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Harouna Ouedraogo*, Dramane Ouedraogo, Idrissa Ibrango, and Aboudramane Guiro

A study of stability of SEIHR model of infectious disease transmission

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Abstract: We develop in this paper a Susceptible Exposed Infectious Hospitalized and Recovered (SEIHR), spread model. In the model studied, we introduce a recruitment constant, to take into account the fact that newborns can transmit disease. The disease-free and endemic equilibrium points are computed and analyzed. The basic reproduction number $\mathcal{R}_0$ is acquired, when $\mathcal{R}_0 \leq 1$, the disease dies out and persists in the community whenever $\mathcal{R}_0 > 1$. From numerical simulation, we illustrate our theoretical analysis.

Keywords: Compartmental modeling, recruitment, infectious disease, reproduction number, equilibria, stability analysis, numerical simulation.

MSC: 34E05, 34D05, 65L20.

1 Introduction

Infectious diseases are a public health problem for populations around the world. For a better understanding of the dynamics of these infectious diseases, several tools are used among which we have the mathematical modeling [3]. Mathematical modeling is a decision support tool used in several fields such as economics, biology, physics and medicine. We have two types of transmission in terms of infectious diseases. The diseases with horizontal transmission whose infection requires the presence of a intermediate host generally a vector and a definitive host which is the human in general and the diseases with vertical transmission of which the infection is inter-human. In this work, we are interested in the study of a continuous model of a vertically transmitted disease, a Susceptible Exposed Infectious Hospitalized and Recovered (SEIHR), spread model. In [12, 21, 22], the authors formulated and studied mathematical models giving the dynamics of the transmission of infectious diseases. They study the stability of steady states when the basic reproduction rate $\mathcal{R}_0$ is less than one and greater than one. Also, they study the impact of quarantine on the dynamics of infectious disease transmission (this method is extensively applied to the outbreak of Corona Virus Diseases 2019 (COVID-19)).

Corona Virus Diseases 2019 (COVID-19) is an infectious disease with vertical transmission, the first case of which appeared towards the end of 2019 in Wuhan, China [13, 20, 25]. There exist a wide variety of models that can be used to describe the evolution of an epidemic. Main standard is held by the so called compartmental models, i.e., the family of *SIR* based models (*SIR*, *SEIR*, *SIRS*, etc)[9, 10]. These models consist of a set of differential equations that take into account transitions between different compartments. The first classical *SIR* model was proposed by Kermack and McKendrick [11]. Due to the recent coronavirus pandemic (COVID-19), most of the scientific community is dedicated to study its behaviour, both by using *SIR* based models and introducing new ones. In [1], a stochastic term is introduced in the system of differential equa-

*Corresponding Author: Harouna Ouedraogo: Université Joseph KI-ZERBO, 03 bp 7021, Ouagadougou 03, Burkina Faso, E-mail: oharounao@gmail.com

Dramane Ouedraogo: Université Nazi Boni, 01 bp 1091, Bobo-Dioulasso 01, Burkina Faso, E-mail: dramaneouedraogo268@yahoo.ca

Idrissa Ibrango: Université Nazi Boni, 01 bp 1091, Bobo-Dioulasso 01, Burkina Faso, E-mail: ibrango2006@yahoo.fr

Aboudramane Guiro: Université Nazi Boni, 01 bp 1091, Bobo-Dioulasso 01, Burkina Faso, E-mail: abouguiro@yahoo.fr

tions to simulate noise in the detection process. D. K. Mamo [18], proposes an ordinary differential equation system of the *SHEIQRD* (Susceptible-Stay-at-Home-Exposed-Infected-Quarantine-Recovered-Death) type to describe the transmission of Corona Virus Disease 2019 (COVID-19). The author calculates the basic reproduction rate $\mathcal{R}_0$ associated with the model and shows that when $\mathcal{R}_0 < 1$, the disease-free equilibrium is globally asymptotically stable and when $\mathcal{R}_0 > 1$ the endemic equilibrium is globally asymptotically stable. In this paper, we formulate and study a mathematical model of the *SEIHR* type of the transmission of the infectious disease. We propose a study of the overall stability of equilibria states when the basic reproduction number associated to the mathematical model is less than 1 or greater than 1.

The paper is organized as it follows:

In the section 2, we are interested in the formulation of the mathematical model. We give a description of the model, the equilibria states of the model as well as the basic reproduction number associated to the model. The section 3 is devoted to the study of the stability of the model. We first of all look at the positivity as well as the boundedness of the solutions and we study the global stability according to the value of the basic reproduction number. In the section 4, we propose a numerical simulation to illustrate our theoretical results. We end with a conclusion and some research perspectives.

2 Mathematical model

In this section, we give in subsection 2.1, the description and the formulation of mathematical model which will be investigate. The subsection 2.2 is devoted to the determination of the equilibria states and the basic reproduction number.

2.1 Description of model

According to the known characteristics of infectious diseases, we assume that each person is in one of the following compartments:

- Susceptible (denoted by S): The person is not infected by the disease pathogen.
- Exposed (denoted by E): The person is in the incubation period after being infected by the disease pathogen and has no visible clinical signs. The individual could infect other people but with a lower probability than people in the infectious compartments.
- Infectious (denoted by I), this compartment will be divided in two compartments which was: Infectious that will be detected (denoted by I_d), and the Infectious that will not be detected (denoted by I_u). The infectious that will be detected starts developing clinical signs and will be detected and reported by the authorities. After this period, people in this compartment are taken in charge by sanitary authorities and we classify them as Hospitalized. The infectious that will not be detected can infect other people and may start developing clinical signs but will not be detected and reported by authorities. After this period, people in this compartment pass to the Recovered state (the person who survive are in the compartment denoted by R_{I_u}).
- Hospitalized or in quarantine at home (but detected and reported by the authorities) that will recover (denoted by H_R). At the end of this state, a person passes to the Recovered state (denoted by R_{I_d}).
- Hospitalized that will die (denoted by H_d): The person is hospitalized and still infect other people.

After the general description given above, we get the transfer diagram at follow.

The function $f(S, X)$ is given by:

$$f(S, X) = \frac{S}{N} (\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_R} H_R + \beta_{H_d} H_d),$$

with

$$X = (E, I, I_u, I_d, H_R, H_d).$$

Figure 1: Transfer diagram for the mathematical model.

By using the Figure 1, we obtain the dynamical system:

$$\begin{cases}
 \dot{S} = \mathcal{A} - \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - dS, \\
 \dot{E} = \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - (\alpha + d)E, \\
 \dot{I} = \alpha E - (\beta + d)I, \\
 \dot{I}_d = p_1 \beta I - (\theta + d)I_d, \\
 \dot{I}_u = (1 - p_1) \beta I - (\eta + d)I_u, \\
 \dot{H}_R = k \theta I_d - (\rho + d)H_R, \\
 \dot{H}_d = (1 - k) \theta I_d + (1 - \xi) \rho H_R - (v + d)H_d, \\
 \dot{R}_{I_d} = \xi \rho H_R + v H_d - dR_{I_d}, \\
 \dot{R}_{I_u} = \eta I_u - dR_{I_u}.
 \end{cases} \quad (1)$$

The parameters used in the System 1 are given in the Table 1.

Remark 2.1. In this paper, we propose a mathematical model describing the transmission dynamics of infectious diseases (such as coronavirus (COVID-19)), with constant recruitment. We note that the disease occurs in several waves and that, like all influenza, immunity is temporary.

2.2 Equilibrium and basic reproduction number

In this subsection, we determine the equilibria points and the basic reproduction number associated with System (1).

Table 1: Parameters of the model.

