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Resonant mixed fractional-order p -Laplacian boundary value problem on the half-line

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Abstract: This study aims at establishing the solvability of a fractional-order p -Laplacian boundary value problem involving both the left Caputo and right Riemann-Liouville fractional derivatives on the half-line. In order to overcome the nonlinearity of the fractional differential operator, we apply the Ge and Ren coincidence degree theorem to obtain existence results for the boundary value problem at resonance. An example is given to demonstrate the established results.

Keywords: Coincidence degree, half-line, fractional derivative, p -Laplacian, resonance.

MSC: 70K30; 34B10; 34B15; 34A08

1 Introduction

In this paper, we obtain existence results for the following fractional-order p -Laplacian boundary value problem at resonance on the half-line with nonlocal boundary conditions

$$D_-^a \phi_p(D_{0+}^b u(t)) = e^{-t} w(t, u(t), D_{0+}^b u(t)), \quad t \in (0, \infty), \quad (1)$$

$$I_{0+}^{1-b} u(0) = 0, \quad \phi_p(D_{0+}^b u(\infty)) = \phi_p(D_{0+}^b u(0)), \quad (2)$$

where D_-^a is the left Caputo fractional derivative on the half line and D_{0+}^b the right Riemann-Liouville fractional derivative on the half-line, $0 < a, b \leq 1$, $1 < a + b \leq 2$, $\phi_p(r) = |r|^{p-2}$, $p \neq 2$, with $\phi_q = \phi_p^{-1}$ and $1/q + 1/p = 1$. $w : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

Recently, fractional differential equations have become the focus of many researches mainly due to the progress made in the development of the theory of fractional calculus and due to its widespread application in engineering and science like in signal processing, viscoelasticity, bioengineering, fluid dynamics [16]. Fractional order derivatives have useful tools for solving integral and differential equations and can handle models more accurately than integer order derivatives [4]. Fractional order derivatives contain a memory term which enables it to describe the memory and hereditary properties of various processes and materials [18].

The presence of the p -Laplacian operator on a boundary value problem causes the fractional differential operator to become nonlinear. Boundary value problems involving p -Laplacian operator occur in combustion theory, nonlinear elasticity, population biology, glaciology, non-Newtonian mechanics, plasma physics and the study of drain flows; see [4, 13]. When $p = 2$ the fractional differential operator becomes linear.

Existence results have been obtained by different researchers for fractional order boundary value problems using different methods. When the corresponding homogeneous problem of the fractional differential operator has a trivial solution, fixed point methods are applied; see [1, 6, 7, 11, 12, 17]. For the case where

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the corresponding homogenous fractional differential equation has a non-trivial solution and the fractional differential operator is nonlinear as a result of the presence of the p -Laplacian operator, the Ge and Ren coincidence degree theorem [5] is used to establish existence results; see [9, 19–21]

The associated homogeneous problem of boundary value problem (1)-(2):

$$\begin{aligned} D_-^\alpha \phi_p(D_{0+}^b u(t)) &= 0, \quad t \in (0, \infty), \\ I_{0+}^{1-b} u(0) &= 0, \quad \phi_p(D_{0+}^b u(\infty)) = \phi_p(D_{0+}^b u(0)), \end{aligned}$$

has a non-trivial solution $u(t) = dt^b$, $d \in \mathbb{R}$, hence (1)-(2) is a resonance problem.

In [4], the authors considered the following fractional p -Laplacian boundary value problem at resonance when $p = 2$

$$D_{0+}^\beta \phi_p(D_{0+}^\alpha x(t)) = f(t, x(t), D_{0+}^\alpha x(t)), \quad t \in [0, 1]$$

subject to the boundary conditions

$$D_{0+}^\alpha x(0) = D_{0+}^\alpha x(1) = 0,$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, D_{0+}^α is a Caputo fractional derivative, and $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

Bai and Zhang [3] obtained existence of solution for the following three point boundary value problem of fractional differential equation with nonlinear growth at resonance by applying Coincidence degree

$$\begin{aligned} D_{0+}^\alpha u(t) &= f(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad 0 < t < 1, \\ u(0) &= 0, \quad u(1) = \sigma u(\eta), \end{aligned}$$

where $1 < \alpha \leq 2$, D_{0+}^α is the Riemann-Liouville derivative, $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, $\sigma \in (0, \infty)$, $\eta \in (0, 1)$ and the resonance condition is $\sigma\eta^{\alpha-1}$.

