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Existence of renormalized solutions for some quasilinear elliptic Neumann problems

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Abstract: This paper is devoted to study some nonlinear elliptic Neumann equations of the type

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{q(\cdot)-2}u = f(x, u, \nabla u) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, u, \nabla u) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases}$$

in the anisotropic variable exponent Sobolev spaces, where A is a Leray-Lions operator and $g(x, u, \nabla u)$, $f(x, u, \nabla u)$ are two Carathéodory functions that verify some growth conditions. We prove the existence of renormalized solutions for our strongly nonlinear elliptic Neumann problem.

Keywords: Renormalized solution, strongly nonlinear elliptic equations, anisotropic variable exponent Sobolev spaces, Neumann problem

MSC: 35J60, 35D05

1 Introduction

Let Ω be a bounded open domain of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$. The study of various mathematical problems with isotropic variable exponent has received considerable attention in recent years. These problems are interesting in applications and raise many difficult and interesting mathematical problems. For example, in [20] the authors established the following stationary $p(x)$ -curl systems arising in electromagnetism:

$$\begin{cases} \nabla \times (|\nabla \times u|^{p(x)-2} \nabla \times u) + a(x)|u|^{p(x)-2}u = f(x, u), & \nabla \cdot u = 0 & \text{in } \Omega, \\ |\nabla \times u|^{p(x)-2} \times u \times n = 0, & u \cdot n = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

In addition in [8], Boccardo and Gallouët have considered the elliptic problem

$$\begin{cases} -\operatorname{div} a(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a(x, u, \nabla u)$ is a Carathéodory function, and the right-hand side f is a bounded Radon measure. They have proved the existence and some regularity results. For more results, we refer the reader to [7, 13, 14]).

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In [5], Ben Cheikh Ali and Guibé have studied some quasilinear elliptic equations of the type,

$$\begin{cases} \lambda(x, u) - \operatorname{div}(a(x, \nabla u) + \Phi(x, u)) = f & \text{in } \Omega, \\ (a(x, \nabla u) + \Phi(x, u)) \cdot n = 0 & \text{on } \Gamma_n, \\ u = 0 & \text{on } \Gamma_d, \end{cases} \quad (1.2)$$

where $Au = -\operatorname{div}(a(x, \nabla u))$ is a Leray-Lions type operator, and $\lambda(x, s)$, $\Phi(x, s)$ are Carathéodory functions. They have proved the existence and uniqueness of renormalized solutions for this problem under some growth conditions.

In the recent years, the interest of scientists has turned towards anisotropic elliptic and parabolic equations. This special interest mainly comes from their applications to the mathematical modeling for some physical processes in an anisotropic continuous medium (see. [2, 19]).

Di Nardo, et al. have considered in [11] the following nonlinear elliptic problems of the type

$$\begin{cases} -\sum_{i=1}^N \partial_{x_i}(a_i(x, u)|\partial_{x_i} u|^{p_i-2} \partial_{x_i} u) = f - \operatorname{div} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and $g \in \Pi_{i=1}^N L^{p'_i}(\Omega)$. They have proved the existence and uniqueness of renormalized solution in the anisotropic Sobolev spaces. In [4], the authors have studied the nonlinear elliptic Dirichlet problem

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{p_0(x)-2} u = f - \operatorname{div} \phi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $g(x, u, \nabla u)$ and $\phi(u)$ are Carathéodory functions, and the data f is assumed to be in $L^1(\Omega)$. They have proved the existence of an entropy solution in the anisotropic variable exponent Sobolev spaces.

In this paper, we will study the existence of renormalized solutions for the following nonlinear elliptic problem

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{q(\cdot)-2} u = f(x, u, \nabla u) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, u, \nabla u) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

with $Au = -\sum_{i=1}^N a_i(x, u, \nabla u)$ is a Leray-Lions operator that acts from $W^{1, \vec{p}(\cdot)}(\Omega)$ into its dual, the Carathéodory functions $g(x, u, \nabla u)$ and $f(x, s, \xi)$ verify only some growth conditions. We will prove the existence of renormalized solutions for our Neumann elliptic problem (1.3).

This paper is organized as follows. In the section 2 we recall some definitions and known results about the anisotropic variable exponent Sobolev spaces. We introduce in the section 3 some assumptions on the Carathéodory functions $a_i(x, s, \xi)$, $g(x, s, \xi)$ and $f(x, s, \xi)$ for which our problem has at least one solution. The section 4 is devoted to prove the existence of renormalized solutions for the considered strongly nonlinear elliptic problem. The proof of Lemma 4.1 is given in the Appendix.

2 Notation and preliminary results.

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot) : \Omega \rightarrow \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x) / x \in \Omega\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x) / x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the generalized Hölder type inequality:

$$\left| \int_{\Omega} uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \quad (2.1)$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [12].

Now, we introduce the anisotropic variable exponent Sobolev space, used in the study of our quasilinear anisotropic elliptic problem.

Let $p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot)$ be $N + 1$ variable exponents in $\mathbb{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot)) \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N,$$

and we define

$$p^+ = \max\{p_1^+, \dots, p_N^+\} \quad \text{and} \quad \underline{p} = \min\{p_1^-, \dots, p_N^-\} \quad \text{then} \quad p^+ \geq \underline{p} > 1. \quad (2.2)$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follows

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for } i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \|u\|_{\underline{p}} + \sum_{i=1}^N \|D^i u\|_{p_i(\cdot)}. \quad (2.3)$$

The space $(W^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)})$ is a separable and reflexive Banach space (cf. [18]).

Lemma 2.1. *We have the following continuous and compact embedding*

- if $\underline{p} < N$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [\underline{p}, \underline{p}^*]$, where $\underline{p}^* = \frac{N\underline{p}}{N-\underline{p}}$,
- if $\underline{p} = N$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [\underline{p}, +\infty[$,

– if $\underline{p} > N$ then $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Definition 2.1. Let $k > 0$, the truncation function $T_k(\cdot) : \mathbf{R} \mapsto \mathbf{R}$ is given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \mapsto \mathbf{R} \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.1. Let $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \mapsto \mathbf{R}$ such that

$$\forall k > 0 \quad D^i T_k(u) = v_i \cdot \chi_{\{|u| < k\}} \quad \text{a.e. } x \in \Omega,$$

where χ_A denotes the characteristic function of a measurable set A . The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u , that is, $v_i = D^i u$.

The proof of the Proposition 2.1 follows the usual techniques developed in [7] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [6] and [11].

3 Essential Assumptions

Let Ω be a bounded open set of $\mathbf{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$.

We consider a Leray-Lions operator A that acts from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$, defined by the formula

$$Au = - \sum_{i=1}^n D^i a_i(x, u, \nabla u) \quad (3.1)$$

where $a_i(x, s, \xi) : \Omega \times \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}^N$ are Carathéodory functions (i.e. measurable with respect to x in Ω for every (s, ξ) in $\mathbf{R} \times \mathbf{R}^N$ and continuous with respect to (s, ξ) in $\mathbf{R} \times \mathbf{R}^N$ for almost every x in Ω) and verifying the following conditions :

$$|a_i(x, s, \xi)| \leq \beta \left(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1} \right) \quad \text{for } i = 1, \dots, N, \quad (3.2)$$

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi'_i) > 0 \quad \text{for } \xi_i \neq \xi'_i, \quad (3.3)$$

$$a_i(x, s, \xi) \xi_i \geq \alpha |\xi_i|^{p_i(\cdot)} \quad \text{for } i = 1, \dots, N, \quad (3.4)$$

for almost every $x \in \Omega$ and all $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$, where $K_i(x) \in L^{p'_i(\cdot)}(\Omega)$ are some positive functions and $\alpha, \beta > 0$.

As a consequence of (3.4) and the continuity of the functions $a_i(x, s, \cdot)$ with respect to ξ , we have

$$a_i(x, s, 0) = 0 \quad \text{for } i = 1, \dots, N.$$

The lower order terms $g(x, s, \xi), f(x, s, \xi) : \Omega \times \mathbf{R} \times \mathbf{R}^N \mapsto \mathbf{R}$ are two Carathéodory functions which satisfy the following growth conditions

$$|g(x, s, \xi)| \leq g_0(x) + d(|s|) \sum_{i=1}^N |\xi_i|^{p_i(x)}, \quad (3.5)$$

and

$$|f(x, s, \xi)| \leq f_0(x) + |s|^{r_0(x)} + \sum_{i=1}^N |\xi_i|^{r_i(x)} \quad (3.6)$$

where $g_0(\cdot)$ and $f_0(\cdot)$ are assumed to be two positive measurable functions in $L^1(\Omega)$ and $d(\cdot) : \mathbf{R} \mapsto \mathbf{R}^+$ is a decreasing function that belongs to $L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ and we assume that $0 \leq r_0(x) < q(x) - 1$ and $0 \leq r_i(x) < \frac{p_i(x)(q(x)-1)}{q(x)+1}$ a. e. in Ω for any $i = 0, 1, \dots, N$.

