; k = 1, \dots, m$) and $h_0(\tau)$ are continuous nonnegative bounded functions. Explicit delay-dependent exponential stability conditions for that equation are established. Applications to integro-differential equations with delay are also discussed.

Keywords: differential-delay equations in a Banach space; stability; integro-differential equations

MSC: 34K30, 34K06, 34K20

1 Introduction and statement of the main result

This paper is devoted to a class of linear nonautonomous functional differential equations in a Banach space with several variable delays, whose coefficients are bounded operators. Such equations include integro-differential equations with delay.

The basic method for the stability analysis of functional differential equations is the Lyapunov-Krasovskij method [5]. By that method, many great results have been obtained. Recently, that method has been extended to functional differential equations in a Hilbert space, cf. [10, 11, 14, 16, 18] and references given therein. Besides, mainly equations with one delay have been considered. To the best of our knowledge, the stability of nonautonomous equations in a Banach space with several delays are not investigated in the available literature. Below we obtain delay-dependent exponential stability conditions for nonautonomous functional-differential equations in a Banach space with several delays.

It should be noted that finding the Lyapunov-Krasovskij type functionals or solving the corresponding operator inequalities are often connected with serious mathematical difficulties, especially regarding nonautonomous equations with many delays. To the contrary, the stability conditions presented in this paper are explicitly formulated in terms of the coefficients and delays. The literature on the delay-dependent stability criteria is rather rich, but mainly equations in a finite dimensional space are considered, cf. [7, 8, 15].

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Introduce the notations. Everywhere below, $\mathcal{X}$ is a complex Banach space with a norm $\|\cdot\|_{\mathcal{X}} = \|\cdot\|$ and the unit operator $I_{\mathcal{X}} = I$. By $\mathcal{B}(\mathcal{X})$, we denote the set of all bounded linear operators in $\mathcal{X}$. For an $A \in \mathcal{B}(\mathcal{X})$, $\sigma(A)$ is the spectrum and $\|A\|$ is the operator norm.

For real $a < b < \infty$, $C([a, b], \mathcal{X})$ is the space of $\mathcal{X}$ -valued functions f defined and continuous on $[a, b]$, and equipped with the finite norm

$$\|f\|_{C(a,b)} = \|f\|_{C([a,b],\mathcal{X})} = \sup_{a \leq t \leq b} \|f(t)\|_{\mathcal{X}},$$

and $C([a, \infty), \mathcal{X})$ is the space of $\mathcal{X}$ -valued functions f defined, bounded and continuous on $[a, \infty)$ with the finite norm $\|f\|_{C(a,\infty)} = \|f\|_{C([a,\infty),\mathcal{X})} = \sup_{t \geq a} \|f(t)\|_{\mathcal{X}}$. Put $R_+ = [0, \infty)$ and $R_{\eta} = [-\eta, \infty)$ with $0 < \eta < \infty$, and introduce on $C(R_{\eta}, \mathcal{X})$ the operator E by

$$(Eu)(t) = \sum_{k=1}^m B_k(t)u(t - h_k(t)) + \int_0^1 B(t, \tau)u(t - h_0(\tau))d\tau \quad (u \in C(R_{\eta}, \mathcal{X}); t \geq 0),$$

where $h_k(t)$ ($k = 1, \dots, m$) and $h_0(\tau)$ are continuous nonnegative functions defined on R_+ and $[0, 1]$, respectively, such that

$$\sup_{t \geq 0} h_k(t) \leq \eta \quad (k = 1, \dots, m) \text{ and } \sup_{0 \leq \tau \leq 1} h_0(\tau) \leq \eta;$$

$B_j(t) : R_+ \rightarrow \mathcal{B}(\mathcal{X})$ ($j = 1, \dots, m$) are continuous in the operator norm, $B(t, \tau) : R_+ \times [0, 1] \rightarrow \mathcal{B}(\mathcal{X})$ is continuous in the operator norm in t and piece-wise continuous in τ .

The present paper is devoted to the equation

$$dv(t)/dt = (Ev)(t) \quad (t \geq 0) \quad (1.1)$$

with the initial condition

$$v(t) = \phi(t) \quad (-\eta \leq t \leq 0, \phi \in C([-\eta, 0], \mathcal{X})), \quad (1.2)$$

where ϕ is given. A solution of problem (1.1), (1.2) is a function $v : R_{\eta} \rightarrow \mathcal{X}$ having a strong derivative $v'(t) \in \mathcal{X}$ for each finite $t > 0$ and satisfying (1.1) and (1.2).