Parameter	Definition
$\beta_E, \beta_I, \beta_{I_u}, \beta_{I_d}, \beta_{H_R}, \beta_{H_d}$	the disease contact rates (day^{-1}) of a person in the corresponding states,
α	the transition rates (day^{-1}) from state E to states I
β	the percentage of infected people that is detected and no
p_1	Rate of infected people that is detected
p_2	Rate of infected people that is not detected
θ	transition rates from states I_d , to H_R and to H_d
ρ	the transition rates from states H_R to R
η	the transition rates from states I_u to R

As the other equations of the System (1) do not depend on compartments R_{I_u} and R_{I_d} , System (1) is equivalent the following system:

$$\begin{cases} \dot{S} = \mathcal{A} - \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - dS, \\ \dot{E} = \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - (\alpha + d)E, \\ \dot{I} = \alpha E - (\beta + d)I, \\ \dot{I}_d = p_1 \beta I - (\theta + d)I_d, \\ \dot{I}_u = (1 - p_1) \beta I - (\eta + d)I_u, \\ \dot{H}_R = k \theta I_d - (\rho + d)H_R, \\ \dot{H}_d = (1 - k) \theta I_d + (1 - \xi) \rho H_R - (v + d)H_d. \end{cases} \quad (2)$$

Let $\mathcal{E} = (S, E, I, I_d, I_u, H_R, H_d)$ be the equilibrium point of System (2). Then, System (2) can be rewritten as follows:

$$\begin{cases} \mathcal{A} - \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - dS = 0, \\ \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - (\alpha + d)E = 0, \\ \alpha E - (\beta + d)I = 0, \\ p_1 \beta I - (\theta + d)I_d = 0, \\ (1 - p_1) \beta I - (\eta + d)I_u = 0, \\ k \theta I_d - (\rho + d)H_R = 0, \\ (1 - k) \theta I_d + (1 - \xi) \rho H_R - (v + d)H_d = 0. \end{cases} \quad (3)$$

From the third equation of System (3), we have:

$$E = \frac{\beta + d}{\alpha} I. \quad (4)$$

By adding the first two equations of System (3) and the relation (4), we get:

$$S = \frac{\mathcal{A}}{d} - \frac{(\alpha + d)(\beta + d)}{\alpha d} I. \quad (5)$$

By using the others equations of System (3) we obtain:

$$I_d = \frac{p_1\beta}{\theta + d}I, \quad (6)$$

$$I_u = \frac{(1-p_1)\beta}{\eta + d}I, \quad (7)$$

$$H_R = \frac{k\theta p_1\beta}{(\rho + d)(\theta + d)}I, \quad (8)$$

$$H_d = \frac{(1-k)(\rho + d)\theta p_1\beta + (1-\xi)\rho k\theta p_1\beta}{(v + d)(\rho + d)(\theta + d)}I. \quad (9)$$

Let $\mathcal{E}_0$ and $\mathcal{E}^*$ be respectively the disease-free and endemic equilibrium points of System (2). The disease-free equilibrium correspond to the case where there is not infected individual. In this case, we have $I = 0$ and the disease-free equilibrium is given by:

$$\mathcal{E}_0 = \left(\frac{\mathcal{A}}{d}, 0, 0, 0, 0, 0, 0\right). \quad (10)$$

We design by I^* the infectious at endemic equilibrium point, so

$$\mathcal{E}^* = (S^*, E^*, I^*, I_d^*, I_u^*, H_R^*, H_d^*); \quad (11)$$

with

$$\begin{aligned} S^* &= \frac{\mathcal{A}}{d} - \frac{(\alpha + d)(\beta + d)}{\alpha d}I^*, \\ E^* &= \frac{\beta + d}{\alpha}I^*, \\ I_d^* &= \frac{p_1\beta}{\theta + d}I^*, \\ I_u^* &= \frac{(1-p_1)\beta}{\eta + d}I^*, \\ H_R^* &= \frac{k\theta p_1\beta}{(\rho + d)(\theta + d)}I^*, \\ H_d^* &= \frac{(1-k)(\rho + d)\theta p_1\beta + (1-\xi)\rho k\theta p_1\beta}{(v + d)(\rho + d)(\theta + d)}I^*. \end{aligned}$$

We also make this assumption:

$$\mathbf{H}: \text{for all } I_d, I_u, H_R, H_d \in \mathbb{R}_+, \begin{pmatrix} I_d \\ I_u \\ H_R \\ H_d \end{pmatrix} \leq \begin{pmatrix} I_d^* \\ I_u^* \\ H_R^* \\ H_d^* \end{pmatrix}.$$

Remark 2.2. In biological view point, hypothesis **H** tells us that at endemic equilibrium the number of severely hospitalized individuals is more than this number at any other time. As well as less severe hospitalized, infected detected and infected undetected. Technically, hypothesis **H** allows us to conclude that the derivative of the candidate Lyapunov function is negative and therefore the overall stability of the endemic equilibrium.

In this work, the basic reproduction number associated to System (2) is denoted by $\mathcal{R}_0$.

Proposition 2.1. The basic reproduction number $\mathcal{R}_0$ associated to the System (2) is define by:

$$\mathcal{R}_0 = \frac{\mathcal{A}}{dN} \left[\frac{\beta_E}{\alpha + d} + \frac{\beta_I \alpha}{(\alpha + d)(\beta + d)} + \frac{\beta_{I_d} \alpha \beta p_1}{(\alpha + d)(\beta + d)(\theta + d)} + \frac{\beta_{I_u} (1 - p_1) \alpha \beta}{(\alpha + d)(\beta + d)(\eta + d)} \right. \\ \left. + \frac{\beta_{H_R} k \alpha \theta}{(\alpha + d)(\beta + d)(\rho + d)} + \frac{\beta_{H_d} [\alpha k \theta (1 - \xi) \rho + \alpha \rho (1 - k)(\rho + d)]}{(\alpha + d)(\beta + d)(\rho + d)(v + d)} \right]. \quad (12)$$

Proof. For this proof we use the technique of Van den Driessche and Watmough [23]. In our model the infected classes correspond to states E, I, I_d, I_u, H_R and H_d . Thus, we can rewrite system (2) as

$$\dot{\chi} = \mathcal{F}(\chi) - \mathcal{V}(\chi),$$

where $\chi = (E, I, I_d, I_u, H_R, H_d)$; $\mathcal{F}$ is the rate of appearance of new infections in each class, and $\mathcal{V}$ is the rate of transfer of individuals out of (for positive values) or into (for negative values) compartment by all other means.

Hence, in our case, we have that

$$\mathcal{F}(\chi) = \begin{pmatrix} \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_d} I_d + \beta_{I_u} I_u + \beta_{H_R} H_R + \beta_{H_d} H_d) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and

$$\mathcal{V}(\chi) = \begin{pmatrix} -(\alpha + d)E \\ \alpha E - (\beta + d)I \\ p_1 \beta I - (\theta + d)I_d \\ (1 - p_1) \beta I - (\eta + d)I_u \\ k \theta I_d - (\rho + d)H_R \\ (1 - k) \theta I_d - (1 - \xi) \rho H_R - (v + d)H_d \end{pmatrix}.$$

The jacobian matrix of $\mathcal{F}$ and $\mathcal{V}$ at the disease-free equilibrium $\mathcal{E}_0$ are given by

$$F = \begin{pmatrix} \frac{\beta_E \mathcal{A}}{dN} & \frac{\beta_I \mathcal{A}}{dN} & \frac{\beta_{I_d} \mathcal{A}}{dN} & \frac{\beta_{I_u} \mathcal{A}}{dN} & \frac{\beta_{H_R} \mathcal{A}}{dN} & \frac{\beta_{H_d} \mathcal{A}}{dN} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} -(\alpha + d) & 0 & 0 & 0 & 0 & 0 \\ \alpha & -(\beta + d) & 0 & 0 & 0 & 0 \\ 0 & p_1\beta & -(\theta + d) & 0 & 0 & 0 \\ 0 & (1 - p_1)\beta & 0 & -(\eta + d) & 0 & 0 \\ 0 & k\theta & 0 & 0 & -(\rho + d) & 0 \\ 0 & (1 - k)\theta & 0 & 0 & (1 - \xi)\rho & -(\nu + d) \end{pmatrix},$$

respectively.