We consider the strongly nonlinear elliptic Neumann problem

$$\begin{cases} Au + g(x, u, \nabla u) + |u|^{q(\cdot)-2}u = f(x, u, \nabla u) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, u, \nabla u) \cdot n_i = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.7)$$

Lemma 3.1. (see [3]) Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \rightarrow g(x)$ a.e. in Ω , then $g_n \rightarrow g$ in $L^{r(\cdot)}(\Omega)$.

Lemma 3.2. (see [4]) Assuming that (3.2) – (3.4) hold, and let $(u_n)_{n \in \mathbf{N}}$ be a sequence in $W^{1, \vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1, \vec{p}(\cdot)}(\Omega)$ and

$$\begin{aligned} & \int_{\Omega} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u) \, dx \\ & + \sum_{i=1}^N \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u) \, dx \rightarrow 0, \end{aligned} \quad (3.8)$$

then $u_n \rightarrow u$ in $W^{1, \vec{p}(\cdot)}(\Omega)$ for a subsequence.

4 Existence of renormalized Solutions

Let Ω be a bounded open subset of $\mathbf{R}^N$ ($N \geq 2$), with smooth boundary $\partial\Omega$.

Let $q(\cdot)$ and $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ for $i = 1, \dots, N$, we set

$$\underline{p} = \min\{p_1^-, p_2^-, \dots, p_N^-\} \quad \text{and} \quad p^+ = \max\{p_1^+, p_2^+, \dots, p_N^+\}. \quad (4.1)$$

Definition 4.1. A measurable function u is called renormalized solution of the strongly nonlinear elliptic problem (3.7) if $u \in \mathcal{T}^{1, \vec{p}(\cdot)}(\Omega)$, $g(x, u, \nabla u) \in L^1(\Omega)$, $f(x, u, \nabla u) \in L^1(\Omega)$ and

$$\lim_{h \rightarrow \infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u| \leq h\}} a_i(x, u, \nabla u) D^i u \, dx = 0, \quad (4.2)$$

such that u verifying the following equality

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) (S'(u) \varphi D^i u + S(u) D^i \varphi) \, dx + \int_{\Omega} g(x, u, \nabla u) S(u) \varphi \, dx \\ & + \int_{\Omega} |u|^{q(x)-2} u S(u) \varphi \, dx = \int_{\Omega} f(x, u, \nabla u) S(u) \varphi \, dx, \end{aligned} \quad (4.3)$$

for every $\varphi \in W^{1, \vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for any smooth function $S(\cdot) \in W^{1, \infty}(\mathbf{R})$ with a compact support.

Theorem 4.1. *Assuming that (3.2) – (3.6) hold true, then there exists at least one renormalized solution u for the strongly nonlinear anisotropic elliptic Neumann problem (3.7).*

Proof of the Theorem 4.1

Step 1 : Approximate problems.

Let $g_n(x, s, \xi) = T_n(g(x, s, \xi))$ and $f_n(x, s, \xi) = T_n(f(x, s, \xi))$, we consider the approximate problem :

$$\begin{cases} -\sum_{i=1}^N D^i a_i(x, T_n(u_n), \nabla u_n) + g_n(x, u_n, \nabla u_n) + |T_n(u_n)|^{q(x)-2} T_n(u_n) \\ \quad + \frac{1}{n} |u_n|^{p-2} u_n = f_n(x, T_n(u_n), \nabla u_n) & \text{in } \Omega, \\ \sum_{i=1}^N a_i(x, T_n(u_n), \nabla u_n) \cdot n_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

We define the operator G_n from $W^{1, \vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1, \vec{p}(\cdot)}(\Omega))'$ by :

$$\begin{aligned} \langle G_n u, v \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i v \, dx + \int_{\Omega} g_n(x, u, \nabla u) v \, dx + \int_{\Omega} |T_n(u)|^{q(x)-2} T_n(u) v \, dx \\ &\quad + \frac{1}{n} \int_{\Omega} |u|^{p-2} u v \, dx - \int_{\partial\Omega} f_n(x, T_n(u), \nabla u) v \, dx \quad \text{for any } u, v \in W^{1, \vec{p}(\cdot)}(\Omega), \end{aligned}$$

Lemma 4.1. *The operator G_n that acts from $W^{1, \vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1, \vec{p}(\cdot)}(\Omega))'$ is bounded and pseudo-monotone. Moreover, G_n is coercive in the following sense:*

$$\frac{\langle G_n v, v \rangle}{\|v\|_{1, \vec{p}(\cdot)}} \rightarrow \infty \quad \text{as } \|v\|_{1, \vec{p}(\cdot)} \rightarrow \infty \quad \text{for } v \in W^{1, \vec{p}(\cdot)}(\Omega).$$

For the proof of Lemma 4.1, see Appendix.

In view of Lemma 4.1 (we refer the reader to [16] and [17]), there exists at least one weak solution $u_n \in W^{1, \vec{p}(\cdot)}(\Omega)$ for the quasilinear elliptic Neumann problem (4.4), i.e. we have:

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i v \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) v \, dx + \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_n(u_n) v \, dx \\ + \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n v \, dx = \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) v \, dx, \quad \text{for any } v \in W^{1, \vec{p}(\cdot)}(\Omega). \end{aligned} \quad (4.5)$$

Step 2: Weak convergence of truncations.

Let $n \in \mathbf{N}^*$ be large enough, and let $1 < \theta < \left(\frac{p_i^-}{r_i^+} - 1\right)(q(x) - 1)$. We define

$$B(s) = \frac{2}{\alpha} \int_0^s d(|\tau|) \, d\tau \quad \text{and} \quad \varphi(s) = \left(2 - \frac{1}{(1 + |s|)^{\theta-1}}\right).$$

Note that, since the function $d(\cdot)$ is integrable on $\mathbf{R}$, we conclude that $0 \leq B(\infty) := \frac{2}{\alpha} \int_0^{+\infty} d(|t|) \, dt$ is a finite real number.

By taking $v = T_k(u_n)\varphi(u_n)e^{B(|u_n|)} \in W^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in the approximate problem (4.5), we have

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i u_n d(|u_n|) |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^{\theta}} |T_k(u_n)| e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} g_n(x, u_n, \nabla u_n) T_k(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_n(u_n) T_k(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n T_k(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & = \sum_{i=1}^N \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) T_k(u_n) \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.6}$$

Having in mind (3.4), (3.5) and (3.6), we conclude that

$$\begin{aligned}
 & \alpha \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} \varphi(u_n) e^{B(|u_n|)} dx + 2 \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{p_i(x)} d(|u_n|) |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \alpha(\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |T_k(u_n)| e^{B(|u_n|)} dx + \sum_{i=1}^N \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \frac{1}{n} \int_{\Omega} |u_n|^{p-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \int_{\Omega} (f_0(x) + |T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)}); dx \\
 & + \int_{\Omega} (g_0(x) + d(|u_n|) \sum_{i=1}^N |D^i u_n|^{p_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq 2ke^{B(\infty)} (\|f_0\|_{L^1(\Omega)} + \|g_0\|_{L^1(\Omega)}) + \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx
 \end{aligned} \tag{4.7}$$

it follows that

$$\begin{aligned}
 & \alpha \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} \varphi(u_n) e^{B(|u_n|)} dx + \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \alpha(\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |T_k(u_n)| e^{B(|u_n|)} dx + \sum_{i=1}^N \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \frac{1}{n} \int_{\Omega} |u_n|^{p-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq 2ke^{B(\infty)} (\|f_0\|_{L^1(\Omega)} + \|g_0\|_{L^1(\Omega)}) + \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.8}$$

For the second and third terms on the right-hand side of (4.8), using Young's inequality we have

$$\begin{aligned}
 \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx & \leq \frac{1}{2} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + C_0 \int_{\Omega} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.9}$$

We have also

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & = \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{r_i(x)}}{(1 + |u_n|)^{\frac{\theta r_i(x)}{p_i(x)}}} (1 + |u_n|)^{\frac{\theta r_i(x)}{p_i(x)}} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{\alpha(\theta - 1)}{2} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + C_1 \sum_{i=1}^N \int_{\Omega} (1 + |u_n|)^{\frac{\theta r_i(x)}{p_i(x)-r_i(x)}} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{(\theta - 1)}{2} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \frac{1}{4} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + C_2 \int_{\Omega} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.10}$$