Below we check the existence of solutions to (1.1) under consideration. Equation (1.1) is said to be *exponentially stable* if there are positive constants m_0, δ_0 independent of the initial function, such that

$$\|v(t)\|_{\mathcal{X}} \leq \|\phi\|_{C([-\eta, 0], \mathcal{X})} m_0 e^{-\delta_0 t} \quad (t \geq 0)$$

for any solution $v(t)$ of problem (1.1), (1.2).

Throughout the paper it is assumed that

$$\chi_0 := \sup_{t \geq 0} \left(\sum_{k=1}^m \|B_k(t)\| + \int_0^1 \|B(t, \tau)\| d\tau \right) < \infty \quad (1.3)$$

and therefore,

$$\psi_0 := \sup_{t \geq 0} \sum_{k=1}^m \|B_k(t)\| h_k(t) + \int_0^1 \|B(t, \tau)\| h_0(\tau) d\tau < \infty.$$

Put

$$M(t) := \sum_{k=1}^m B_k(t) + \int_0^1 B(t, \tau) d\tau$$

and introduce the operator V_M by

$$(V_M f)(t) = \int_0^t U(t, s) f(s) ds \quad (f \in C(R_+, \mathcal{X})),$$

where $U(t, s)$ is the evolution operator of the differential equation

$$z'(t) = M(t)z(t). \quad (1.4)$$

Now we are in a position to formulate our main result.

Theorem 1.1. *Let V_M be bounded in $C(R_+, \mathcal{X})$ with a norm $\|V_M\|_{C(R_+, \mathcal{X})}$. Let the conditions (1.3) and*

$$\|V_M\|_{C(R_+, \mathcal{X})} \chi_0 \psi_0 < 1 \quad (1.5)$$

hold. Then equation (1.1) is exponentially stable.

The proof of this theorem is presented in the next section.

As it follows from Theorem 1.1 our stability conditions are based, in particular, on the norm estimates for V_M . Below we consider such estimates under various conditions. In particular, in Section 3 we assume that $M(t)$ is dissipative. In Section 4 it is supposed that $M(t)$ satisfied the so called generalized Lipschitz condition. Section 5 is devoted to equations with differentiable operators a Hilbert space.

In Section 6 Theorem 1.1 is applied to integro-differential equations.

2 Proof of Theorem 1.1

Consider the non-homogeneous equation corresponding to (1.1):

$$w'(t) = (Ew)(t) + f(t) \quad (f \in C(R_+) = C(R_+, \mathcal{X}), t \geq 0) \quad (2.1)$$

with the zero initial condition

$$w(t) = 0, t \leq 0. \quad (2.2)$$

A solution of problem (2.1), (2.2) is a strongly differentiable function $w : R_\eta \rightarrow \mathcal{X}$ satisfying (2.1) for all finite $t > 0$ and (2.2). Integrating (2.1), we have

$$w(t) = \int_0^t (Ew)(s)ds + f_1(t),$$

where

$$f_1(t) = \int_0^t f(s)ds.$$

For each finite $T > 0$ the operator W defined in space $C(0, T) = C([0, T], \mathcal{X})$ by

$$(Wf)(t) = \int_0^t (Ef)(s)ds \quad (f \in C(0, T))$$

is a Volterra one and therefore, problem (2.1), (2.2) has a unique solution representable by

$$w = \sum_{k=0}^{\infty} W^k f_1.$$

Furthermore, for a finite $T > 0$ for the brevity put $|w|_T = \|w\|_{C(0, T)}$. Due to (1.3) for a solution of (2.1), (2.2) we have.

$$|Ew|_T \leq \chi_0 |w|_T. \quad (2.3)$$

Now (2.1) implies

$$|w'|_T \leq \chi_0 |w|_T + \|f\|_{C(R_+)} \quad (0 < T < \infty). \quad (2.4)$$

We have $(Ew)(t) = M(t)w(t) + (Zw)(t)$ ($t \geq 0$), where

$$(Zw)(t) = \sum_{k=1}^m B_k(t)(w(t - h_k(t)) - w(t)) + \int_0^1 B(t, \tau)(w(t - h_0(\tau)) - w(t))d\tau.$$

Thus (2.1) can be written as

$$w'(t) = M(t)w(t) + (Zw)(t) + f(t) \quad (t \geq 0). \quad (2.5)$$

By the Variation of Constants formula problem (2.5), (2.2) takes the form

$$w(t) = \int_0^t U(t, s)((Zw)(s) + f(s))ds. \quad (2.6)$$

Observe that

$$w(t) - w(t - \tau) = \int_{t-\tau}^t w'(s)ds \quad (t \geq 0, \tau > 0).$$

Hence, by (2.4)

$$\|w(t) - w(t - \tau)\|_{C(0, T)} \leq \tau \|w'\|_T \leq \chi_0 \tau \|w\|_T + \tau \|f\|_{C(R_+)}. \quad (2.7)$$