The matrix V is invertible (non-zero determinant) and its inverse V^{-1} is defined by:

$$V^{-1} = \begin{pmatrix} -\frac{1}{\alpha + d} & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{(\alpha + d)(\beta + d)} & -\frac{1}{\beta + d} & 0 & 0 & 0 & 0 \\ B_1 & -\frac{p_1\beta}{(\beta + d)(\theta + d)} & -\frac{1}{\theta + d} & 0 & 0 & 0 \\ B_2 & -\frac{p_1\beta}{(\beta + d)(\theta + d)} & 0 & -\frac{1}{\eta + d} & 0 & 0 \\ B_3 & -\frac{k\theta}{(\beta + d)(\rho + d)} & 0 & 0 & -\frac{1}{\rho + d} & 0 \\ B_4 & B_5 & 0 & 0 & -\frac{(1 - \xi)\rho}{(\rho + d)(\nu + d)} & -\frac{1}{\nu + d} \end{pmatrix}, \quad (13)$$

were

$$\begin{aligned} B_1 &= -\frac{\alpha\beta p_1}{(\alpha + d)(\beta + d)(\theta + d)}, \\ B_2 &= -\frac{(1 - p_1)\alpha\beta}{(\alpha + d)(\beta + d)(\eta + d)}, \\ B_3 &= -\frac{k\alpha\theta}{(\alpha + d)(\beta + d)(\rho + d)}, \\ B_4 &= -\frac{\alpha k\theta(1 - \xi)\rho + \alpha\theta(1 - k)(\rho + d)}{(\alpha + d)(\beta + d)(\rho + d)(\nu + d)}, \\ B_5 &= -\frac{k\theta(1 - \xi)\rho + (1 - k)(\rho + d)\theta}{(\beta + d)(\rho + d)(\nu + d)}. \end{aligned}$$

So the next-generation matrix $-FV^{-1}$ is given by:

$$-FV^{-1} = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{aligned}
 A_1 &= \frac{\mathcal{A}}{dN} \left[\frac{\beta_E}{\alpha + d} + \frac{\beta_I \alpha}{(\alpha + d)(\beta + d)} + \frac{\beta_{I_d} \alpha \beta p_1}{(\alpha + d)(\beta + d)(\theta + d)} + \frac{\beta_{I_u}(1 - p_1) \alpha \beta}{(\alpha + d)(\beta + d)(\eta + d)} \right. \\
 &\quad \left. + \frac{\beta_{H_R} k \alpha \theta}{(\alpha + d)(\beta + d)(\rho + d)} + \frac{\beta_{H_d} [\alpha k \theta (1 - \xi) \rho + \alpha \rho (1 - k)(\rho + d)]}{(\alpha + d)(\beta + d)(\rho + d)(v + d)} \right], \\
 A_2 &= \frac{\mathcal{A}}{dN} \left[\frac{\beta_I}{\beta + d} + \frac{\beta_{I_d} p_1 \beta}{(\beta + d)(\theta + d)} + \frac{\beta_{I_u}(1 - p_1) \beta}{(\beta + d)(\eta + d)} + \frac{\beta_{H_R} k \theta}{(\beta + d)(\rho + d)} \right. \\
 &\quad \left. + \frac{\beta_{H_d} [k \theta (1 - \xi) \rho + (1 - k)(\rho + d) \theta]}{(\beta + d)(\rho + d)(v + d)} \right], \\
 A_3 &= \frac{\beta_{I_d} \mathcal{A}}{(\theta + d) dN}, \\
 A_4 &= \frac{\beta_{I_u} \mathcal{A}}{(\eta + d) dN}, \\
 A_5 &= \frac{\mathcal{A}}{dN} \left[\frac{\beta_{H_R}}{\rho + d} + \frac{\beta_{H_d}(1 - \xi) \rho}{(\rho + d)(v + d)} \right], \\
 A_6 &= \frac{\beta_{H_d} \mathcal{A}}{(v + d) dN}.
 \end{aligned}$$

The basic reproduction ratio is given by $\mathcal{R}_0 = \rho(-FV^{-1})$, the spectral radius of next-generation matrix $-FV^{-1}$. In our case, we have that

$$\begin{aligned}
 \mathcal{R}_0 &= \frac{\mathcal{A}}{dN} \left[\frac{\beta_E}{\alpha + d} + \frac{\beta_I \alpha}{(\alpha + d)(\beta + d)} + \frac{\beta_{I_d} \alpha \beta p_1}{(\alpha + d)(\beta + d)(\theta + d)} + \frac{\beta_{I_u}(1 - p_1) \alpha \beta}{(\alpha + d)(\beta + d)(\eta + d)} \right. \\
 &\quad \left. + \frac{\beta_{H_R} k \alpha \theta}{(\alpha + d)(\beta + d)(\rho + d)} + \frac{\beta_{H_d} [\alpha k \theta (1 - \xi) \rho + \alpha \rho (1 - k)(\rho + d)]}{(\alpha + d)(\beta + d)(\rho + d)(v + d)} \right].
 \end{aligned}$$

□

Remark 2.3. The basic reproduction number $\mathcal{R}_0$ evaluate the average number of new infections generated by a single infected individual in a completely susceptible population.

3 Stability analysis of the model

In this section, we prove in subsection 3.1 the positivity and boundedness of the the solution of System (1) and in subsection 3.2 we prove the global stability of disease-free equilibrium $\mathcal{E}_0$, when $\mathcal{R}_0 < 1$ and the global stability of endemic equilibrium $\mathcal{E}^*$, when $\mathcal{R}_0 > 1$.

Based on the biological meaning of model (1), we only consider solutions starting at $t = 0$ from initial values satisfying

$$\begin{aligned}
 S(0) &> 0, E(0) \geq 0, I(0) \geq 0, I_d(0) \geq 0, I_u(0) \geq 0, H_R(0) \geq 0, \\
 H_d(0) &\geq 0, R_{I_d}(0) \geq 0 \text{ and } R_{I_u}(0) \geq 0.
 \end{aligned} \tag{14}$$

3.1 Positivity and boundness of solution

In this subsection we prove the positivity and boundedness of the solutions of system (1) with the initial conditions (14).

Lemma 3.1. [26] Suppose $\Omega \subset \mathbb{R} \times \mathbb{C}^n$ is open, $f_i \in C(\Omega, \mathbb{R})$, $i = 1, 2, \dots, n$.

If $f_i|_{x_i(t)=0, X_t \in \mathbb{C}^n} \geq 0$, $X_t = (x_1, x_2, \dots, x_n)^T$, $i = 1, 2, \dots, n$, then $\mathbb{C}_{+0}^n = \{\phi = (\phi_1, \dots, \phi_n) : \phi \in \mathbb{C}([-\tau, 0], \mathbb{R}_{+0}^n)\}$ is the invariant domain of the following equations

$$\frac{dx_i(t)}{dt} = f_i(t, X_t), t \geq \sigma, i = 1, 2, \dots, n. \quad (15)$$

Where $\mathbb{R}_{+0}^n = \{(x_1, \dots, x_n) : x_i \geq 0, i = 1, 2, \dots, n\}$.

Proposition 3.1. Let $(S, E, I, I_d, I_u, H_R, H_d, R_{I_d}, R_{I_u}) \in \mathbb{R}^9$ be the solution of System (1) with initial conditions (14). Then, $(S(t) > 0, E(t) > 0, I(t) > 0, I_d(t) > 0, I_u(t) > 0, H_R(t) > 0, H_d(t) > 0, R_{I_d}(t), R_{I_u}(t) > 0)$ is non-negative for all $t \geq 0$.

Proof. The system (1) can be rewrite as follows:

$$\frac{d\mathcal{X}}{dt} = f(\mathcal{X}(t)), \mathcal{X}(0) = \mathcal{X}_0 \geq 0, \quad (16)$$

where $f(\mathcal{X}(t)) = (f_1(\mathcal{X}), \dots, f_9(\mathcal{X}))^T$.