By combining (4.8) and (4.9) – (4.10), we conclude that

$$\begin{aligned}
 & \alpha \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx + \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{p_i(x)} d(|u_n|) |T_k(u_n)| dx \\
 & + \frac{\alpha(\theta - 1)}{2} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |T_k(u_n)| dx + \frac{1}{4} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_k(u_n)| dx \\
 & + \frac{1}{n} \int_{\Omega} |u_n|^{p-1} |T_k(u_n)| dx \\
 & \leq (C_0 + C_2) \int_{\Omega} |T_k(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq C_3 k.
 \end{aligned} \tag{4.11}$$

It follows that

$$\sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i} dx + N|\Omega| \leq C_4 k. \quad (4.12)$$

Moreover, thanks to (4.11) we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\{|u_n|>k\}} |D^i u_n|^{p_i(x)} d(|u_n|) dx + \sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} dx \\ & \quad + \int_{\{|u_n|>k\}} |T_n(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_{\{|u_n|>k\}} |u_n|^{p-1} dx \\ & \leq C_5. \end{aligned} \quad (4.13)$$

On the one hand, thanks to Young's inequality we have

$$\begin{aligned} \|T_k(u_n)\|_{1,\vec{p}(\cdot)} &= \|T_k(u_n)\|_{\underline{p}} + \sum_{i=1}^N \|D^i T_k(u_n)\|_{p_i} \\ &\leq \int_{\Omega} |T_k(u_n)|^{\underline{p}} dx + \sum_{i=1}^N \left(\int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx \right)^{\frac{1}{p_i}} + N \\ &\leq k^{\underline{p}} \cdot \text{meas}(\Omega) + \sum_{i=1}^N \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx + 2N \\ &\leq C_6 k^{\underline{p}}, \end{aligned} \quad (4.14)$$

where C_6 is a positive constant that does not depend on k and n . Thus $(T_k(u_n))_n$ is bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$ uniformly in n , and there exists a subsequence still denoted $(T_k(u_n))_n$ and $v_k \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup v_k & \text{weakly in } W^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \rightarrow v_k & \text{strongly in } L^1(\Omega) \text{ and a.e in } \Omega. \end{cases} \quad (4.15)$$

Moreover, thanks to (4.13) it follows that

$$\begin{aligned} k^{q^- - 1} \cdot \text{meas}(\{k < |u_n|\}) &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{q^- - 1} dx \leq \int_{\{|u_n|>k\}} |T_n(u_n)|^{q(x)-1} dx \\ &\leq C_5 \quad \text{for any } k \geq 1, \end{aligned}$$

and since $q^- > 1$, it follows necessary that

$$\text{meas}(\{k < |u_n|\}) \leq \frac{C_5}{k^{q^- - 1}} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (4.16)$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure.

For all $\lambda > 0$, we have

$$\text{meas}\{|u_n - u_m| > \lambda\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}.$$

Let $\varepsilon > 0$, using (4.16) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \quad (4.17)$$

On the other hand, thanks to (4.15) we have $T_k(u_n) \rightarrow v_k$ in $L^1(\Omega)$ and a.e. in Ω . Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure, and for all $k > 0$ and $\varepsilon, \lambda > 0$, there exists $n_0 = n_0(k, \varepsilon, \lambda)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\} \leq \frac{\varepsilon}{3} \quad \text{for all } m, n \geq n_0(k, \varepsilon, \lambda). \quad (4.18)$$

By combining (4.17) – (4.18), we conclude that

$$\forall \varepsilon, \lambda > 0 \text{ there exists } n_0 = n_0(\varepsilon, \lambda) \text{ such that } \text{meas}\{|u_n - u_m| > \lambda\} \leq \varepsilon,$$

for any $n, m \geq n_0(\varepsilon, \lambda)$. It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u . We conclude that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W^{1, \vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \rightarrow T_k(u) & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{cases} \quad (4.19)$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \rightarrow T_k(u) \quad \text{in } L^{p_i(\cdot)}(\Omega) \quad \text{and a.e. in } \Omega \quad \text{for } i = 1, \dots, N. \quad (4.20)$$

Moreover, thanks to (4.16) we conclude that $\frac{T_k(u_n)}{k} \rightarrow 0$ strongly in $L^1(\Omega)$ and weak* in $L^\infty(\Omega)$.

Step 3 : Some a priori estimates.

Let $h \geq 1$, in this section we will prove that :

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^N \frac{1}{h} \int_{\{|u_n| \leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0.$$

By taking $v = \frac{T_h(u_n)}{h} \varphi(u_n) e^{B(|u_n|)}$ as a test function in the approximate problem (4.5), we have:

$$\begin{aligned} & \frac{1}{h} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i T_h(u_n) \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \frac{2}{\alpha h} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i u_n d(|u_n|) |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \frac{\theta - 1}{h} \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta} |T_h(u_n)| e^{B(|u_n|)} \, dx \\ & + \frac{1}{h} \int_{\Omega} g_n(x, u_n, \nabla u_n) T_h(u_n) \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_n(u_n) T_h(u_n) \varphi(u_n) e^{B(|u_n|)} \, dx + \frac{1}{nh} \int_{\Omega} |u_n|^{p-2} u_n T_h(u_n) \varphi(u_n) e^{B(|u_n|)} \, dx \\ & = \frac{1}{h} \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) T_h(u_n) \varphi(u_n) e^{B(|u_n|)} \, dx. \end{aligned} \quad (4.21)$$

In view of (3.4), (3.5), and (3.6), we conclude that

$$\begin{aligned}
 & \frac{1}{h} \sum_{i=1}^N \int_{\Omega} a(x, T_n(u_n), \nabla u_n) D^i T_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx + \frac{2}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{p_i(x)} d(|u_n|) |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{\alpha(\theta-1)}{h} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} |T_h(u_n)| e^{B(|u_n|)} dx + \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{nh} \int_{\Omega} |u_n|^{p-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{1}{h} \int_{\Omega} (f_0(x) + |T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)}) |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{h} \int_{\Omega} (g_0(x) + d(|u_n|) \sum_{i=1}^N |D^i u_n|^{p_i(x)}) |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \int_{\Omega} (|f_0(x)| + |g_0(x)|) \frac{|T_h(u_n)|}{h} \varphi(u_n) e^{B(|u_n|)} dx + \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx + \frac{1}{h} \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.22}$$

It follows that

$$\begin{aligned}
 & \frac{1}{h} \sum_{i=1}^N \int_{\Omega} a(x, T_h(u_n), \nabla T_h(u_n)) D^i T_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx + \frac{1}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{p_i(x)} d(|u_n|) |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{\alpha(\theta-1)}{h} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} |T_h(u_n)| e^{B(|u_n|)} dx + \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{nh} \int_{\Omega} |u_n|^{p-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \int_{\Omega} (|f_0(x)| + |g_0(x)|) \frac{|T_h(u_n)|}{h} \varphi(u_n) e^{B(|u_n|)} dx + \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.23}$$

Concerning the two last terms on the right-hand side of (4.23), similarly to (4.9) and (4.10) we have

$$\begin{aligned}
 \frac{1}{h} \int_{\Omega} |T_n(u_n)|^{r_0(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx & \leq \frac{1}{2h} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{C_0}{h} \int_{\Omega} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.24}$$

Since $1 \leq \varphi(u_n) \leq 2$, then we have

$$\begin{aligned}
 & \frac{1}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{(\theta-1)}{2h} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} |T_h(u_n)| e^{B(|u_n|)} dx \\
 & \quad + \frac{1}{4h} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \quad + \frac{C_1}{h} \int_{\Omega} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.25}$$

By combining (4.23) and (4.24) – (4.25), we conclude that

$$\begin{aligned} & \frac{1}{h} \sum_{i=1}^N \int_{\Omega} a(x, T_n(u_n), \nabla u_n) D^i u_n \, dx + \frac{1}{h} \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{p_i(x)} d(|u_n|) |T_h(u_n)| \, dx \\ & + \frac{(\theta-1)}{2h} \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} |T_h(u_n)| \, dx + \frac{1}{4h} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_h(u_n)| \, dx \\ & + \frac{1}{nh} \int_{\Omega} |u_n|^{p-1} |T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} \, dx \\ & \leq 2e^{B(\infty)} \int_{\Omega} (|f_0(x)| + |g_0(x)| + C_0 + C_1) \frac{|T_h(u_n)|}{h} \, dx. \end{aligned} \quad (4.26)$$

Thanks to (4.16), we have : $\text{meas} \{|u_n| > h\} \rightarrow 0$ as h tends to infinity, thus $\frac{|T_h(u_n)|}{h} \rightharpoonup 0$ weak* in $L^\infty(\Omega)$. Using the Lebesgue's dominated convergence theorem, we have

$$\lim_{h \rightarrow \infty} 2e^{B(\infty)} \int_{\Omega} (|f_0(x)| + |g_0(x)| + C_0 + C_1) \frac{|T_h(u_n)|}{h} \, dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty. \quad (4.27)$$

Thus, by letting h tends to infinity in (4.26) we conclude that

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0. \quad (4.28)$$

Moreover, we have

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^N \int_{\{|u_n| > h\}} |D^i u_n|^{p_i(x)} d(|u_n|) \, dx = 0, \quad (4.29)$$

then

$$\lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_{i=1}^N \int_{\{|u_n| > h\}} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}} \, dx = 0. \quad (4.30)$$

Step 4 : The equi-integrability of $(|T_n(u_n)|^{q(x)-2} T_n(u_n))_n$ and $(\frac{1}{n} |u_n|^{p-2} u_n)_n$.