Thus

$$\begin{aligned} |Zw|_T &\leq \sup_{0 \leq t \leq T} \left(\sum_{k=1}^m \|B_k(t)\| \|w(t - h_k(t)) - w(t)\| + \int_0^1 \|B(t, \tau)\| \|w(t - h_0(\tau)) - w(t)\| d\tau \right) \\ &\leq \sup_{t \geq 0} \left(\sum_{k=1}^m \|B_k(t)\| h_k(t) (\chi_0 \|w\|_T + \|f\|_{C(R_+)}) + \int_0^1 \|B(t, \tau)\| h_0(\tau) (\chi_0 \|w\|_T + \|f\|_{C(R_+)}) d\tau \right). \end{aligned}$$

Therefore,

$$|Zw|_T \leq \psi_0 (\chi_0 \|w\|_T + \|f\|_{C(R_+)}). \quad (2.8)$$

From (2.6) and (2.8) it follows

$$\begin{aligned} \|w\|_T &\leq \|V_M\|_{C(0, T)} \|Zw + f\|_{C(0, T)} \leq \|V_M\|_{C(0, T)} (\|Zw\|_T + \|f\|_{C(R_+)}) \\ &\leq \|V_M\|_{C(0, T)} (\psi_0 \chi_0 \|w\|_T + (\psi_0 + 1) \|f\|_{C(R_+)}). \end{aligned}$$

If (1.5) holds, then

$$\|w\|_T \leq (1 - \chi_0 \psi_0 \|V_M\|_{C(R_+)})^{-1} (\psi_0 + 1) \|V_M\|_{C(R_+)} \|f\|_{C(R_+)}.$$

Hence, letting $T \rightarrow \infty$, we get

$$\|w\|_{C(R_+)} \leq (1 - \chi_0 \psi_0 \|V_M\|_{C(R_+)})^{-1} (\psi_0 + 1) \|V_M\|_{C(R_+)} \|f\|_{C(R_+)}. \quad (2.9)$$

We thus have proved the following result.

Lemma 2.1. *Under the hypothesis of Theorem 1.1, for any $f \in C(R_+)$ problem (2.1), (2.2) has a unique solution w , which satisfies inequality (2.9).*

Let $v(t)$ be a solution of problem (1.1), (1.2). Put

$$\hat{\phi}(t) = \begin{cases} \phi(0) & \text{if } t \geq 0, \\ \phi(t) & \text{if } -\eta \leq t < 0 \end{cases}$$

and $x_0(t) = v(t) - \hat{\phi}(t)$. Since $d\hat{\phi}(t)/dt = 0$ ($t \geq 0$), from (1.1) it follows

$$x_0'(t) = (Ex_0)(t) + f_0(t) \quad (t > 0),$$

where $f_0 = E\hat{\phi}$. Besides, the zero initial condition holds. Under conditions (1.3), (1.5) Lemma 2.1 implies

$$\|x_0\|_{C(R_+)} \leq (1 - \psi_0 \chi_0 \|V_M\|_{C(R_+)})^{-1} (1 + \psi_0) \|V_M\|_{C(R_+)} \|f_0\|_{C(R_+)}.$$

Note that $\|\hat{\phi}\|_{C(R_+)} \leq \|\phi\|_{C(-\eta, 0)}$ and due to (2.3)

$$\|f_0\|_{C(R_+)} \leq \chi_0 \|\hat{\phi}\|_{C(R_+)} \leq \chi_0 \|\phi\|_{C(-\eta, 0)}.$$

Since,

$$\|v\|_{C(R_+)} \leq \|x_0\|_{C(R_+)} + \|\hat{\phi}\|_{C(R_+)},$$

we arrive at the following result.

Lemma 2.2. *Under the hypothesis of Theorem 1.1 any solution v of problem (1.1), (1.2) satisfies the inequality*

$$\|v\|_{C(R_+)} \leq \|\phi\|_{C(-\eta, 0)} [(1 - \psi_0 \chi_0 \|V_M\|_{C(R_+)})^{-1} (1 + \psi_0) \|V_M\|_{C(R_+)} \chi_0 + 1].$$

For an $\epsilon > 0$ put

$$M_\epsilon(t) := \sum_{k=0}^m B_k(t) e^{\epsilon h_k(t)} + \int_0^1 B(t, \tau) e^{\epsilon h_0(\tau)} d\tau + \epsilon I$$

and

$$(V_{M, \epsilon} f)(t) = \int_0^t U_\epsilon(t, s) f(s) ds \quad (f \in C(R_+)),$$

where $U_\epsilon(t, s)$ is the evolution operator of the differential equation $z'(t) = M_\epsilon(t)z(t)$.