In [21], the authors studied the following fractional order boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) = f(t, u(t), D_{0+}^{\alpha-1} u(t)), & t \in (0, \infty), \\ u(0) = 0, \quad \lim_{t \rightarrow \infty} D_{0+} u(t) = \beta u(\eta), \end{cases}$$

where $0 < \alpha, b \leq 1$, $1 < \alpha + b \leq 2$, $p = 2$, f is a continuous function, D_{0+}^α and D_{0+}^b are Caputo fractional derivatives.

In [10], Imaga and Iyase considered a second-order p -Laplacian boundary value problem. at resonance:

$$\begin{aligned} (\phi_p(u'(t)))' + g(t, u(t), u'(t)) &= 0, \quad t \in (0, +\infty), \\ \phi_p(u'(0)) &= \int_0^{+\infty} v(t) \phi_p(u'(t)) dt, \quad \phi_p(u'(+\infty)) = \sum_{j=1}^m \beta_j \int_0^{\eta_j} \phi_p(u'(t)) dt, \end{aligned}$$

where $g : [0, +\infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is an L^1 -Caratheodory function, $0 < \eta_1 < \eta_2 < \dots < \eta_m < +\infty$, $\beta_j \in \mathbb{R}$, $j = 1, 2, \dots, m$, $v \in L^1[0, \infty)$, $v(t) > 0$ and $\phi_p(s) = |s|^{p-2}s$, $p \geq 2$. They used the Ge and Ren continuation theory to obtain existence results.

Also, in [20], the authors used the Ge and Ren continuation theorem to obtain existence result for the following fractional p -Laplacian problem at resonance

$$(\phi_p(D_{0+}^\alpha x(t)))' + f(t, x(t), D_{0+}^\alpha x(t)) = 0, \quad 0 < t < +\infty, \quad (3)$$

$$x(0) = x'(0) = 0, \quad \phi_p(D_{0+}^\alpha x(+\infty)) = \sum_{i=1}^n \alpha_i \phi_p(D_{0+}^\alpha x(\xi_i)), \quad (4)$$

where $1 < \alpha \leq 2$, D_{0+}^α is the standard Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \dots < \xi_n < +\infty$, $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$, $\phi_p^{-1} = \phi_q$ with $1/p + 1/q = 1$.

Motivated by the above results, we study the solvability for a resonant fractional-order p -Laplacian boundary value problem at resonance on the half-line. Though, some researchers have considered mixed

fractional-order boundary value problem; see [2, 8], to the best of our knowledge this is the first work to consider p -Laplacian mixed fractional boundary value problems on the half-line. In Section 2 of this work, required lemmas, theorem and definitions will be presented, Section 3 contains conditions for existence of solutions. An example will be given in Section 4 to illustrate the results obtained.

2 Preliminaries

In this section, we will give definitions, lemmas and theorems that will be used in this work.

Definition 2.1. ([5]) Let $(U, \|\cdot\|_U)$ and $(Z, \|\cdot\|_Z)$ be any two Banach spaces and $M : U \cap \text{dom } M \rightarrow Z$ a continuous operator. If $M(X \cap \text{dom } M)$ is a closed subset of Z and $\ker M = \{u \in U \cap \text{dom } M : Mu = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$, then, M is quasi-linear.

Let $U_1 = \ker M$ and U_2 be the complement of U_1 in U , such that $U = U_1 \oplus U_2$. Also, let $Z_1 \subset Z$ and Z_2 , the complement of Z_1 in Z such that $Z = Z_1 \oplus Z_2$. Similarly, let $E : U \rightarrow U$, $F : Z \rightarrow Z$ be continuous projectors and $\Omega \subset U$ be open and bounded with the origin $\theta \in \Omega$.

Definition 2.2. ([5]) Let $N_k : \overline{\Omega} \rightarrow Z$, $k \in [0, 1]$ be a continuous operator and let $\bigvee_k = \{u \in \overline{\Omega} : Mu = N_k u\}$. We denote N_1 by N . N_k is said to be M -compact in $\overline{\Omega}$ if there exists a vector subspace Z_1 of Z with $\dim Z_1 = \dim U_1$ and a continuous and compact operator, $P : \overline{\Omega} \times [0, 1] \rightarrow U_2$ such that, for $k \in [0, 1]$

- $(\rho_1) (I - F)N_k(\overline{\Omega}) \subset \text{Im } M \subset (I - F)Z$;
- $(\rho_2) FN_k u = \theta, k \in (0, 1) \Leftrightarrow FNu = \theta$;
- $(\rho_3) P(\cdot, k)$ is the zero operator and $P(\cdot, k)|_{\bigvee_k} = (I - E)|_{\bigvee_k}$;
- $(\rho_4) M[E + P(\cdot, k)] = (I - F)N_k$.