Now, we shall show that

$$|T_n(u_n)|^{q(x)-2} T_n(u_n) \rightarrow |u|^{q(x)-2} u \quad \text{and} \quad \frac{1}{n} |u_n|^{p-2} u_n \rightarrow 0 \quad \text{strongly in } L^1(\Omega). \quad (4.31)$$

Thanks to (4.28) we have $|T_n(u_n)|^{q(x)-2} T_n(u_n) \rightarrow |u|^{q(x)-2} u$ and $\frac{1}{n} |u_n|^{p-2} u_n \rightarrow 0$ a.e. in Ω . Then, in view of Vitali's theorem, it suffices to prove that $(|u_n|^{q(x)-2} u_n)_n$ and $(\frac{1}{n} |u_n|^{p-2} u_n)_n$ are uniformly equi-integrable.

Firstly, thanks to (4.26) we have

$$\int_{\{|u_n| > h\}} |T_n(u_n)|^{q(x)-1} \, dx + \frac{1}{n} \int_{\{|u_n| > h\}} |u_n|^{p-1} \, dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty, \quad (4.32)$$

it follows that : for any $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{|u_n| > h(\eta)\}} |T_n(u_n)|^{q(x)-1} \, dx + \frac{1}{n} \int_{\{|u_n| > h(\eta)\}} |u_n|^{p-1} \, dx \leq \frac{\eta}{2}. \quad (4.33)$$

On the other hand, for any measurable subset E of Ω , we have

$$\begin{aligned} & \int_E |T_n(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_E |u_n|^{p-1} dx \\ & \leq \int_E |T_{h(\eta)}(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_E |T_{h(\eta)}(u_n)|^{p-1} dx \\ & \quad + \int_{\{|u_n| > h(\eta)\}} |T_n(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_{\{|u_n| > h(\eta)\}} |u_n|^{p-1} dx. \end{aligned} \quad (4.34)$$

It's clear that, there exists $\mu(\eta) > 0$, such that for all $E \subset \Omega$, we have

$$\int_E |T_{h(\eta)}(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_E |T_{h(\eta)}(u_n)|^{p-1} dx \leq \frac{\eta}{2} \quad \text{for } \text{meas}(E) \leq \mu(\eta). \quad (4.35)$$

By combining the inequalities (4.33) and (4.35), we conclude that : For any $\eta > 0$, there exists $\mu(\eta) > 0$ such that

$$\int_E |T_n(u_n)|^{q(x)-1} dx + \frac{1}{n} \int_E |u_n|^{p-1} dx \leq \eta, \quad \text{for any } E \subset \Omega, \text{ such that } \text{meas}(E) \leq \mu(\eta). \quad (4.36)$$

Thus, the sequence $(|T_n(u_n)|^{q(x)-2} u_n)_n$ and $(\frac{1}{n} |u_n|^{p-2} u_n)_n$ are equi-integrable, which conclude the proof of (4.31).

Step 4: Strong convergence of truncations.

Let $h \geq k \geq 1$, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$, various real-valued functions of real variables that converge to 0 as n tends to infinity, similarly for $\varepsilon_i(h)$ and $\varepsilon_i(n, h)$.

In this step, we will prove the convergence of the sequence $(D^i u_n)_n$ to $D^i u$ almost everywhere in Ω , for any $i = 1, \dots, N$. We set

$$S_h(\tau) = 1 - \frac{|T_{2h}(\tau) - T_h(\tau)|}{h} \quad \text{and} \quad \psi(s) = s \cdot \exp\left(\frac{\gamma^2 s^2}{2}\right),$$

where $\gamma = \left(2(\theta - 1) + \frac{6\|d(\cdot)\|_{L^\infty(\mathbf{R})}}{\alpha}\right)$, note that $\psi'(s) - \gamma|\psi(s)| \geq \frac{1}{2} \quad \forall s \in \mathbf{R}$.

By taking $v = \psi(T_k(u_n) - T_k(u))S_h(u_n)\varphi(u_n)e^{B(|u_n|)}$ as a test function in the approximate problem (4.5), we

have

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& - \frac{1}{h} \sum_{i=1}^N \int_{\{h < |u_n| \leq 2h\}} a_i(x, T_n(u_n), D^i u_n) D^i u_n |\psi(T_k(u_n) - T_k(u))| \varphi(u_n) e^{B(|u_n|)} dx \\
& + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), D^i u_n) D^i u_n}{(1 + |u_n|)^{\theta}} \text{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) e^{B(|u_n|)} dx \\
& + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) D^i u_n d(|u_n|) \text{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \int_{\Omega} g_n(x, u_n, D^i u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_h(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& = \int_{\Omega} f_n(x, T_n(u_n), D^i u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), D^i u_n) D^i u_n}{(1 + |u_n|)^{\theta}} \text{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) e^{B(|u_n|)} dx \\
& + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) D^i u_n d(|u_n|) \text{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& \leq \int_{\Omega} |f_n(x, T_n(u_n), D^i u_n)| |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \int_{\Omega} |g_n(x, u_n, D^i u_n)| |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_h(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
& + \frac{1}{h} \sum_{i=1}^N \int_{\{h < |u_n| \leq 2h\}} a_i(x, T_n(u_n), D^i u_n) D^i u_n |\psi(T_k(u_n) - T_k(u))| \varphi(u_n) e^{B(|u_n|)} dx.
\end{aligned}$$

In view of (3.5) and (3.6), we conclude that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), D^i u_n) D^i u_n \operatorname{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) e^{B(|u_n|)}}{(1 + |u_n|)^{\theta}} dx \\
 & + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) D^i u_n d(|u_n|) \operatorname{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq 2e^{B(\infty)} \int_{\Omega} (f_0(x) + g_0(x)) |\psi(T_k(u_n) - T_k(u))| dx + 2e^{B(\infty)} \int_{\Omega} |T_n(u_n)|^{r_0(x)} |\psi(T_k(u_n) - T_k(u))| dx \\
 & + \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + 2e^{B(\infty)} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |\psi(T_k(u_n) - T_k(u))| dx + \frac{2e^{B(\infty)}}{n} \int_{\Omega} |u_n|^{p-1} |\psi(T_k(u_n) - T_k(u))| dx \\
 & + \frac{2e^{B(\infty)} \psi(2k)}{h} \sum_{i=1}^N \int_{\{h < |u_n| \leq 2h\}} a_i(x, T_n(u_n), D^i u_n) D^i u_n dx.
 \end{aligned} \tag{4.37}$$

For the first term of the right-hand side of (4.37), we have $\psi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak* in $L^\infty(\Omega)$, and since $f_0(x)$ and $g_0(x)$ belong to $L^1(\Omega)$, it follows that

$$\varepsilon_0(n) = 2e^{B(\infty)} \int_{\Omega} (f_0(x) + g_0(x)) |\psi(T_k(u_n) - T_k(u))| dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.38}$$

Similarly, since $r_0(x) \leq q(x) - 1$ a. e. in Ω and thanks to (4.31), we conclude that

$$\varepsilon_1(n) = 2e^{B(\infty)} \int_{\Omega} |T_{2h}(u_n)|^{r_0(x)} |\psi(T_k(u_n) - T_k(u))| dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.39}$$

$$\varepsilon_2(n) = 2e^{B(\infty)} \int_{\{|u_n| \leq k\}} |T_k(u_n)|^{q(x)-1} |\psi(T_k(u_n) - T_k(u))| dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.40}$$

In addition,

$$\varepsilon_3(n) = \frac{2e^{B(\infty)}}{n} \int_{\Omega} |u_n|^{p-2} u_n \psi(T_k(u_n) - T_k(u)) S_h(u_n) dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.41}$$