We note that

$$\begin{aligned} \frac{dS}{dt}|_{S=0} &= \mathcal{A} \geq 0, \quad \frac{dE}{dt}|_{E=0} = \frac{S}{N}(\beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_R} H_R + \beta_{H_d} H_d) \geq 0 \\ \frac{dI}{dt}|_{I=0} &= \alpha E \geq 0, \quad \frac{dI_d}{dt}|_{I_d=0} = p_1 \beta I \geq 0, \quad \frac{dI_u}{dt}|_{I_u=0} = (1 - p_1) \beta I \geq 0 \\ \frac{dH_R}{dt}|_{H_R=0} &= k \theta I_d \geq 0, \quad \frac{dH_R}{dt}|_{H_R=0} = (1 - k) \theta I_d + (1 - \xi) \rho H_R \geq 0 \\ \frac{dR_{I_d}}{dt}|_{R_{I_d}=0} &= \xi \rho H_R + \nu H_d \geq 0, \quad \frac{dR_{I_u}}{dt}|_{R_{I_u}=0} = \eta I_u \geq 0. \end{aligned}$$

Then it follows from the Lemma 3.1 that $\mathbb{R}_+^9$ is an invariant set. $\square$

Proposition 3.2. Assume that $\mathcal{W} = S + E + I + I_d + I_u + H_R + H_d + R_{I_d} + R_{I_u}$. Then all solution of system (1) enter the set

$$\tilde{\Gamma} = \left\{ (S, E, I, I_d, I_u, H_R, H_d, R_{I_d}, R_{I_u}) \in \mathbb{R}^9 : \mathcal{W} \leq \frac{\mathcal{A}}{d} \right\}. \quad (17)$$

Proof. By adding the equations of system (1), we obtain

$$\begin{aligned} \dot{S} + \dot{E} + \dot{I} + \dot{I_d} + \dot{I_u} + \dot{H_R} + \dot{H_d} + \dot{R_{I_d}} + \dot{R_{I_u}} &= \mathcal{A} - d(S + E + I + I_d + I_u + H_R \\ &\quad + H_d + R_{I_d} + R_{I_u}) \\ \dot{\mathcal{W}} &= \mathcal{A} - d\mathcal{W} \\ \dot{\mathcal{W}} + d\mathcal{W} &= \mathcal{A}. \end{aligned} \quad (18)$$

According to [4], from (18) we have:

$$\mathcal{W}(t) = \frac{\mathcal{A}}{d} + \left(\mathcal{W}(0) - \frac{\mathcal{A}}{d} \right) e^{-dt},$$

where $\mathcal{W}(0) = S(0) + E(0) + I(0) + I_d(0) + I_u(0) + H_R(0) + H_d(0) + R_{I_d}(0) + R_{I_u}(0)$.

Thus, as $t \rightarrow +\infty$, $\mathcal{W}(t) \leq \frac{\mathcal{A}}{d}$. $\square$

3.2 Global stability of the equilibria

This subsection is devoted to the studied of global stability of disease-free equilibrium $\mathcal{E}_0$, when $\mathcal{R}_0 < 1$ and the global stability of endemic equilibrium $\mathcal{E}^*$, if $\mathcal{R}_0 > 1$.

Theorem 3.1. *The disease-free equilibrium $\mathcal{E}_0$ of system (2) is globally asymptotically stable, when $\mathcal{R}_0 < 1$.*

Proof. Let us consider the infected classes E, I, I_d, I_u, H_R and H_d . By the equations corresponding to these states, we have the linearization system at $\mathcal{E}_0$ given by:

$$\begin{cases} \dot{E} = \left(\frac{\beta_E \mathcal{A}}{dN} - (\alpha + d) \right) E + \frac{\beta_I \mathcal{A}}{dN} I + \frac{\beta_{I_d} \mathcal{A}}{dN} I_d + \frac{\beta_{I_u} \mathcal{A}}{dN} I_u + \frac{\beta_{H_R} \mathcal{A}}{dN} H_R + \frac{\beta_{H_d} \mathcal{A}}{dN} H_d, \\ \dot{I} = \alpha E - (\beta + d) I, \\ \dot{I}_d = p_1 \beta I - (\theta + d) I_d, \\ \dot{I}_u = (1 - p_1) \beta I - (\eta + d) I_u, \\ \dot{H}_R = k \theta I_d - (\rho + d) H_R, \\ \dot{H}_d = (1 - k) \theta I_d + (1 - \xi) \rho H_R - (\nu + d) H_d \end{cases} \quad (19)$$

The matrix $\mathcal{M}_1$ associate to the linearised system (19) is given by:

$$\mathcal{M}_1 = \begin{pmatrix} \frac{\beta_E \mathcal{A}}{dN} - (\alpha + d) & \frac{\beta_I \mathcal{A}}{dN} & \frac{\beta_{I_d} \mathcal{A}}{dN} & \frac{\beta_{I_u} \mathcal{A}}{dN} & \frac{\beta_{H_R} \mathcal{A}}{dN} & \frac{\beta_{H_d} \mathcal{A}}{dN} \\ \alpha & -(\beta + d) & 0 & 0 & 0 & 0 \\ 0 & p_1 \beta & -(\theta + d) & 0 & 0 & 0 \\ 0 & (1 - p_1) \beta & 0 & -(\eta + d) & 0 & 0 \\ 0 & 0 & k \theta & 0 & -(\rho + d) & 0 \\ 0 & 0 & (1 - k) \theta & 0 & (1 - \xi) \rho & -(\nu + d) \end{pmatrix}, \quad (20)$$

and the linearization system (19) can be rewrite at follows

$$\dot{\mathcal{Y}} \leq \mathcal{M}_1 \mathcal{Y}, \quad (21)$$

where $\mathcal{Y} = (E, I, I_d, I_u, H_R, H_d)^t$.

Let $\mathcal{M}_1 = F_1 + V_1$, with :

$$F_1 = \begin{pmatrix} \frac{\beta_E \mathcal{A}}{dN} & \frac{\beta_I \mathcal{A}}{dN} & \frac{\beta_{I_d} \mathcal{A}}{dN} & \frac{\beta_{I_u} \mathcal{A}}{dN} & \frac{\beta_{H_R} \mathcal{A}}{dN} & \frac{\beta_{H_d} \mathcal{A}}{dN} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V_1 = \begin{pmatrix} -(\alpha + d) & 0 & 0 & 0 & 0 & 0 \\ \alpha & -(\beta + d) & 0 & 0 & 0 & 0 \\ 0 & p_1 \beta & -(\theta + d) & 0 & 0 & 0 \\ 0 & (1 - p_1) \beta & 0 & -(\eta + d) & 0 & 0 \\ 0 & k \theta & 0 & 0 & -(\rho + d) & 0 \\ 0 & (1 - k) \theta & 0 & 0 & (1 - \xi) \rho & -(\nu + d) \end{pmatrix},$$

The matrix V_1 is an invertible matrix and it invert V_1^{-1} is given by the relation (13). We can also see that $F_1 \geq 0$ and $V_1^{-1} \geq 0$.

Thus, $\mathcal{R}_0 = \rho(-F_1 V_1^{-1}) < 1$ and from the theorem of Varga (see [24]) the matrix $\mathcal{M}_1$ is asymptotically stable. The eigenvalue of matrix $\mathcal{M}_1$ has negative real part, by a standard comparison theorem [14], when $t \rightarrow +\infty$, $E \rightarrow 0$, $I \rightarrow 0$, $I_d \rightarrow 0$, $I_u \rightarrow 0$, $H_R \rightarrow 0$ and $H_d \rightarrow 0$ for system (19) and substituting $E = 0$, $I = 0$, $I_d = 0$, $I_u = 0$, $H_R = 0$ and $H_d = 0$ in (2) we get $S \rightarrow \frac{\mathcal{A}}{d}$ as well as $t \rightarrow +\infty$.