Theorem 2.1. ([5]). Let $(U, \|\cdot\|_U)$ and $(Z, \|\cdot\|_Z)$ be any two Banach spaces and $\Omega \subset U$ be bounded, open and nonempty. Assume that $M : U \cap \text{dom } M \rightarrow Z$ is a quasi-linear operator and $N_k : \overline{\Omega} \rightarrow Z$, $k \in [0, 1]$ is M -compact on $\overline{\Omega}$. Suppose the following hold:

- $(\tau_1) Mu \neq N_k u, \forall (u, k) \in \partial\Omega \times (0, 1)$,
- $(\tau_2) FNu \neq 0, \forall u \in \ker M \cap \partial\Omega$,
- $(\tau_3) \deg(JFN, \ker M \cap \Omega, 0) \neq 0$, where $F : Z \rightarrow \text{Im } F$ is a projector and $J : \text{Im } F \rightarrow \ker M$ is a homeomorphism with $J(\theta) = \theta$.

Then at least one solution exists for the abstract equation $Mu = Nu$ in $\text{dom } M \cap \overline{\Omega}$ where $N = N_1$.

Definition 2.3. [4]. The set $X \subset U$ defined by $X = \{u \in C[0, \infty), \lim_{t \rightarrow \infty} u(t) \text{ exist}\}$ is relatively compact if

$$X_1 = \left\{ \frac{u(t)}{1 + t^{a-1}} : u \in X \right\}, \quad X_2 = \{D_{0+}^a u(t) : u \in X\}$$

are uniformly bounded; equicontinuous on any compact subinterval of $[0, \infty)$ and equiconvergent at ∞ .

Definition 2.4. . Let $\alpha > 0$, the left-sided Caputo and right-sided Riemann-Liouville fractional integral of a function $x : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$I_-^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \frac{x(r)}{(r-t)^{1-\alpha}} dr, \quad t \in [0, \infty)$$

and

$$I_{0+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(r)}{(t-r)^{1-\alpha}} dr, \quad t \in [0, \infty)$$

respectively.

Definition 2.5. Let $\alpha > 0$, the left-sided Caputo and right-sided Riemann-Liouville fractional derivative of a function $x : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$D_-^\alpha x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^\infty \frac{x(r)}{(r-t)^{\alpha-n+1}} dr, \quad t \in [0, \infty),$$

and

$$D_{0+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{x(r)}{(t-r)^{\alpha-n+1}} dr, \quad t \in [0, \infty)$$

respectively where $n = [a] + 1$.

Definition 2.6. Let $a > 0$, then the Euler Gamma function is given as

$$\Gamma(a) = \int_0^\infty r^{a-1} e^{-r} dr.$$

Lemma 2.1. [14]. Let $a \in (0, \infty)$. The general solution of the Riemann-Liouville fractional differential equation

$$D_{0+}^a w(t) = 0$$

is $w(t) = b_1 t^{a-1} + b_2 t^{a-2} + \dots + b_n t^{a-n}$, where $b_j \in \mathbb{R}$, $j = 1, 2, \dots, n$ while, the general solution of the Caputo fractional differential equation

$$D_{0+}^a w(t) = 0$$

is $w(t) = d_0 + d_1 t + \dots + d_n t^n$, where $d_i \in \mathbb{R}$, $i = 0, 1, \dots, n$ and $n = [a] + 1$ is the smallest integer greater than or equal to a .

Lemma 2.2. [14]. Let $a \in (0, \infty)$ and $i = 1, 2, \dots, n$, $n = [a] + 1$ then

$$(I_{0+}^a D_{0+}^a w)(t) = (I_-^a D_-^a w)(t) w(t) + d_1 t^{a-1} + d_2 t^{a-2} + \dots + d_n t^{a-n}$$

holds almost everywhere on $[0, \infty)$ for some $d_i \in \mathbb{R}$. Similarly,

$$(I_{0+}^a D_{0+}^a w)(t) = (I_-^a D_-^a w)(t) w(t) + d_0 + d_1 t^1 + d_2 t^2 + \dots + d_n t^n$$

holds almost everywhere on $[0, \infty)$ for some $d_i \in \mathbb{R}$, $i = 0, 1, \dots, n$.