Concerning the last term of the right-hand side of (4.37) and using (4.28), we obtain

$$\varepsilon_4(h) = \frac{2ke^{B(\infty)} \psi(2k)}{h} \sum_{i=1}^N \int_{\{h < |u_n| \leq 2h\}} a_i(x, T_n(u_n), D^i u_n) D^i u_n dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty. \tag{4.42}$$

By combining (4.37) and (4.38) – (4.42), we conclude that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), D^i u_n) D^i u_n \operatorname{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) e^{B(|u_n|)}}{(1 + |u_n|)^{\theta}} dx \\
 & + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), D^i u_n) D^i u_n d(|u_n|) \operatorname{sign}(u_n) \psi(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \quad (4.43) \\
 & \leq 2 \sum_{i=1}^N \int_{\Omega} |D^i u_n|^{r_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx + \varepsilon_5(n, h).
 \end{aligned}$$

We have $a_i(x, s, 0) = 0$, and $S_h(u_n) = 1$ on the set $\{|u_n| \leq h\}$. Moreover, $\psi(T_k(u_n) - T_k(u))$ have the same sign as u_n on the set $\{|u_n| > k\}$. Thus, in view of (3.4) and using Young's inequality, one has

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n)) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) \varphi(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & - \sum_{i=1}^N \int_{\{k < |u_n| \leq 2h\}} a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n)) D^i T_k(u) \psi'(T_k(u_n) - T_k(u)) S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & - (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_k(u_n), D^i T_k(u_n)) D^i T_k(u_n)}{(1 + |u_n|)^{\theta}} |\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & + \alpha(\theta - 1) \sum_{i=1}^N \int_{\{k < |u_n| \leq 2h\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) e^{B(|u_n|)} dx \\
 & - \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n)) D^i T_k(u_n) d(|u_n|) \psi(T_k(u_n) - T_k(u)) \varphi(u_n) e^{B(|u_n|)} dx \quad (4.44) \\
 & + 2 \sum_{i=1}^N \int_{\{k < |u_n| \leq 2h\}} d(|u_n|) |D^i u_n|^{p_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \alpha(\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) e^{B(|u_n|)} dx \\
 & + \frac{1}{2} \int_{\Omega} |T_n(u_n)|^{q(x)-1} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) e^{B(|u_n|)} dx \\
 & + 2C_0 \int_{\Omega} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} d(|u_n|) |D^i u_n|^{p_i(x)} |\psi(T_k(u_n) - T_k(u))| S_h(u_n) \varphi(u_n) e^{B(|u_n|)} dx + \varepsilon_5(n, h).
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))\psi'(T_k(u_n) - T_k(u))\varphi(u_n)e^{B(|u_n|)} dx \\
 & - 2 \sum_{i=1}^N \int_{\{k < |u_n| \leq 2h\}} a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n))D^i T_k(u)\psi'(T_k(u_n) - T_k(u))S_h(u_n)e^{B(|u_n|)} dx \\
 & - 2(\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_k(u_n), D^i T_k(u_n))D^i T_k(u_n)}{(1 + |u_n|)^{\theta}} |\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & - \frac{3}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n))D^i T_k(u_n)d(|u_n|) \psi(T_k(u_n) - T_k(u))\varphi(u_n)e^{B(|u_n|)} dx \\
 & \leq \varepsilon_6(n, h).
 \end{aligned} \tag{4.45}$$

Concerning the second term of the left-hand side of (4.45), we have $(a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n)))_n$ is bounded in $L^{p_i(\cdot)}(\Omega)$, then there exists $\vartheta_i \in L^{p_i(\cdot)}(\Omega)$ such that $a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n)) \rightharpoonup \vartheta_i$ in $L^{p_i(\cdot)}(\Omega)$ for any $i = 1, \dots, N$, it follows that

$$\begin{aligned}
 \varepsilon_7(n) &= \sum_{i=1}^N \left| \int_{\{k < |u_n| \leq 2h\}} a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n))D^i T_k(u)\psi'(T_k(u_n) - T_k(u))S_h(u_n)\varphi(u_n)e^{B(|u_n|)} dx \right| \\
 &\leq 2e^{B(\infty)}\psi'(2k) \sum_{i=1}^N \int_{\{k < |u_n| \leq 2h\}} |a_i(x, T_{2h}(u_n), D^i T_{2h}(u_n))| |D^i T_k(u)| dx \\
 &\longrightarrow 2e^{B(\infty)}\psi'(2k) \sum_{i=1}^N \int_{\{k < |u| \leq 2h\}} |\vartheta_i| |D^i T_k(u)| dx = 0 \quad \text{as } n \rightarrow \infty.
 \end{aligned} \tag{4.46}$$

We conclude that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))\psi'(T_k(u_n) - T_k(u))\varphi(u_n)e^{B(|u_n|)} dx \\
 & - \left(2(\theta - 1) + \frac{6\|d(\cdot)\|_{L^\infty(R)}}{\alpha}\right) \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n))D^i T_k(u_n)|\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & \leq \varepsilon_8(n, h),
 \end{aligned} \tag{4.47}$$

since $\gamma = \left(2(\theta - 1) + \frac{6\|d(\cdot)\|_{L^\infty(R)}}{\alpha}\right)$, it follows that

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u)) \right) (D^i T_k(u_n) - D^i T_k(u))\psi'(T_k(u_n) - T_k(u))e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u))(D^i T_k(u_n) - D^i T_k(u))\psi'(T_k(u_n) - T_k(u))\varphi(u_n)e^{B(|u_n|)} dx \\
 & - \gamma \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u)) \right) (D^i T_k(u_n) - D^i T_k(u))|\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & - \gamma \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u))(D^i T_k(u_n) - D^i T_k(u))|\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & - \gamma \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n))D^i T_k(u)|\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\
 & \leq \varepsilon_9(n, h).
 \end{aligned} \tag{4.48}$$

For the second term of the right-hand side of (4.48), in view of (4.20) we have $T_k(u_n) \rightarrow T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, then

$$a_i(x, T_k(u_n), \nabla T_k(u)) \rightarrow a_i(x, T_k(u), \nabla T_k(u)) \quad \text{strongly in } L^{p_i(\cdot)}(\Omega),$$

and since $D^i T_k(u_n)$ tends to $D^i T_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\begin{aligned} \varepsilon_{10}(n) &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) \psi'(T_k(u_n) - T_k(u)) e^{B(|u_n|)} dx \\ &\leq e^{B(\infty)} \psi'(2k) \sum_{i=1}^N \int_{\Omega} |a_i(x, T_k(u_n), D^i T_k(u))| |D^i T_k(u_n) - D^i T_k(u)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.49)$$

Similarly, we have

$$\begin{aligned} \varepsilon_{11}(n) &= \left| \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u)) (D^i T_k(u_n) - D^i T_k(u)) |\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \right| \\ &\leq \psi(2k) e^{B(\infty)} \sum_{i=1}^N \int_{\Omega} |a_i(x, T_k(u_n), D^i T_k(u))| |D^i T_k(u_n) - D^i T_k(u)| dx \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.50)$$

Concerning the last term of the left-hand side of (4.48), we have $(|a_i(x, T_k(u_n), D^i T_k(u_n))|)_n$ is bounded in $L^{p_i(\cdot)}(\Omega)$, then there exists $v_i \in L^{p_i(\cdot)}(\Omega)$ such that $|a_i(x, T_k(u_n), D^i T_k(u_n))| \rightharpoonup v_i$ weakly in $L^{p_i(\cdot)}(\Omega)$, and since $D^i T_k(u) |\psi(T_k(u_n) - T_k(u))|$ tends strongly to 0 in $L^{p_i(\cdot)}(\Omega)$ for any $i = 1, \dots, N$, it follows that

$$\begin{aligned} \varepsilon_{12}(n) &= \left| \sum_{i=1}^N \int_{\Omega} a_i(x, T_k(u_n), D^i T_k(u_n)) D^i T_k(u) |\psi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \right| \\ &\leq e^{B(\infty)} \sum_{i=1}^N \int_{\Omega} |a_i(x, T_k(u_n), D^i T_k(u_n))| |D^i T_k(u)| |\psi(T_k(u_n) - T_k(u))| dx \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.51)$$