Thus, $(E, I, I_d, I_u, H_R, H_d) \rightarrow (\frac{\mathcal{A}}{d}, 0, 0, 0, 0, 0)$ as $t \rightarrow +\infty$ for system (2), when $\mathcal{R}_0 < 1$. Therefore disease-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable in the positively set $\tilde{T}$ when $\mathcal{R}_0 < 1$. $\square$

Theorem 3.2. *The endemic equilibrium $\mathcal{E}^*$ of system (2) is globally asymptotically stable, when $\mathcal{R}_0 > 1$.*

Proof. Let $\mathcal{E}^* = (S^*, E^*, I^*, I_d^*, I_u^*, H_R^*, H_d^*)$ be the endemic equilibrium of system (2). From system (2), we have:

$$\begin{cases} \mathcal{A} = \frac{S^*}{N}(\beta_E E^* + \beta_I I^* + \beta_{I_u} I_u^* + \beta_{I_d} I_d^* + \beta_{H_d} H_d^* + \beta_{H_R} H_R^*) + dS^*, \\ \frac{S^*}{N}(\beta_E E^* + \beta_I I^* + \beta_{I_u} I_u^* + \beta_{I_d} I_d^* + \beta_{H_d} H_d^* + \beta_{H_R} H_R^*) = (\alpha + d)E^*, \\ \alpha E^* = (\beta + d)I^*, \\ p_1 \beta I^* = (\theta + d)I_d^*, \\ (1 - p_1) \beta I^* = (\eta + d)I_u^*, \\ k \theta I_d^* = (\rho + d)H_R^*, \\ (1 - k) \theta I_d^* + (1 - \xi) \rho H_R^* = (\nu + d)H_d^*. \end{cases} \quad (22)$$

Let define the function ψ on $\mathbb{R}_+^*$ by

$$\psi(x) = x - 1 - \ln x. \quad (23)$$

The function ψ is non-negative for all $x \in \mathbb{R}_+^*$. Let us consider the Lyapunov candidate function define by:

$$V = V_S + V_E + V_I + V_{I_d} + V_{I_u} + V_{H_R} + V_{H_d},$$

where

$$\begin{aligned} V_S &= S^* \psi\left(\frac{S}{S^*}\right), \quad V_E = E^* \psi\left(\frac{E}{E^*}\right), \\ V_I &= I^* \psi\left(\frac{I}{I^*}\right), \quad V_{I_d} = I_d^* \psi\left(\frac{I_d}{I_d^*}\right), \\ V_{I_u} &= I_u^* \psi\left(\frac{I_u}{I_u^*}\right), \quad V_{H_R} = H_R^* \psi\left(\frac{H_R}{H_R^*}\right), \\ V_{H_d} &= H_d^* \psi\left(\frac{H_d}{H_d^*}\right). \end{aligned}$$

Now, we have to differentiate the function V with respect to the time.

Let us compute $\dot{V}_S$:

$$\begin{aligned} \dot{V}_S &= \left(1 - \frac{S^*}{S}\right) \dot{S} \\ &= \left(1 - \frac{S^*}{S}\right) \left(\mathcal{A} - \frac{S}{N}(\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_R} H_R + \beta_{H_d} H_d) - dS\right). \end{aligned}$$

By using the first equation of (22), we have:

$$\begin{aligned}
\dot{V}_S &= -d \frac{(S-S^*)^2}{S} + \frac{\beta_E S^* E^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SE}{S^* E^*}\right) \right] + \frac{\beta_I S^* I^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SI}{S^* I^*}\right) \right] \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SI_u}{S^* I_u^*}\right) \right] + \frac{\beta_{I_d} S^* I_d^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SI_d}{S^* I_d^*}\right) \right] \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SH_R}{S^* H_R^*}\right) \right] + \frac{\beta_{H_d} S^* H_d^*}{N} \left[\left(1 - \frac{S^*}{S}\right) \left(1 - \frac{SH_d}{S^* H_d^*}\right) \right], \\
&= -d \frac{(S-S^*)^2}{S} + \frac{\beta_E S^* E^*}{N} \left(1 - \frac{SE}{S^* E^*} - \frac{S^*}{S} + \frac{E}{E^*}\right) + \frac{\beta_I S^* I^*}{N} \left(1 - \frac{SI}{S^* I^*} - \frac{S^*}{S} + \frac{I}{I^*}\right) \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left(1 - \frac{SI_u}{S^* I_u^*} - \frac{S^*}{S} + \frac{I_u}{I_u^*}\right) + \frac{\beta_{I_d} S^* I_d^*}{N} \left(1 - \frac{SI_d}{S^* I_d^*} - \frac{S^*}{S} + \frac{I_d}{I_d^*}\right) \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left(1 - \frac{SH_R}{S^* H_R^*} - \frac{S^*}{S} + \frac{H_R}{H_R^*}\right) + \frac{\beta_{H_d} S^* H_d^*}{N} \left(1 - \frac{SH_d}{S^* H_d^*} - \frac{S^*}{S} + \frac{H_d}{H_d^*}\right),
\end{aligned}$$

$$\begin{aligned}
\dot{V}_S &= -d \frac{(S-S^*)^2}{S} + \frac{\beta_E S^* E^*}{N} \left(-\frac{SE}{S^* E^*} + 1 + \ln \frac{SE}{S^* E^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{E}{E^*} - 1 - \ln \frac{E}{E^*} \right) \\
&\quad + \frac{\beta_I S^* I^*}{N} \left(-\frac{SI}{S^* I^*} + 1 + \ln \frac{SI}{S^* I^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{I}{I^*} - 1 - \ln \frac{I}{I^*} \right) \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left(-\frac{SI_u}{S^* I_u^*} + 1 + \ln \frac{SI_u}{S^* I_u^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{I_u}{I_u^*} - 1 - \ln \frac{I_u}{I_u^*} \right) \\
&\quad + \frac{\beta_{I_d} S^* I_d^*}{N} \left(-\frac{SI_d}{S^* I_d^*} + 1 + \ln \frac{SI_d}{S^* I_d^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{I_d}{I_d^*} - 1 - \ln \frac{I_d}{I_d^*} \right) \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left(-\frac{SH_R}{S^* H_R^*} + 1 + \ln \frac{SH_R}{S^* H_R^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{H_R}{H_R^*} - 1 - \ln \frac{H_R}{H_R^*} \right) \\
&\quad + \frac{\beta_{H_d} S^* H_d^*}{N} \left(-\frac{SH_d}{S^* H_d^*} + 1 + \ln \frac{SH_d}{S^* H_d^*} - \frac{S^*}{S} + 1 + \ln \frac{S^*}{S} + \frac{H_d}{H_d^*} - 1 - \ln \frac{H_d}{H_d^*} \right),
\end{aligned}$$

$$\begin{aligned}
\dot{V}_S &= -d \frac{(S-S^*)^2}{S} + \frac{\beta_E S^* E^*}{N} \left[-\psi\left(\frac{SE}{S^* E^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_I S^* I^*}{N} \left[-\psi\left(\frac{SI}{S^* I^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I}{I^*}\right) \right] \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left[-\psi\left(\frac{SI_u}{S^* I_u^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I_u}{I_u^*}\right) \right] \\
&\quad + \frac{\beta_{I_d} S^* I_d^*}{N} \left[-\psi\left(\frac{SI_d}{S^* I_d^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I_d}{I_d^*}\right) \right] \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left[-\psi\left(\frac{SH_R}{S^* H_R^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{H_R}{H_R^*}\right) \right] \\
&\quad + \frac{\beta_{H_d} S^* H_d^*}{N} \left[-\psi\left(\frac{SH_d}{S^* H_d^*}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{H_d}{H_d^*}\right) \right]. \tag{24}
\end{aligned}$$

Let us compute $\dot{V}_E$:

$$\begin{aligned}
\dot{V}_E &= \left(1 - \frac{E^*}{E}\right) \dot{E} \\
&= \left(1 - \frac{E^*}{E}\right) \left[\frac{S}{N} (\beta_E E + \beta_I I + \beta_{I_u} I_u + \beta_{I_d} I_d + \beta_{H_d} H_d + \beta_{H_R} H_R) - (\alpha + d)E \right].
\end{aligned}$$