Lemma 2.3. [14] Let $a > 0$, $\rho > -1$, $t > 0$, then

$$I_{0+}^a t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1+a)} t^{a+\rho}, \quad D_{0+}^a t^\rho = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-a)} t^{a-\rho},$$

where $n = [a] + 1$.

Lemma 2.4. [15] Let $a > 0$. If $w(t) \in (0, \infty)$, then

$$D_{0+}^a I_{0+}^a w(t) = w(t), \quad D_-^a I_-^a w(t) = w(t).$$

Let

$$U = \left\{ C([0, +\infty), \mathbb{R}), \lim_{t \rightarrow \infty} \frac{|u(t)|}{1+t^{a-1}}, \text{ and } \lim_{t \rightarrow \infty} |D_{0+}^a u(t)| \text{ exists} \right\},$$

with norm $\|u\| = \max\{\|u\|_0, \|D_{0+}^a u(t)\|_\infty\}$ defined on U where

$$\|u\|_0 = \sup_{t \in [0, \infty)} \frac{|u(t)|}{1+t^{a-1}}, \quad \|u\|_\infty = \sup_{t \in [0, \infty)} |D_{0+}^a u|.$$

Let $Z = L^1[0, \infty)$ with the norm $\|z\| = \int_0^\infty |z(s)|ds$. Then, the spaces $(U, \|\cdot\|)$ and $(Z, \|\cdot\|)$ by standard argument are Banach Spaces.

We will define the operators $M : \text{dom } M \subset U \rightarrow Z$ as $Mu = e^t D_-^a(\phi_p(D_{0+}^b u))$ and $N_k : U \rightarrow Z$ as $= kw(t, u(t), D_{0+}^b u(t))$, $k \in [0, 1]$ where

$$\text{dom } M = \left\{ u \in U : D_-^a \phi_p(D_{0+}^b u(t)) \in AC[0, \infty), I_{0+}^{1-b} u(0) = 0, \right. \\ \left. \phi_p(D_{0+}^b u(\infty)) = \phi_p(D_{0+}^b u(0)) \right\}$$

then, (1) can be written as $Mu = N_k u$, $u \in \text{dom } M$.

Lemma 2.5. For $Mu = D_-^a(\phi_p(D_{0+}^b u))$,

$$\ker M = \{u \in \text{dom } M : u(t) = dt^b, d \in \mathbb{R}, t \in [0, \infty)\}, \quad (5)$$

$$\text{Im } M = \left\{ z \in Z : \int_0^\infty r^{a-1} z(r) dr = 0 \right\}, \quad (6)$$

and M is a quasi-linear operator.

Proof. Consider the homogeneous problem $D_-^a \phi_p(D_{0+}^b u(t)) = 0$, by Lemma 2.1 this becomes

$$u(t) = d_2 t^{b-1} + I_{0+}^b \phi_q(d_1) = d_2 t^{b-1} + \frac{\phi_q(d_1)}{\Gamma(b+1)} t^b, \quad d_1, d_2 \in \mathbb{R}. \quad (7)$$

Applying boundary condition $I_{0+}^{1-b} u(0) = 0$ gives (5). Conversely, if (5) holds for $u \in \text{dom } M$ and u be as defined in (7) then, $D_-^a \phi_p(D_{0+}^b u(t)) = 0$.

Let $z \in \text{Im } M$, then there exists $u \in \text{dom } M$ such that $D_-^a \phi_p(D_{0+}^b u(t)) = e^{-t} z(t)$. By Lemma 2.2,

$$\phi_p(D_{0+}^b u(t)) = I_-^a e^{-t} z(t) + d_1.$$

From the boundary condition, we have $I_{0+}^{1-a}(D_{0+}^b u(\infty)) = d_1$ while $\phi_p(D_{0+}^b u(0)) = I_-^a z(0) + d_1$, hence

$$\int_0^\infty r^{a-1} z(r) dr = 0. \quad (8)$$

Conversely, Let $z \in Z$ and satisfies (8). Set $u(t) = I_{0+}^b \phi_q(I_-^a e^{-t} z(t))$, then $u \in \text{dom } M$, and $Mu(t) = D_{0+}^a \phi_p(D_{0+}^b u(t)) = e^{-t} z(t)$. Therefore, $z \in \text{Im } M$, hence i.e. (6) is true. Since $\dim \ker M = 1 < \infty$ and $M(\text{dom } M \cap U)$ is a closed subset of Z , the M is a quasi-linear operator. $\square$

Lemma 2.6. The operators $E : U \rightarrow U_1$ and $F : Z \rightarrow Z_1$ defined as

$$Eu(t) = \frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b, \quad t \in [0, \infty),$$

and

$$Fz(t) = \frac{e^{-t} \int_0^\infty r^{a-1} z(r) dr}{\Gamma(a)}$$

respectively are projectors.