By combining (4.48) and (4.49) – (4.51), we conclude that

$$\begin{aligned} 0 &\leq \frac{1}{2} \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u)) \right) (D^i T_k(u_n) - D^i T_k(u)) dx \\ &\leq \sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), D^i T_k(u_n)) - a_i(x, T_k(u_n), D^i T_k(u)) \right) (D^i T_k(u_n) - D^i T_k(u)) \\ &\quad \times \left(\psi'(T_k(u_n) - T_k(u)) - \gamma |\psi(T_k(u_n) - T_k(u))| \right) e^{B(|u_n|)} dx \\ &\leq \varepsilon_{14}(n, h) \rightarrow 0 \quad \text{as } n, h \rightarrow \infty. \end{aligned} \quad (4.52)$$

In view of Lebesgue dominated convergence theorem, we have $T_k(u_n) \rightarrow T_k(u)$ strongly in $L^p(\Omega)$. Thus, by letting n then h tend to infinity, we deduce that

$$\begin{aligned} &\sum_{i=1}^N \int_{\Omega} \left(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) (D^i T_k(u_n) - D^i T_k(u)) dx \\ &+ \int_{\Omega} (|T_k(u_n)|^{p-2} T_k(u_n) - |T_k(u)|^{p-2} T_k(u)) (T_k(u_n) - T_k(u)) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.53)$$

Thanks to Lemma 3.2, we conclude that

$$\begin{cases} T_k(u_n) \rightarrow T_k(u) & \text{strongly in } W^{1, \vec{p}(\cdot)}(\Omega), \\ D^i u_n \rightarrow D^i u & \text{a.e. in } \Omega \quad \text{for } i = 1, \dots, N. \end{cases} \quad (4.54)$$

Moreover, we have $a_i(x, T_n(u_n), \nabla u_n) D^i u_n$ tends to $a_i(x, u, \nabla u) D^i u$ almost everywhere in Ω , in view of Fatou's lemma and (4.28), we conclude that

$$\begin{aligned} & \lim_{h \rightarrow \infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u| \leq h\}} a_i(x, u, \nabla u) D^i u \, dx \\ & \leq \lim_{h \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u_n| \leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx \\ & \leq \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \sum_{i=1}^N \int_{\{|u_n| \leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0, \end{aligned} \quad (4.55)$$

which prove (4.2).

Step 4 : The equi-integrability of $(g_n(x, u_n, \nabla u_n))_n$ and $(f_n(x, T_n(u_n), \nabla u_n))_n$.

In order to pass to the limit in the approximate problem (4.5), we will show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \quad \text{and} \quad f_n(x, T_n(u_n), \nabla u_n) \longrightarrow f(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \quad (4.56)$$

Thanks to (4.54), we have $g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)$ and $f_n(x, T_n(u_n), \nabla u_n) \rightarrow f(x, u, \nabla u)$ a.e. in Ω . Then, in view of Vitali's theorem, it suffices to prove that $(g_n(x, u_n, \nabla u_n))_n$ and $(f_n(x, T_n(u_n), \nabla u_n))_n$ are uniformly equi-integrable.

By taking $(T_{h+1}(u_n) - T_h(u_n))\varphi(u_n)e^{B(|u_n|)}$ as a test function in the approximate problem (4.5), we have

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i (T_{h+1}(u_n) - T_h(u_n)) \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \frac{2}{\alpha} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i u_n d(|u_n|) \varphi(u_n) |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} \, dx \\ & + (\theta - 1) \sum_{i=1}^N \int_{\Omega} \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^{\theta}} |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} \, dx \\ & + \int_{\Omega} g_n(x, u_n, \nabla u_n) (T_{h+1}(u_n) - T_h(u_n)) \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \int_{\Omega} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} \, dx \\ & + \frac{1}{n} \int_{\Omega} |u_n|^{\frac{p-1}{2}} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} \, dx \\ & = \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) (T_{h+1}(u_n) - T_h(u_n)) \varphi(u_n) e^{B(|u_n|)} \, dx. \end{aligned}$$

According to (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}
 & \alpha \sum_{i=1}^N \int_{\{h \leq |u_n| \leq 2h\}} |D^i u_n|^{p_i(x)} dx + 2 \sum_{i=1}^N \int_{\{h \leq |u_n|\}} |D^i u_n|^{p_i(x)} d(|u_n|) \varphi(u_n) |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\
 & + \alpha(\theta - 1) \sum_{i=1}^N \int_{\{h \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\
 & + \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \int_{\{h \leq |u_n|\}} \left(f_0(x) + |T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} \right) |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \int_{\{h \leq |u_n|\}} \left(g_0(x) + d(|u_n|) \sum_{i=1}^N |D^i u_n|^{p_i(x)} \right) |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & \alpha \sum_{i=1}^N \int_{\{h \leq |u_n| \leq h+1\}} |D^i u_n|^{p_i(x)} dx + \sum_{i=1}^N \int_{\{h \leq |u_n|\}} |D^i u_n|^{p_i(x)} d(|u_n|) \varphi(u_n) |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\
 & + \alpha(\theta - 1) \sum_{i=1}^N \int_{\{h \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\
 & + \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \int_{\{h \leq |u_n|\}} (f_0(x) + g_0(x)) |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{r_0(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + \sum_{i=1}^N \int_{\{h \leq |u_n|\}} |D^i u_n|^{r_i(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.57}$$

Using Young's inequality, we have

$$\begin{aligned}
 & \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{r_0(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{1}{2} \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + C_0 \int_{\{h \leq |u_n|\}} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx,
 \end{aligned} \tag{4.58}$$

and

$$\begin{aligned}
 & \sum_{i=1}^N \int_{\{h \leq |u_n|\}} |D^i u_n|^{r_i(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & \leq \frac{\alpha(\theta - 1)}{2} \sum_{i=1}^N \int_{\{h \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\
 & + \frac{1}{4} \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\
 & + C_1 \int_{\{h \leq |u_n|\}} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx.
 \end{aligned} \tag{4.59}$$

It follows that

$$\begin{aligned} & \alpha \sum_{i=1}^N \int_{\{h \leq |u_n| \leq h+1\}} |D^i u_n|^{p_i(x)} dx + \sum_{i=1}^N \int_{\{h \leq |u_n|\}} |D^i u_n|^{p_i(x)} d(|u_n|) \varphi(u_n) |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\ & + \frac{\alpha(\theta-1)}{2} \sum_{i=1}^N \int_{\{h \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| e^{B(|u_n|)} dx \\ & + \frac{1}{4} \int_{\{h \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx \\ & \leq \int_{\{h \leq |u_n|\}} (f_0(x) + g_0(x) + C_0 + C_1) |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) e^{B(|u_n|)} dx. \end{aligned}$$

Thanks to (4.16), we conclude that

$$\begin{aligned} & e^{B(\infty)} \sum_{i=1}^N \int_{\{h+1 \leq |u_n|\}} |D^i u_n|^{p_i(x)} d(|u_n|) dx + \frac{\alpha(\theta-1)e^{B(\infty)}}{2} \sum_{i=1}^N \int_{\{h+1 \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta} dx \\ & + \frac{e^{B(\infty)}}{4} \int_{\{h+1 \leq |u_n|\}} |T_n(u_n)|^{q(x)-1} dx \\ & \leq 2e^{B(\infty)} \int_{\{h \leq |u_n|\}} (f_0(x) + g_0(x) + C_0 + C_1) dx \longrightarrow 0 \quad \text{as } h \rightarrow \infty. \end{aligned} \quad (4.60)$$

In view of (4.58) – (4.59) and (4.60), we conclude that : for any $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\int_{\{h(\eta) < |u_n|\}} \left(|T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} \right) dx \leq \frac{\eta}{4}, \quad (4.61)$$

and

$$\sum_{i=1}^N \int_{\{h(\eta) < |u_n|\}} d(|u_n|) |D^i u_n|^{p_i(x)} dx \leq \frac{\eta}{4}. \quad (4.62)$$

On the other hand, for any measurable subset $E \subseteq \Omega$, we have

$$\begin{aligned} & \int_E \left(|T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} \right) dx + \sum_{i=1}^N \int_E d(|u_n|) |D^i u_n|^{p_i(x)} dx \\ & \leq \int_E \left(|T_{h(\eta)}(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{r_i(x)} \right) dx + \sum_{i=1}^N \int_E d(|T_{h(\eta)}(u_n)|) |D^i T_{h(\eta)}(u_n)|^{p_i(x)} dx \\ & + \int_{\{h(\eta) < |u_n|\}} \left(|T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} \right) dx + \sum_{i=1}^N \int_{\{h(\eta) < |u_n|\}} d(|u_n|) |D^i u_n|^{p_i(x)} dx. \end{aligned} \quad (4.63)$$

From (4.54), there exists $\mu(\eta) > 0$ such that

$$\int_E \left(|T_{h(\eta)}(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i T_{h(\eta)}(u_n)|^{r_i(x)} \right) dx + \sum_{i=1}^N \int_E d(|T_{h(\eta)}(u_n)|) |D^i T_{h(\eta)}(u_n)|^{p_i(x)} dx \leq \frac{\eta}{2}, \quad (4.64)$$

for all E such that $\text{meas}(E) \leq \mu(\eta)$.