From the second equation of system (22), we get:

$$\begin{aligned}
\dot{V}_E &= \frac{\beta_E S^* E^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SE}{S^* E^*} - \frac{E}{E^*}\right) + \frac{\beta_I S^* I^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SI}{S^* I^*} - \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SI_u}{S^* I_u^*} - \frac{E}{E^*}\right) + \frac{\beta_{I_d} S^* I_d^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SI_d}{S^* I_d^*} - \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SH_R}{S^* H_R^*} - \frac{E}{E^*}\right) + \frac{\beta_{H_d} S^* H_d^*}{N} \left(1 - \frac{E^*}{E}\right) \left(\frac{SH_d}{S^* H_d^*} - \frac{E}{E^*}\right), \\
\dot{V}_E &= \frac{\beta_E S^* E^*}{N} \left(\frac{SE}{S^* E^*} - 1 - \ln \frac{SE}{S^* E^*} - \frac{S}{S^*} + 1 \ln \frac{S}{S^*} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_I S^* I^*}{N} \left(\frac{SI}{S^* I^*} - 1 - \ln \frac{SI}{S^* I^*} - \frac{SIE^*}{S^* I^* E} + 1 \ln \frac{SIE^*}{S^* I^* E} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left(\frac{SI_u}{S^* I_u^*} - 1 - \ln \frac{SI_u}{S^* I_u^*} - \frac{SI_u E^*}{S^* I_u^* E} + 1 \ln \frac{SI_u E^*}{S^* I_u^* E} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{I_d} S^* I_d^*}{N} \left(\frac{SI_d}{S^* I_d^*} - 1 - \ln \frac{SI_d}{S^* I_d^*} - \frac{SI_d E^*}{S^* I_d^* E} + 1 \ln \frac{SI_d E^*}{S^* I_d^* E} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left(\frac{SH_R}{S^* H_R^*} - 1 - \ln \frac{SH_R}{S^* H_R^*} - \frac{SH_R E^*}{S^* H_R^* E} + 1 \ln \frac{SH_R E^*}{S^* H_R^* E} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right) \\
&\quad + \frac{\beta_{H_d} S^* H_d^*}{N} \left(\frac{SH_d}{S^* H_d^*} - 1 - \ln \frac{SH_d}{S^* H_d^*} - \frac{SH_d E^*}{S^* H_d^* E} + 1 \ln \frac{SH_d E^*}{S^* H_d^* E} - \frac{E}{E^*} + 1 + \ln \frac{E}{E^*}\right), \\
\dot{V}_E &= \frac{\beta_E S^* E^*}{N} \left[\psi\left(\frac{SE}{S^* E^*}\right) - \psi\left(\frac{S}{S^*}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_I S^* I^*}{N} \left[\psi\left(\frac{SI}{S^* I^*}\right) - \psi\left(\frac{SIE^*}{S^* I^* E}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left[\psi\left(\frac{SI_u}{S^* I_u^*}\right) - \psi\left(\frac{SI_u E^*}{S^* I_u^* E}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{I_d} S^* I_d^*}{N} \left[\psi\left(\frac{SI_d}{S^* I_d^*}\right) - \psi\left(\frac{SI_d E^*}{S^* I_d^* E}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left[\psi\left(\frac{SH_R}{S^* H_R^*}\right) - \psi\left(\frac{SH_R E^*}{S^* H_R^* E}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{H_d} S^* H_d^*}{N} \left[\psi\left(\frac{SH_d}{S^* H_d^*}\right) - \psi\left(\frac{SH_d E^*}{S^* H_d^* E}\right) - \psi\left(\frac{E}{E^*}\right) \right]. \tag{25}
\end{aligned}$$

By adding the equations (24) and (25), we obtain:

$$\begin{aligned}
\dot{V}_S + \dot{V}_E &= -d \frac{(S - S^*)^2}{S} + \frac{\beta_E S^* E^*}{N} \left[-\psi\left(\frac{S^*}{S}\right) - \psi\left(\frac{S}{S^*}\right) \right] \\
&\quad + \frac{\beta_I S^* I^*}{N} \left[-\psi\left(\frac{SIE^*}{S^* I^* E}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I}{I^*}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{I_u} S^* I_u^*}{N} \left[-\psi\left(\frac{SI_u E^*}{S^* I_u^* E}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I_u}{I_u^*}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{I_d} S^* I_d^*}{N} \left[-\psi\left(\frac{SI_d E^*}{S^* I_d^* E}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{I_d}{I_d^*}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{H_R} S^* H_R^*}{N} \left[-\psi\left(\frac{SH_R E^*}{S^* H_R^* E}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{H_R}{H_R^*}\right) - \psi\left(\frac{E}{E^*}\right) \right] \\
&\quad + \frac{\beta_{H_d} S^* H_d^*}{N} \left[-\psi\left(\frac{SH_d E^*}{S^* H_d^* E}\right) - \psi\left(\frac{S^*}{S}\right) + \psi\left(\frac{H_d}{H_d^*}\right) - \psi\left(\frac{E}{E^*}\right) \right]. \tag{26}
\end{aligned}$$

Let us compute $\dot{V}_I$:

$$\begin{aligned}\dot{V}_I &= \left(1 - \frac{I^*}{I}\right)\dot{I} \\ &= \left(1 - \frac{I^*}{I}\right)(\alpha E - (\beta + d)I).\end{aligned}$$

By using the third equation of (22), we get:

$$\begin{aligned}\dot{V}_I &= \alpha E^* \left(1 - \frac{I^*}{I}\right) \left(\frac{E}{E^*} - \frac{I}{I^*}\right), \\ &= \alpha E^* \left(\frac{E}{E^*} - 1 - \ln \frac{E}{E^*} - \frac{I}{I^*} + 1 + \ln \frac{I}{I^*} - \frac{EI^*}{E^*I} + 1 + \ln \frac{EI^*}{E^*I}\right), \\ &= \alpha E^* \left[\psi\left(\frac{E}{E^*}\right) - \psi\left(\frac{I}{I^*}\right) - \psi\left(\frac{EI^*}{E^*I}\right)\right].\end{aligned}\quad (27)$$

Let us compute $\dot{V}_{I_d}$:

$$\begin{aligned}\dot{V}_{I_d} &= \left(1 - \frac{I_d^*}{I_d}\right)\dot{I}_d, \\ &= \left(1 - \frac{I_d^*}{I_d}\right)(p_1\beta I - (\theta + d)I_d).\end{aligned}$$

By using the fourth equation of (22), we obtain:

$$\begin{aligned}\dot{V}_{I_d} &= p_1\beta I^* \left(1 - \frac{I_d^*}{I_d}\right) \left(\frac{I}{I^*} - \frac{I_d}{I_d^*}\right), \\ &= p_1\beta I^* \left(\frac{I}{I^*} - 1 - \ln \frac{I}{I^*} - \frac{I_d}{I_d^*} + 1 + \ln \frac{I_d}{I_d^*} - \frac{II_d^*}{I^*I_d} + 1 + \ln \frac{II_d^*}{I^*I_d}\right), \\ &= p_1\beta I^* \left[\psi\left(\frac{I}{I^*}\right) - \psi\left(\frac{I_d}{I_d^*}\right) - \psi\left(\frac{II_d^*}{I^*I_d}\right)\right].\end{aligned}\quad (28)$$

Let us compute $\dot{V}_{I_u}$:

$$\begin{aligned}\dot{V}_{I_u} &= \left(1 - \frac{I_u^*}{I_u}\right)\dot{I}_d, \\ &= \left(1 - \frac{I_u^*}{I_u}\right)((1 - p_1)\beta I - (\eta + d)I_d).\end{aligned}$$

By using the five equation of (22), we obtain:

$$\begin{aligned}\dot{V}_{I_u} &= (1 - p_1)\beta I^* \left(1 - \frac{I_u^*}{I_u}\right) \left(\frac{I}{I^*} - \frac{I_u}{I_u^*}\right), \\ &= (1 - p_1)\beta I^* \left(\frac{I}{I^*} - 1 - \ln \frac{I}{I^*} - \frac{I_u}{I_u^*} + 1 + \ln \frac{I_u}{I_u^*} - \frac{II_u^*}{I^*I_u} + 1 + \ln \frac{II_u^*}{I^*I_u}\right), \\ &= (1 - p_1)\beta I^* \left[\psi\left(\frac{I}{I^*}\right) - \psi\left(\frac{I_u}{I_u^*}\right) - \psi\left(\frac{II_u^*}{I^*I_u}\right)\right].\end{aligned}\quad (29)$$