Lemma 2.3. *If V_M is bounded in $C(R_+)$, then for all sufficiently small $\epsilon > 0$ $V_{M, \epsilon}$ is also bounded in $C(R_+)$ and $\|V_{M, \epsilon}\|_{C(R_+)} \rightarrow \|V\|_{C(R_+)}$ as $\epsilon \downarrow 0$.*

Proof. Define the operators

$$(Rz)(t) = z'(t) - M(t)z(t), \quad (R_\epsilon z)(t) = z'(t) - M_\epsilon(t)z(t)$$

with the joint domain $D_0 = \{z \in C(R_+) : z' \in C(R_+), z(0) = 0\}$. By the Variation of Constants formula we have $R^{-1} = V_M$ and $R_\epsilon^{-1} = V_{M, \epsilon}$. Clearly, $\|M_\epsilon(\cdot) - M(\cdot)\|_{C(R_+)} \rightarrow 0$ ($\epsilon \rightarrow 0$). Hence, $\|R_\epsilon - R\|_{C(R_+)} = \|M_\epsilon(\cdot) - M(\cdot)\|_{C(R_+)} \rightarrow 0$. Obviously

$$R_\epsilon = R + M_\epsilon(\cdot) - M(\cdot) = R(I + R^{-1}(M_\epsilon(\cdot) - M(\cdot))).$$

R is boundedly invertible, since $R^{-1} = V_M$ is bounded. So if $\|R^{-1}(M_\epsilon(\cdot) - M(\cdot))\|_{C(R_+)} < 1$, then R_ϵ is boundedly invertible. Since $R_\epsilon^{-1} = V_{M, \epsilon}$, this proves the lemma. $\square$

Proof of Theorem 1.1: Substituting

$$v(t) = y_\epsilon(t) e^{-\epsilon t} \quad (\epsilon > 0) \tag{2.10}$$

into (1.1), we obtain the equation

$$y'_\epsilon(t) = (E_\epsilon y_\epsilon)(t), \tag{2.11}$$

where

$$(E_\epsilon u)(t) = \sum_{k=1}^m B_k(t) e^{\epsilon h_k(t)} u(t - h_k(t)) + \int_0^1 B(t, \tau) e^{\epsilon h_0(\tau)} u(t - h_0(\tau)) d\tau + \epsilon I$$

$$(u \in C(R_\eta); t \geq 0).$$

We have

$$\sup_{t \geq 0} \left(\sum_{k=1}^m e^{\epsilon h_k(t)} \|B_k(t)\| + \int_0^1 e^{\epsilon h_0(\tau)} \|B(t, \tau)\| d\tau + \epsilon I \right) \rightarrow \chi_0$$

as $\epsilon \rightarrow 0$. So according to Lemma 2.3 conditions (1.3) and (1.5) hold for all sufficiently small ϵ with $V_{M,\epsilon}$ instead of V_M and E_ϵ instead of E . Besides, due to Lemmas 2.2 solutions of (2.11) are bounded. Now (2.10) yields the exponential stability. As claimed. $\square$

3 Equations with dissipative operators

We will say that $A \in \mathcal{B}(\mathcal{X})$ is *dissipative* if $\|I + A\delta\| < 1$ for all sufficiently small $\delta > 0$. In this section we consider (1.1), assuming that $M(t)$ is dissipative for sufficiently large t . To this end introduce the operator products

$$\prod_{1 \leq k \leq m}^{\leftarrow} (I + M(t_k^{(m)})\delta_k) := (I + M(t_m^{(m)})\delta_m)(I + M(t_{m-1}^{(m)})\delta_{m-1}) \cdots (I + M(t_1^{(m)})\delta_1),$$

where

$$s = t_1^{(m)} < t_2^{(m)} < \dots < t_m^{(m)} = t \text{ and } \delta_k = t_k^{(m)} - t_{k-1}^{(m)} \quad (k = 1, \dots, m).$$

I.e., the arrow over the symbol of the product means that the indexes of the co-factors increase from right to left. The strong limit of these products as $\max_k \delta_k \rightarrow 0$ (if it exists) is called *the left multiplicative integral* and is denoted by $\int_{[s,t]}^{\leftarrow} (I + M(s_1)ds_1)$. As it is well-known,

$$U(t, s) = \int_{[s,t]}^{\leftarrow} (I + M(s_1)ds_1). \quad (3.1)$$

This equality is proved in [9, Chapter 1].