Finally, by combining (4.61) – (4.64), we conclude that

$$\int_E \left(|T_n(u_n)|^{r_0(x)} + \sum_{i=1}^N |D^i u_n|^{r_i(x)} \right) dx + \sum_{i=1}^N \int_E d(|u_n|) |D^i u_n|^{p_i(x)} dx \leq \eta \quad \text{for all } E \text{ with } \text{meas}(E) \leq \mu(\eta). \quad (4.65)$$

Then, the sequences $(|T_n(u_n)|^{r_0(x)})_n$, $(\sum_{i=1}^N |D^i u_n|^{r_i(x)})_n$ and $(\sum_{i=1}^N d(|u_n|) |D^i u_n|^{p_i(x)})_n$ are uniformly equi-integrable, thanks to (3.5) and (3.6) we conclude that $(g_n(x, u_n, \nabla u_n))_n$ and $(f_n(x, T_n(u_n), \nabla u_n))_n$ are equi-integrable and by using Vitali's theorem, the convergence (4.56) is concluded.

Step 5 : Passage to the limit

Let $\varphi \in W^{1, \vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and let $S(\cdot)$ be a smooth function in $W^{1, \infty}(\mathbb{R})$ such that $\text{supp}(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.

By choosing $S(u_n)\varphi \in W^{1, \vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (4.5), we obtain

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) (D^i u_n S'(u_n) \varphi + S(u_n) D^i \varphi) dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) S(u_n) \varphi dx \\ & + \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_n(u_n) S(u_n) \varphi dx + \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n S(u_n) \varphi dx = \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) S(u_n) \varphi dx. \end{aligned} \quad (4.66)$$

We begin by the first term of the left-hand side of (4.66), we have

$$\begin{aligned} & \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) (S'(u_n) \varphi D^i u_n + S(u_n) D^i \varphi) dx \\ & = \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) (S'(u_n) \varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi) dx, \end{aligned}$$

in view of (4.54), we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i(\cdot)}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω , it follows that

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \quad \text{in } L^{p_i(\cdot)}(\Omega),$$

and since $S'(u_n) \varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi$ tends strongly to $S'(u) \varphi D^i T_M(u) + S(T_M(u)) D^i \varphi$ in $L^{p_i(\cdot)}(\Omega)$, we deduce that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) (S'(u_n) \varphi D^i u_n + S(u_n) D^i \varphi) dx \\ & = \lim_{n \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) (S'(u_n) \varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi) dx \\ & = \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u), \nabla T_M(u)) (S'(u) \varphi D^i T_M(u) + S(T_M(u)) D^i \varphi) dx \\ & = \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) (S'(u) \varphi D^i u + S(u) D^i \varphi) dx. \end{aligned} \quad (4.67)$$

Concerning the second term of the left-hand side of (4.66), we have $S(T_M(u_n))\varphi \rightarrow S(T_M(u))\varphi$ weak* in $L^\infty(\Omega)$, and thanks to (4.56), we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} g_n(x, u_n, \nabla u_n) S(T_M(u_n)) \varphi dx = \int_{\Omega} g(x, u, \nabla u) S(T_M(u)) \varphi dx = \int_{\Omega} g(x, u, \nabla u) S(u) \varphi dx, \quad (4.68)$$

furthermore, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n(x, T_n(u_n), \nabla u_n) S(T_M(u_n)) \varphi dx = \int_{\Omega} f(x, u, \nabla u) S(T_M(u)) \varphi dx = \int_{\Omega} f(x, u, \nabla u) S(u) \varphi dx. \quad (4.69)$$

Moreover, thanks to (4.31), we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} |T_n(u_n)|^{q(x)-2} T_h(u_n) S(T_M(u_n)) \varphi \, dx = \int_{\Omega} |u|^{q(x)-2} u S(T_M(u)) \varphi \, dx = \int_{\Omega} |u|^{q(x)-2} u S(u) \varphi \, dx, \quad (4.70)$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_{\Omega} |u_n|^{p-2} u_n S(u_n) \varphi \, dx = 0. \quad (4.71)$$

Hence, putting all the terms (4.66) and (4.67) – (4.71) together, we obtain

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) (D^i u S'(u) \varphi + S(u) D^i \varphi) \, dx + \int_{\Omega} g(x, u, \nabla u) S(u) \varphi \, dx \\ + \int_{\Omega} |u|^{q(x)-2} u S(u) \varphi \, dx = \int_{\Omega} f(x, u, \nabla u) S(u) \varphi \, dx. \end{aligned} \quad (4.72)$$

which conclude the proof of Theorem 4.1.

5 Appendix

Proof of Lemma 4.1

Using the Hölder and Poincaré type inequalities, and (3.2), we have for any $u, v \in W^{1, \vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned} |\langle G_n u, v \rangle| &\leq \sum_{i=1}^N \int_{\Omega} |a_i(x, T_n(u), \nabla u)| |D^i v| \, dx + \int_{\Omega} |g_n(x, u, \nabla u)| |v| \, dx \\ &\quad + \int_{\Omega} |T_n(u)|^{q(x)-1} |v| \, dx + \frac{1}{n} \int_{\Omega} |u|^{p-1} |v| \, dx + \int_{\Omega} |f_n(x, T_n(u), \nabla u)| |v| \, dx \\ &\leq \beta \sum_{i=1}^N \int_{\Omega} (K_i(x) + |T_n(u)|^{p_i(x)-1} + |D^i u|^{p_i(x)-1}) |D^i v| \, dx \\ &\quad + \frac{1}{n} \|u\|_{\vec{p}}^{p-1} \|v\|_{\vec{p}} + (n^{q^+-1} + 2n) \int_{\Omega} |v| \, dx \\ &\leq \beta \sum_{i=1}^N \left(\|K_i(x)\|_{p'_i(x)} + \|n\|_{p_i(x)}^{p_i^+-1} + \|D^i u\|_{p_i(x)}^{p_i^+-1} + 2 \right) \|D^i v\|_{p_i(x)} \\ &\quad + \frac{1}{n} \|u\|_{1, \vec{p}(\cdot)}^{p-1} \|v\|_{1, \vec{p}(\cdot)} + (n^{q^+-1} + 2n) \|v\|_{L^1(\Omega)} \\ &\leq C_0 (1 + n^{p^+-1} + \|u\|_{1, \vec{p}(\cdot)}^{p^+-1}) \|v\|_{1, \vec{p}(\cdot)} + \frac{1}{n} \|u\|_{1, \vec{p}(\cdot)}^{p-1} \|v\|_{1, \vec{p}(\cdot)} + (n^{q^+-1} + 2n) \|v\|_{1, \vec{p}(\cdot)}, \end{aligned} \quad (5.1)$$

thus, the operator G_n is bounded. For the coercivity, we have for any $u \in W^{1, \vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned} \langle G_n u, u \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u), \nabla u) D^i u \, dx + \int_{\Omega} g_n(x, u, \nabla u) u \, dx \\ &\quad + \int_{\Omega} |T_n(u)|^{q(x)-1} |u| \, dx + \frac{1}{n} \int_{\Omega} |u|^p \, dx - \int_{\Omega} f_n(x, T_n(u), \nabla u) u \, dx \\ &\geq \alpha \sum_{i=1}^N \int_{\Omega} |D^i u|^{p_i(x)} \, dx + \frac{1}{n} \int_{\Omega} |u|^p \, dx - 2n \int_{\Omega} |u| \, dx \\ &\geq C_1 \|u\|_{1, \vec{p}(\cdot)}^p - \alpha N - 2n \|u\|_{1, \vec{p}(\cdot)}. \end{aligned}$$

It follows that

$$\frac{\langle G_n u, u \rangle}{\|u\|_{1, \vec{p}(\cdot)}} = \frac{C_1 \|u\|_{1, \vec{p}(\cdot)}^p - \alpha N - 2n \|u\|_{1, \vec{p}(\cdot)}}{\|u\|_{1, \vec{p}(\cdot)}} \longrightarrow +\infty \quad \text{as} \quad \|u\|_{1, \vec{p}(\cdot)} \rightarrow \infty.$$

It remains to show that G_n is pseudo-monotone. Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $W^{1, \vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W^{1, \vec{p}(\cdot)}(\Omega), \\ G_n u_k \rightharpoonup \chi_n & \text{in } (W^{1, \vec{p}(\cdot)}(\Omega))', \\ \limsup_{k \rightarrow \infty} \langle G_n u_k, u_k \rangle \leq \langle \chi_n, u \rangle. \end{cases} \quad (5.2)$$

We will show that

$$\chi_n = G_n u \quad \text{and} \quad \langle G_n u_k, u_k \rangle \longrightarrow \langle \chi_n, u \rangle \quad \text{as } k \rightarrow +\infty.$$

In view of the compact embedding $W^{1, \vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\vec{p}(\cdot)}(\Omega)$, there exists a subsequence still denoted $(u_k)_{k \in \mathbb{N}^*}$ such that $u_k \rightarrow u$ in $L^{\vec{p}(\cdot)}(\Omega)$.