Computation of $\dot{V}_{H_R}$:

$$\begin{aligned}\dot{V}_{H_R} &= \left(1 - \frac{H_R^*}{H_R}\right) \dot{H}_R, \\ &= \left(1 - \frac{H_R^*}{H_R}\right) (k\theta I_d - (\rho + d)H_R).\end{aligned}$$

From the six equation of (22), we obtain:

$$\begin{aligned}\dot{V}_{H_R} &= k\theta I_d^* \left(1 - \frac{H_R^*}{H_R}\right) \left(\frac{I_d}{I_d^*} - \frac{H_R}{H_R^*}\right), \\ &= k\theta I_d^* \left(\frac{I_d}{I_d^*} - 1 - \ln \frac{I_d}{I_d^*} - \frac{H_R}{H_R^*} + 1 + \ln \frac{H_R}{H_R^*} - \frac{I_d H_R^*}{I_d^* H_R} + 1 + \ln \frac{I_d H_R^*}{I_d^* H_R}\right), \\ &= k\theta I_d^* \left[\psi\left(\frac{I_d}{I_d^*}\right) - \psi\left(\frac{H_R}{H_R^*}\right) - \psi\left(\frac{I_d H_R^*}{I_d^* H_R}\right)\right].\end{aligned}\quad (30)$$

Computation of $\dot{V}_{H_d}$:

$$\begin{aligned}\dot{V}_{H_d} &= \left(1 - \frac{H_d^*}{H_d}\right) \dot{H}_d, \\ &= \left(1 - \frac{H_d^*}{H_d}\right) [(1-k)\theta I_d + (1-\xi)\rho H_R - (v+d)H_d].\end{aligned}$$

By using the last equation of (22), we get:

$$\begin{aligned}\dot{V}_{H_d} &= (1-k)\theta I_d^* \left(1 - \frac{H_d^*}{H_d}\right) \left(\frac{I_d}{I_d^*} - \frac{H_d}{H_d^*}\right) + (1-\xi)\rho H_R^* \left(1 - \frac{H_d^*}{H_d}\right) \left(\frac{H_R}{H_R^*} - \frac{H_d}{H_d^*}\right), \\ &= (1-k)\theta I_d^* \left(\frac{I_d}{I_d^*} - 1 - \ln \frac{I_d}{I_d^*} - \frac{H_d}{H_d^*} + 1 + \ln \frac{H_d}{H_d^*} - \frac{I_d H_d^*}{I_d^* H_d} + 1 + \ln \frac{I_d H_d^*}{I_d^* H_d}\right) \\ &\quad + (1-\xi)\rho H_R^* \left(\frac{H_R}{H_R^*} - 1 - \ln \frac{H_R}{H_R^*} - \frac{H_d}{H_d^*} + 1 + \ln \frac{H_d}{H_d^*} - \frac{H_R H_d^*}{H_R^* H_d} + 1 + \ln \frac{H_R H_d^*}{H_R^* H_d}\right), \\ &= (1-k)\theta I_d^* \left[\psi\left(\frac{I_d}{I_d^*}\right) - \psi\left(\frac{H_d}{H_d^*}\right) - \psi\left(\frac{I_d H_d^*}{I_d^* H_d}\right)\right] \\ &\quad + (1-\xi)\rho H_R^* \left[\psi\left(\frac{H_R}{H_R^*}\right) - \psi\left(\frac{H_d}{H_d^*}\right) - \psi\left(\frac{H_R H_d^*}{H_R^* H_d}\right)\right].\end{aligned}\quad (31)$$

$$\text{Let } \kappa_1 = \max \left\{ \frac{\beta_E S^* E^*}{N}; \frac{\beta_E S^* I^*}{N}; \frac{\beta_E S^* I_u^*}{N}; \frac{\beta_E S^* I_d^*}{N}; \frac{\beta_E S^* H_R^*}{N}; \frac{\beta_E S^* H_d^*}{N} \right\},$$

$$\kappa_2 = \max \left\{ \alpha E^*; p_1 \beta I^*; (1-p_1) \beta I^*; k\theta I_d^*; (1-k)\theta I_d^*; (1-\xi)\rho H_R^* \right\}$$

$$\text{and } \kappa = \max \left\{ \kappa_1; \kappa_2 \right\}.$$

From the relations (26), (27), (28), (29), (30), (31) and κ , we get:

$$\begin{aligned}\dot{V} &\leq -d \frac{(S-S^*)}{S} + \kappa \left[-\psi\left(\frac{S}{S^*}\right) - 6\psi\left(\frac{S^*}{S}\right) - 4\psi\left(\frac{E}{E^*}\right) - \psi\left(\frac{SIE^*}{S^* I^* E}\right) \right. \\ &\quad - \psi\left(\frac{S I_u E^*}{S^* I_u^* E}\right) - \psi\left(\frac{S I_d E^*}{S^* I_d^* E}\right) - \psi\left(\frac{S H_R E^*}{S^* H_R^* E}\right) - \psi\left(\frac{S H_d E^*}{S^* H_d^* E}\right) \\ &\quad - \psi\left(\frac{E I^*}{E^* I}\right) + 2\psi\left(\frac{I}{I^*}\right) - \psi\left(\frac{I_d^*}{I^* I_d}\right) - \psi\left(\frac{I_u^*}{I^* I_u}\right) + 2\psi\left(\frac{I_d}{I_d^*}\right) - \psi\left(\frac{H_R^* I_d}{H_R I_d^*}\right) \\ &\quad \left. - \psi\left(\frac{H_d}{H_d^*}\right) - \psi\left(\frac{I_d H_d^*}{I_d^* H_d}\right) + \psi\left(\frac{H_R}{H_R^*}\right) - \psi\left(\frac{H_R H_d^*}{H_R^* H_d}\right) \right].\end{aligned}$$

By using the assumption **H**, we obtain

$$\dot{V} \leq 0. \quad (32)$$

Also, we have $V > 0$ for all $S, E, I, I_d, I_u, H_R, H_d \in \mathbb{R}_+$ and $\dot{V} = 0$ for $S = S^*, E = E^*, I = I^*, I_d = I_d^*, I_u = I_u^*, H_R = H_R^*, H_d = H_d^*$. Then, by the asymptotic stability theorem [15], the endemic equilibrium $\mathcal{E}^*$ of System (2) is globally asymptotically stable. $\square$

4 Numerical result

In this section, our aim is to discuss numerically, the dynamic of the different compartment of system (1). We have to distinguish two, at first we illustrate numerically the dynamic of the different class when the basic reproduction number $\mathcal{R}_0$ is less than 1 and we also present the dynamic of the different class when the basic reproduction number $\mathcal{R}_0$ is more than 1.

Table 2: Parameters values of the model.

Symbols	Values for extinction	Source	Values for persistence	Source
$\mathcal{A}$	1000	estimated	1000000	estimated
β_E	0.2	[8, 16]	0.8	estimated
β_I	0.2850	[8, 16]	0.4	estimated
β_{I_d}	0.3373	[8, 16]	0.2	estimated
β_{I_u}	0.1222	[8, 16]	0.3	estimated
β_{H_R}	0.126	[8, 16]	0.6	estimated
β_{H_d}	0.126	[8, 16]	0.2	estimated
β	0.2	[8, 16]	0.03	estimated
α	0.1	[8]	0.4	estimated
θ	0.5	[8]	0.4	estimated
ρ	0.143	[8]	0.5	estimated
k	0.95	[8]	0.4	estimated
p	0.2	[8]	0.6	estimated
ξ	0.33	[8]	0.7	estimated
η	0.143	[8]	0.9	estimated
ν	0.05	[8]	0.1	estimated

Figure 2 gives the dynamics of susceptible and exposed humans, when the basic reproduction $\mathcal{R}_0 < 1$. The black and green curves, respectively give the dynamics of susceptible humans and exposed humans. We see the decrease in exposures and an increase in susceptible, which means the disease tends to disappear from the population.