Lemma 3.1. *Let there be a real Riemann-integrable function $v(t)$, such that*

$$\|I + M(t)\delta\| \leq 1 + v(t)\delta + o(\delta) \quad (t \geq 0) \quad (3.2)$$

for all sufficiently small $\delta > 0$. Then

$$\|U(t, s)\| \leq \exp\left[\int_s^t v(s_1)ds_1\right] \quad (t \geq s \geq 0).$$

Proof. We have

$$\left\| \prod_{1 \leq k \leq m}^{\leftarrow} e^{M(t_k^{(m)})\delta_k} \right\| \leq \prod_{k=1}^m e^{v(t_k^{(m)})\delta_k} = \exp\left[\sum_{k=1}^m v(t_k^{(m)})\delta_k\right].$$

The passage to the limit as $m \rightarrow \infty$ and representation (3.1) give the required estimate. $\square$

Assume that

$$\theta_M := \sup_{t \geq 0} \int_0^t \exp\left[\int_s^t v(s_1)ds_1\right] ds < \infty. \quad (3.3)$$

Then due to Lemma 3.1 $\|V_M\|_{C(R_+)} = \|V_M\|_{C(R_+, \mathcal{X})} \leq \theta_M$, provided (3.2) holds.

Now Theorem 1.1 yields.

Corollary 3.2. *Let the conditions (1.3), (3.2), (3.3) and $\psi_0 \chi_0 \theta_M < 1$ hold. Then (1.1) is exponentially stable.*

Let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and $\Lambda(M_R(t)) = \sup \sigma(M_R(t))$, where $M_R(t) = \frac{1}{2}(M(t) + M^*(t))$ and the asterisk means the adjointness. Since

$$\begin{aligned} \|(I + M(t)\delta)h\|^2 &= \|(I + M(t)\delta)h\|^2 = \|(I + 2M_R(t)\delta + M^*(t)M(t)\delta^2)h\| \\ &\leq 1 + 2\Lambda(M_R(t))\delta + o(\delta) \quad (h \in \mathcal{H}, \|h\| = 1), \end{aligned}$$

we can take $v(t) = \Lambda(M_R(t))$. Hence we arrive at the Wintner inequality

$$\|U(t, s)\| \leq \exp\left[\int_s^t \Lambda(M_R(s_1))ds_1\right] \quad (t \geq s \geq 0), \quad (3.4)$$

cf [6, Theorem III.4.7]. So in the case $\mathcal{X} = \mathcal{H}$ Corollary 3.2 can be applied with

$$\theta_M = \sup_{t \geq 0} \int_0^t \exp\left[\int_s^t \Lambda(M_R(s_1))ds_1\right]ds, \quad (3.5)$$

provided $\theta_M < \infty$.

4 Equations in a Banach space with the generalized Lipschitz property

In this section we consider equation (1.1), assuming that $M(t)$ satisfies the generalized Lipschitz condition

$$\|M(t) - M(\tau)\| \leq a(|t - \tau|) \quad (t, \tau \geq 0), \quad (4.1)$$

where $a(t)$ is a non-negative piece-wise continuous function defined on $[0, \infty)$ and $a(0) = 0$. A particular case of (4.1) is the traditional Lipschitz condition

$$\|M(t) - M(\tau)\| \leq q_0|t - \tau| \quad (q_0 = \text{const} > 0; \quad t, \tau \geq 0).$$

In addition to (4.1) suppose that there is a positive integrable on $[0, \infty)$ function $p(t)$ independent of s , such that

$$\|\exp[M(s)t]\| \leq p(t) \quad (t, s \geq 0) \text{ and } J_0 := \int_0^\infty p(t)dt < \infty. \quad (4.2)$$

Lemma 4.1. *Let the conditions (4.1), (4.2) and*

$$\zeta_0 := \int_0^\infty a(s)p(s)ds < 1 \quad (4.3)$$

hold. Then a solution $u(t)$ to the non-homogeneous problem

$$\frac{du(t)}{dt} = M(t)u(t) + f(t), \quad u(0) = 0 \quad (f \in C(R_+, \mathcal{X})) \quad (4.4)$$

satisfies the inequality

$$\|u\|_{C(R_+, \mathcal{X})} \leq \frac{J_0 \|f\|_{C(R_+, \mathcal{X})}}{1 - \zeta_0}.$$