As $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence in $W^{1, \vec{p}(\cdot)}(\Omega)$, using the growth condition (3.2), it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k \in \mathbb{N}^*}$ is bounded in $L^{p'_i(\cdot)}(\Omega)$, and there exists a function $\varphi_i \in L^{p'_i(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \text{ weakly in } L^{p'_i(\cdot)}(\Omega) \text{ as } k \rightarrow \infty. \quad (5.3)$$

Similarly, since $(H_n(x, u_k, \nabla u_k))_{k \in \mathbb{N}^*}$ and $(f_n(x, T_n(u_k), \nabla u_k))_{k \in \mathbb{N}^*}$ are bounded in $L^{\vec{p}(\cdot)}(\Omega)$, then there exists two measurable functions ψ_n and ϕ_n in $L^{\vec{p}(\cdot)}(\Omega)$, such that

$$g_n(x, u_k, \nabla u_k) \longrightarrow \psi_n \quad \text{and} \quad f_n(x, T_n(u_k), \nabla u_k) \longrightarrow \phi_n \quad \text{weakly in } L^{\vec{p}(\cdot)}(\Omega). \quad (5.4)$$

Also, we have

$$|T_n(u_k)|^{q(x)-2} T_n(u_k) \rightarrow |T_n(u)|^{q(x)-2} T_n(u) \quad \text{and} \quad \frac{1}{n} |u_k|^{\vec{p}-2} u_k \rightarrow \frac{1}{n} |u|^{\vec{p}-2} u \quad \text{strongly in } L^{\vec{p}(\cdot)}(\Omega). \quad (5.5)$$

Thus, for any $v \in W^{1, \vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned} \langle \chi_n, v \rangle &= \lim_{k \rightarrow \infty} \langle G_n u_k, v \rangle \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i v \, dx + \lim_{k \rightarrow \infty} \int_{\Omega} g_n(x, u_k, \nabla u_k) v \, dx \\ &\quad + \lim_{k \rightarrow \infty} \int_{\Omega} |T_n(u_k)|^{q(x)-2} T_n(u_k) v \, dx + \lim_{k \rightarrow \infty} \frac{1}{n} \int_{\Omega} |u_k|^{\vec{p}-2} u_k v \, dx \\ &\quad - \lim_{k \rightarrow \infty} \int_{\Omega} f_n(x, T_n(u_k), \nabla u_k) v \, dx \\ &= \sum_{i=1}^N \int_{\Omega} \varphi_i D^i v \, dx + \int_{\Omega} \psi_n v \, dx + \int_{\Omega} |T_n(u)|^{q(x)-2} T_n(u) v \, dx \\ &\quad + \frac{1}{n} \int_{\Omega} |u|^{\vec{p}-2} u v \, dx - \int_{\Omega} \phi_n v \, dx. \end{aligned} \quad (5.6)$$

Having in mind (5.2) and (5.6), we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle G_n(u_k), u_k \rangle &= \limsup_{k \rightarrow \infty} \left(\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx + \int_{\Omega} g_n(x, u_k, \nabla u_k) u_k \, dx \right. \\ &\quad \left. + \int_{\Omega} |T_n(u_k)|^{q(x)} \, dx + \frac{1}{n} \int_{\Omega} |u_k|^{\vec{p}} \, dx - \int_{\Omega} f_n(x, T_n(u_k), \nabla u_k) u_k \, dx \right) \\ &\leq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx + \int_{\Omega} \psi_n u \, dx + \int_{\Omega} |T_n(u)|^{q(x)} \, dx \\ &\quad + \frac{1}{n} \int_{\Omega} |u|^{\vec{p}} \, dx - \int_{\Omega} \phi_n u \, dx. \end{aligned} \quad (5.7)$$

Thanks to (5.4) – (5.5), we have

$$\int_{\Omega} g_n(x, u_k, \nabla u_k) u_k \, dx - \int_{\Omega} f_n(x, T_n(u_k), \nabla u_k) u_k \, dx \longrightarrow \int_{\Omega} \psi_n u \, dx - \int_{\Omega} \phi_n u \, dx \quad \text{as } k \rightarrow \infty, \quad (5.8)$$

and

$$\int_{\Omega} |T_n(u_k)|^{q(x)} \, dx + \frac{1}{n} \int_{\Omega} |u_k|^p \, dx \longrightarrow \int_{\Omega} |T_n(u)|^{q(x)} \, dx + \frac{1}{n} \int_{\Omega} |u|^p \, dx \quad \text{as } k \rightarrow \infty. \quad (5.9)$$

It follows that

$$\limsup_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \leq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx. \quad (5.10)$$

On the other hand, in view of (3.4), we have

$$\sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx \geq 0, \quad (5.11)$$

then

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx &\geq \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u \, dx \\ &\quad + \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u) \, dx. \end{aligned}$$

In view of Lebesgue's dominated convergence theorem, we have $T_n(u_k) \rightarrow T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x, T_n(u_k), \nabla u) \rightarrow a_i(x, T_n(u), \nabla u)$ strongly in $L^{p_i(\cdot)}(\Omega)$, and using (5.3), we get

$$\liminf_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx \geq \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx. \quad (5.12)$$

Having in mind (5.10), we conclude that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_k), \nabla u_k) D^i u_k \, dx = \sum_{i=1}^N \int_{\Omega} \varphi_i D^i u \, dx. \quad (5.13)$$

Therefore, by combining (5.8) and (5.9), we obtain

$$\langle G_n u_k, u_k \rangle \longrightarrow \langle \chi_n, u \rangle \quad \text{as } k \rightarrow \infty. \quad (5.14)$$

On the other hand, thanks to (5.13), we can show that

$$\lim_{k \rightarrow \infty} \sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx = 0.$$

Now, since $u_k \rightarrow u$ strongly in $L^p(\Omega)$, it follows that

$$\begin{aligned} &\int_{\Omega} (|u_k|^{p-2} u_k - |u|^{p-2} u)(u_k - u) \, dx \\ &\quad + \sum_{i=1}^N \int_{\Omega} (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx \rightarrow 0, \quad \text{as } k \rightarrow \infty, \end{aligned} \quad (5.15)$$

in view of Lemma 3.2, we conclude that

$$u_k \rightarrow u \quad \text{in } W^{1, \vec{p}(\cdot)}(\Omega) \quad \text{and} \quad D^i u_k \rightarrow D^i u \quad \text{a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{in } L^{p'_i(\cdot)}(\Omega) \quad \text{for } i = 1, \dots, N,$$

and

$$g_n(x, u_k, \nabla u_k) \rightharpoonup g_n(x, u, \nabla u) \quad \text{and} \quad f_n(x, T_n(u_k), \nabla u_k) \rightharpoonup f_n(x, T_n(u), \nabla u) \quad \text{in } L^{p'}(\Omega),$$

having in mind (5.5), we obtain $\chi_n = G_n u$. Thus, the proof of the Lemma 4.1 is concluded.

As models example of applications for problem (3.7), we state the following model:

Example 5.1. Let $q_0(x) = 2$ with $p_i(x) = 2$ and $0 \leq r_i(x) < \frac{2}{3}$ for $i = 1, \dots, N$. We consider the nonlinear Neumann elliptic problem

$$\begin{cases} -\Delta u + \alpha \sum_{i=1}^N \frac{|D^i u|^2}{(1 + |u|)^{\frac{3}{2}}} + u = f(x) + \beta |s|^{r_0(x)} + \gamma \sum_{i=1}^N |\xi_i|^{r_i(x)} & \text{in } \Omega \\ \nabla u \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.16)$$

with α, β and γ are some real constants and the data $f(x)$ belongs to $L^1(\Omega)$.

In view of theorem 4.1, there exists at least one renormalized solution for the quasilinear elliptic problem (5.16).

Moreover, we have $|u|^{r_0(x)} \in L^1(\Omega)$ and $|D^i u|^{r_i(x)} \in L^1(\Omega)$ for $i = 1, \dots, N$.

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