Figure 3 shows us the dynamics of infectious humans, when $\mathcal{R}_0 < 1$. The black, green and blue curves respectively give the dynamics of infectious humans, detected infectious and undetected infectious. We see the decrease of the infectious class, which means that the disease is tending to disappear from the population.

Figures 4 and 5 respectively give the dynamics of hospitalized and recovered humans when $\mathcal{R}_0 < 1$. In the Figure 4, the black and green curves give respectively dynamic hospitalized H_R and the hospitalized H_d , in the figure 5 the black and green curves give respectively the dynamic of recovered R_d and recovered R_u . The decrease in the curves observed at the level of the different figures means that the disease tends to disappear from the population.

Figure 2: Dynamic of susceptible and exposed humans, when $\mathcal{R}_0 = 0.37$.

Figure 3: Dynamic of infectious humans, when $\mathcal{R}_0 = 0.37$.

Figure 6 gives the dynamics of susceptible and exposed humans, when the basic reproduction $\mathcal{R}_0 > 1$. The black and green curves, respectively give the dynamics of susceptible humans and exposed human. The state of the curves tells us that the disease remains persistent in the population.

Figure 7 shows us the dynamics of infectious humans, when $\mathcal{R}_0 > 1$. The black, green and blue curves respectively give the dynamics of infectious humans, detected infectious and undetected infectious. We see the increase in infectious class, which means that the disease remains persistent in the population.

Figures 8 and 9 respectively give the dynamics of hospitalized and recovered humans when $\mathcal{R}_0 > 1$. In the Figure 8, the black and green curves give respectively dynamic hospitalized H_R and the hospitalized H_d , in the figure 9 the black and green curves give respectively the dynamic of recovered R_d and recovered R_u . The increase in the curves observed at the level of the different figures means that the disease remains persistent in the population.

Figure 4: Dynamic of hospitalized humans, when $\mathcal{R}_0 = 0.37$.

Figure 5: Dynamic of recover humans, when $\mathcal{R}_0 = 0.37$.

5 Conclusion

In this paper, we have studied a continuous mathematical model which modeling an infectious disease. The mathematical model studied in this work is the *SEIHR* (Susceptible-Exposed-Infectious-Hospitalized-Recover) type. We have calculated the basic reproduction number ($\mathcal{R}_0$) associated with the model, the basic properties of the model (positivity of the solutions and their bounditude). We also prove the stability of the model according to the values of $\mathcal{R}_0$. We used the Varga theorem [24] to proved that the disease-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable, when $\mathcal{R}_0 < 1$. The technique of Lyapunov function is used to obtain the global stability of endemic equilibrium $\mathcal{E}^*$, when $\mathcal{R}_0 > 1$. In addition we illustrate our theoretical results by the numerical simulation. In the following we would like to propose a control system to which we will associate a problem of minimising the cost related to the use of the different candidate vaccines and the other preventive measures such as the wearing of masks, hand washing etc... To solve the control problem we use the Pontryagin maximum principle.

Figure 6: Dynamic of susceptible and exposed humans, when $\mathcal{R}_0 = 12.45$.

Figure 7: Dynamic of infectious humans, when $\mathcal{R}_0 = 12.45$.

Figure 8: Dynamic of hospitalized humans, when $\mathcal{R}_0 = 12.45$.

Availability of material: There is no use of material in this paper.

Figure 9: Dynamic of recover humans, when $\mathcal{R}_0 = 12.45$.

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References

- [1] I. Ali and S. K. Ullah, Analysis of stochastic delayed SIRS model with exponential birth and saturated incidence rate, *Chaos Solitons Fractal*, (2020).
- [2] H. Amann, *Ordinary differential equations: An introduction to nonlinear analysis*, Walter de Gruyter, Berlin. New York (1990).
- [3] R.M. Anderson and R.M. May, *Infectious Diseases of Humans. Dynamics and Control*, Oxford Science Publications, (1991).
- [4] G. Birkhoff and G. C. Rota, *Ordinary Differential Equations*: Ginn, Boston, (1982).
- [5] F. Brauer, *Mathematical epidemiology: Past, present, and future*, *Infectious Disease Modelling*, 2, (2017) pp.113–127.
- [6] O. Diekmann, J.A.P. Heesterbeek and J.A.J. Metz: On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations, *Journal of Mathematical Biology*, 28, (1990) pp.365–382.
- [7] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic Optimal control*, Springer, New York, NY, USA, (1975).
- [8] A. Guiro, B. Kone and S. Ouaro, *Mathematical Model of the Spread of the Coronavirus Disease 2019 (COVID-19) in Burkina Faso*, *Applied Mathematics*, 11, (2020), pp. 1204–1218.
- [9] A. Guiro, D. Ouedraogo, H. Ouedraogo, *Stability Analysis for a Discrete SIR Epidemic Model with Delay and General Nonlinear Incidence Function*, *Applied Mathematics*, 9 (2018), pp. 1039–1054.
- [10] B. Ivorra, D. Ngom and A. M. Ramos, *Be-CoDis: A Mathematical Model to Predict the Risk of Human Diseases Spread Between Countries Validation and Application to the 2014-2015 Ebola Virus Disease Epidemic*, *Bulletin of Mathematical Biology*, 17, 9 (2015) pp.1668–1704.
- [11] O. W. Kermack and G. A. McKendrick, *A contribution to the mathematical theory of epidemics*, *Proc. R. Soc. Lond. Ser. A. Contain Paper Math Phys Charact*, (1927).
- [12] R. Kiran, M. Roy, S. Abbas and A. Taraphder, *Effect of population migration and punctuated lockdown on the spread of infectious diseases*, *arXiv:2006.15010v2*, (2021).

- [13] C. C. Lai, et al. severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) and corona virus disease-2019 (COVID-19): the epidemic and the challenges, *Int. J. Antimicrob. Agents*, 55,3 (2020), pp.924–934.
- [14] V. Lakshmikantham, S. Leela and A. A. Martynyuk, *Stability Analysis of Nonlinear Systems*, Marcel Dekker, New York, (1989).
- [15] J. P. Lasalle, *The stability of Dynamical Systems*, SIAM and Philadelphia, (1976).
- [16] T. Liu, J. Kang, L. Lin, H. Zhong and J. Xiao, *Transmission Dynamics of 2019 Novel Coronavirus (2019-nCoV)*, (2020).
- [17] D. L. Lukes, *Differential Equations: Classical To Controlled*, Mathematics in Science and Engineering, Academic Press, New York, NY, USA, 162 (1982).
- [18] D. K. Mamo, *Model the transmission dynamics of COVID-19 propagation with public health intervention*, *Applied Mathematics*, (2020).
- [19] R. Ross, *The Prevention of Malaria*, John Murray, (1911).
- [20] B. Tang, et al. Estimation of the transmission risk of the 2019-nCoV and its implication for public health interventions, *J. Clin. Med.*, 9, 2, (2020), pp.462–474.
- [21] S. Tyagi, S. Gupta, S. Abbas, K. P. Das and B. Riadh, *Analysis of infectious disease transmission and prediction through SEIQR epidemic model*, *Nonauto. Dyn. Sys.*, (2021), no.1, 75–86.
- [22] S. Tyagi, S. C. Martha, S. Abbas and A. Debbouche, *Mathematical modeling and analysis for controlling the spread of infectious diseases*, *Chaos, Solitons and Fractals*, 144 (2021), Paper No.110707.
- [23] P. Van den Driessche, J. Watmough, *Reproduction numbers and subthreshold endemic equilibria for the compartmental models of disease transmission*, *Mathematical Biosciences*, 180 (2002) pp. 29–48.
- [24] R. Varga, *matrix iterative analysis*, Prentice-Hall, (1962).
- [25] J. T. Wu, K. Leung and G. M. Leung, *Nowcasting and forecasting the potential domestic and international spread of the 2019-nCoV outbreak originating in Wuhan, China: a modeling study*, *Lancet*, 395, (2020) pp.261–269.
- [26] Y. Xia, C. Lansun and C. Jufang, *Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models*, *Computers Mathematics with Applications*, 32, 4 (1996) pp.109–116.