Proof. Rewrite (4.4) as

$$\frac{du(t)}{dt} = M(\tau)u(t) + [M(t) - M(\tau)]u(t) + f(t)$$

with an arbitrary fixed $\tau \geq 0$. So problem (4.4) is equivalent to the equation

$$u(t) = \int_0^t \exp[M(\tau)(t-s)]([M(s) - M(\tau)]u(s) + f(s))ds.$$

Hence,

$$\|u(t)\| \leq \int_0^t \exp[M(\tau)(t-s)](\|M(s) - M(\tau)\|\|u(s)\| + \|f(s)\|)ds.$$

According to (4.1) and (4.2),

$$\|u(t)\| \leq \int_0^t p(t-s)(a(|s-\tau|)\|u(s)\| + \|f(s)\|)ds.$$

Taking $\tau = t$, we obtain

$$\|u(t)\| \leq \int_0^t p(t-s)(a(t-s)\|u(s)\| + \|f(s)\|)ds.$$

Hence for any positive finite T , with $\sup_{t \leq T} \|u(t)\| = |u|_T$ we have

$$|u|_T \leq |u|_T \int_0^T p(t_1)a(t_1)dt_1 + \int_0^T p(t_1)dt_1 \|f\|_{C(R_+)}$$

Consequently, $|u|_T \leq |u|_T \zeta_0 + J_0 \|f\|_{C(R_+)}$. According to (4.3) we get

$$|u|_T \leq (1 - \zeta_0)^{-1} J_0 \|f\|_{C(R_+)}.$$

Extending this result to all $T \geq 0$, we prove the lemma. $\square$

From this lemma it follows.

Corollary 4.2. *Let conditions (4.1) and (4.2) hold. Then*

$$\|V_M\|_{C(R_+)} \leq \frac{J_0}{1 - \zeta_0}.$$

Due to this corollary condition (1.5) under the hypothesis of the previous lemma is provided by the inequality $\chi_0 \psi_0 \frac{J_0}{(1-\zeta_0)} < 1$. Or

$$\chi_0 \psi_0 J_0 + \zeta_0 < 1. \quad (4.5)$$

Now Theorem 1.1 yields.

Corollary 4.3. *Let the conditions (1.3), (4.1), (4.2) and (4.5) hold. Then equation (1.1) is exponentially stable.*

5 Equations with differentiable coefficients in a Hilbert space

5.1 Statement of the result

In this section $\mathcal{X} = \mathcal{H}$ is a separable Hilbert space with a scalar product $(\cdot, \cdot)$. By $\mathcal{B}(\mathcal{H})$ the set of all linear bounded operators acting in $\mathcal{H}$ is denoted.

Consider equation (1.1) in $\mathcal{H}$, assuming that operator $M(t)$ defined in Section 1 is uniformly bounded on R_+ , has a measurable strong derivative uniformly bounded on R_+ and

$$\alpha_M := \sup_{t \geq 0} \alpha(M(t)) = \sup_{t \geq 0} \sup \operatorname{Re} \sigma(M(t)) < 0. \quad (5.1)$$

Introduce quantity

$$q(t) := 2 \int_0^\infty \|e^{M(t)s}\|^2 ds.$$

By the classical Dunford-Cauchy formula for operator functions (see for instance [6, Ch. 1]),

$$e^{M(t)t} = \frac{1}{2\pi i} \int_C e^{tz} (Iz - M(t))^{-1} dz,$$

where C is a Jordan contour surrounding $\sigma(M(t))$. Hence, for each $t \geq 0$ we have

$$\|e^{M(t)s}\| \leq \operatorname{const} e^{s(\alpha_M + \epsilon)} \quad (\epsilon > 0, s \geq 0)$$

and therefore $q(t) < \infty$. It is assumed that

$$\sup_{t \geq 0} q^2(t) \|M'(t)\| < 2 \quad (5.2)$$

and

$$\gamma_M := \sup_{t \geq 0} \sqrt{|\lambda_{\inf}(M_R(t))|} \int_0^t \sqrt{q(s)} \exp \left[- \int_s^t \frac{(1 - \frac{1}{2} q^2(t_1) \|M'(t_1)\|)}{q(t_1)} dt_1 \right] ds < \infty,$$

where $\lambda_{\inf}(M_R(t)) := \inf \sigma(M_R(t))$. Recall that $M_R(t) = \frac{1}{2}(M(t) + M^*(t))$.

Theorem 5.1. *Let the conditions (1.3), (5.1), (5.2) and*

$$\chi_0 \gamma_M \psi_0 < 1 \quad (5.3)$$

hold. Then (1.1) is exponentially stable.

The proof of this theorem is presented in the next subsection.

5.2 Proof of Theorem 5.1

An operator $A_0 \in \mathcal{B}(\mathcal{H})$ is said to be stable if $\alpha(A_0) = \sup \operatorname{Re} \sigma(A_0) < 0$. As is well known the Lyapunov equation

$$A_0^* Y + Y A_0 = -2I$$

with a stable operator A_0 has a solution Y which is represented as

$$Y = 2 \int_0^\infty e^{A_0^* s} e^{A_0 s} ds, \quad (5.4)$$

cf. [6, Section I.4.4]. Put $y(t) = e^{A_0 t} z$ ($z \in \mathcal{H}$). Then $y'(t) = A_0 y(t)$, and

$$\frac{d(y(t), y(t))}{dt} = ((A_0 + A_0^*)y(t), y(t)).$$

With $\operatorname{Re} A_0 = (A_0 + A_0^*)/2$ and $\lambda_{\inf}(\operatorname{Re} A_0) = \inf \sigma(\operatorname{Re} A_0)$, we conclude that

$$\frac{d(y(t), y(t))}{dt} \geq 2\lambda_{\inf}(\operatorname{Re} A_0)(y(t), y(t)) \text{ and therefore } \|e^{A_0 t} z\| \geq e^{t\lambda_{\inf}(\operatorname{Re} A_0)} \|z\| \quad (t \geq 0).$$

Since A_0 is stable, $\|e^{A_0 t} z\| \rightarrow 0$ and therefore $e^{t\lambda_{\inf}(\operatorname{Re} A_0)} \rightarrow 0$ ($t \rightarrow \infty$). Hence it follows that $\lambda_{\inf}(\operatorname{Re} A_0) < 0$. We thus get

$$(Yz, z) = 2 \int_0^\infty (e^{A_0^* s} e^{A_0 s} z, z) ds \geq 2 \int_0^\infty e^{2\lambda_{\inf}(\operatorname{Re} A_0)s} ds \|z\|^2 = \frac{\|z\|^2}{|\lambda_{\inf}(\operatorname{Re} A_0)|} \quad (z \in \mathcal{H}). \quad (5.5)$$

Furthermore, according to (5.4) the operator

$$Q(t) := 2 \int_0^\infty e^{M^*(t)s} e^{M(t)s} ds$$

is a unique solution of the equation

$$M^*(t)Q(t) + Q(t)M(t) = -2I \quad (t \geq 0). \quad (5.6)$$

Clearly, $\|Q(t)\| \leq q(t)$. Due to (5.5)

$$(Q(t)z, z) \geq \frac{\|z\|^2}{|\lambda_{\inf}(M_R(t))|} \quad (z \in \mathcal{H}, t \geq 0). \quad (5.7)$$

Lemma 5.2. *Let condition (1.3) hold and $M(t)$ be strongly differentiable. Then $Q(t)$ is strongly differentiable and $\|Q'(t)\| \leq q^2(t)\|M'(t)\|$.*

For the proof see Lemma 2.1 from [13].

Lemma 5.3. *Let $M(t)$ be strongly differentiable. Then*

$$\frac{d}{dt}(Q(t)u(t), u(t)) \leq (-2 + \|Q'(t)\|)(u(t), u(t))$$

for any solution to (1.4).

Proof. Multiplying equation (1.4) by $Q(t)$ and doing the scalar product, we can write

$$(Q(t)u'(t), u(t)) = (Q(t)M(t)u(t), u(t)).$$

Since

$$\frac{d}{dt}(Q(t)u(t), u(t)) = (Q(t)u'(t), u(t)) + (u(t), Q(t)u'(t)) + (Q'(t)u(t), u(t)),$$

in view of (5.6) we get

$$\begin{aligned} \frac{d}{dt}(Q(t)u(t), u(t)) &= (Q(t)M(t)u(t), u(t)) + (u(t), Q(t)M(t)u(t)) + (Q'(t)u(t), u(t)) \\ &= ((Q(t)M(t) + M^*(t)Q(t))u(t), u(t)) + (Q'(t)u(t), u(t)) \\ &= -2(u(t), u(t)) + (Q'(t)u(t), u(t)). \end{aligned}$$

Hence, we get the required result. $\square$

Let

$$\sup_{t \geq 0} \|Q'(t)\| < 2. \quad (5.8)$$

Then according to the previous lemma

$$\frac{d}{dt}(Q(t)u(t), u(t)) = ((-2I + Q'(t))u(t), u(t)) \leq -\frac{(2 - \|Q'(t)\|)}{\|Q(t)\|}(Q(t)u(t), u(t)).$$

Solving this inequality and taking into account that $\|Q(t)\| \leq q(t)$, we obtain

$$(Q(t)u(t), u(t)) \leq (Q(s)u(s), u(s)) \exp \left[-\int_s^t \frac{(2 - \|Q'(t_1)\|)}{q(t_1)} dt_1 \right].$$

Now (5.7) implies

$$\|u(t)\|^2 \leq |\lambda_{\inf}(M_R(t))|q(s)\|u(s)\|^2 \exp \left[- \int_s^t \frac{(2 - \|Q'(t_1)\|)}{q(t_1)} dt_1 \right].$$

Making use of Lemma 5.2, we arrive at

Lemma 5.4. *Let conditions (5.1) and (5.2) hold. Then the evolution operator of (1.4) satisfies the inequality*

$$\|U(t, s)\|^2 \leq |\lambda_{\inf}(M_R(t))|q(s) \exp \left[- \int_s^t \frac{(2 - q^2(t_1)\|M'(t_1)\|)}{q(t_1)} dt_1 \right] \quad (t \geq s \geq 0).$$

Proof of Theorem 5.1: Obviously,

$$\|V\|_{C(R_+)} \leq \sup_t \int_0^t \|U(t, s)\| ds.$$

Hence, due to Lemma 5.4 $\|V\|_{C(R_+)} \leq \gamma_M$. Now Theorem 1.1 yields the required result.

6 Integro-differential equations with delays

Our main object in this section is the equation

$$\frac{\partial u(t, x)}{\partial t} = c(t, x)u(t - h_1(t), x) + \int_0^x k(t, x, x_1)u(t - h_2(t), x_1)dx_1 \quad (t > 0; 0 \leq x \leq 1), \quad (6.1)$$

where $c(., .) : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ is continuous in t and piece-wise continuous in x ; $k(., ., .) : [0, \infty) \times [0, 1]^2 \rightarrow \mathbb{R}$ has the property: the integral $\int_0^1 |k(t, x, x_1)|dx_1$ is continuous in $t \geq 0$.

We consider equation (6.1) in the space $C(0, 1)$ of scalar continuous functions defined on $[0, 1]$ with the sup-norm. Equation (6.1) has the form (1.1) with $m = 2$; $h_1(t), h_2(t)$ are the same as above; $B(t, \tau) = 0$,

$$(B_1(t)z)(x) = c(t, x)z(x), (B_2(t)z)(x) = \int_0^1 k(t, x, x_1)z(x_1)dx_1 \quad (z \in C(0, 1)),$$

and $M(t) = B_1(t) + B_2(t)$. Condition (1.3) takes the form

$$\chi_1 = \sup_{t \geq 0} (\sup_x |c(t, x)| + \|B_2(t)\|) < \infty. \quad (6.2)$$

So in the considered case $\chi_0 = \chi_1$ and $\psi_0 = \psi_1$, where

$$\psi_1 = \sup_{t \geq 0} (h_1(t) \sup_x |c(t, x)| + h_2(t) \|B_2(t)\|).$$

Condition (3.2) takes the form

$$\|I + M(t)\delta\| = \max_x |z(x) + c(t, x)z(x)\delta + (B_2(t)z)(x)\delta| \leq 1 + v(t)\delta + o(\delta) \quad (\|z\| = 1). \quad (6.3)$$

For example, if

$$c(t, x) + \|B_2(t)\| \leq -v_1(t) \quad (0 \leq x \leq 1, t \geq 0),$$

where $v_1(t)$ is a positive continuous function, then (6.3) holds with $v(t) = -v_1(t)$. If

$$\Lambda_1 := \sup_{t \geq 0} v(t) < 0, \quad (6.4)$$

then according to (3.3)

$$\theta_M = \sup_{t \geq 0} \int_0^t \exp\left[\int_s^t v(s_1) ds_1\right] ds \leq \sup_{t \geq 0} \int_0^t \exp[(t-s)\Lambda_1] ds = \frac{1}{|\Lambda_1|}.$$

Now Corollary 3.2 implies.

Corollary 6.1. *Let the conditions (6.2)-(6.4) and $\psi_1 \chi_1 < |\Lambda_1|$ hold. Then (6.1) is exponentially stable.*

About other approaches to integro-differential equations see [1–4, 17, 19].

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