In addition, let $P : U \times [0, 1] \rightarrow U_2$ be defined as

$$P(u, k) = I_{0+}^b \left[\phi_q \left(I_-^a e^{-t} (I - F) N_k u(r) \right) - D_{0+}^b u(0) \right],$$

then, P is M -compact.

Proof. For all $u \in U$ and $t \in [0, \infty)$, we have

$$\begin{aligned} E^2 u(t) &= E(Eu(t)) \\ &= D_{0+}^b \left(\frac{t^b}{\Gamma(b+1)} \right) \frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b \\ &= \frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b \\ &= Eu(t). \end{aligned}$$

Hence, $E : U \rightarrow U_1$ is a projector. For all $z \in Z$ and $t \in [0, \infty)$, then

$$\begin{aligned} F^2 z(t) &= F(Fz(t)) \\ &= \frac{e^{-t} \int_0^\infty s^{a-1} Fz(r) dr}{\Gamma(a)} \\ &= \left(\frac{e^{-t} \int_0^\infty s^{a-1} z(r) dr}{\Gamma(a)} \right) \left(\frac{\int_0^\infty r^{a-1} e^{-r} dr}{\Gamma(a)} \right) \\ &= Fz(r). \end{aligned}$$

Hence, $F : Z \rightarrow Z_1$ is a projector. Let $\Omega \subset U$, then by the continuity of w we can conclude that $N_k u(t)$ and $FN_k u(t)$ are continuous for all $u \in \overline{\Omega}$, $t \in [0, \infty)$. Therefore, there exists $\tau > 0$ such that $I_-^a |e^{-t}| |(I-F)N_k u(t)| \leq \tau$, $|u| \leq \tau$ for all $u \in \overline{\Omega}$, $t \in [0, \infty)$.

Hence, for $u \in \overline{\Omega}$,

$$\begin{aligned} \|P(u, k)\|_0 &= \sup_{t \in [0, \infty)} \frac{|P(u, k)(t)|}{1 + t^{a-1}} \\ &= \sup_{t \in [0, \infty)} \frac{\left| I_{0+}^b \left[\phi_q \left(\frac{1}{\Gamma(a)} \int_s^\infty (s-r)^{a-1} e^{-r} (I-F)N_k u(r) dr \right) ds - D_{0+}^b u(0) \right] \right|}{1 + t^{a-1}} \\ &\leq \frac{\phi_q(\tau) + \tau}{\Gamma(b)} \sup_{t \in [0, \infty)} \int_0^t \frac{(t-s)^{a-1}}{1 + t^{a-1}} ds \\ &\leq \frac{\phi_q(\tau) + \tau}{\Gamma(b+1)} \leq \phi_q(\tau) + \tau, \end{aligned}$$

and

$$\begin{aligned} \|D_{0+}^a P(u, k)\|_\infty &= \sup_{t \in [0, \infty)} |D_{0+}^a P(u, k)u(t)| \\ &= \sup_{t \in [0, \infty)} \left| \phi_q \left(\frac{1}{\Gamma(a)} \int_t^\infty (t-r)^{a-1} e^{-r} (I-F)N_k u(r) dr \right) - D_{0+}^b u(0) \right| \\ &\leq \phi_q(\tau) + \tau. \end{aligned}$$

Since $\|P(u, k)\|_0 = \|D_{0+}^a P(u, k)\|_\infty \leq \phi_q(\tau) + \tau$, then, $P(u, k)\overline{\Omega}$ is uniformly bounded in U . We will next show the equicontinuity of $P(u, k)$ on any-compact interval of $[0, \infty)$. Let $H > 0$, $t_1, t_2 \in [0, H]$, with $t_1 < t_2$, $u \in \overline{\Omega}$ and $k \in [0, 1]$, we get

$$\begin{aligned} &\left| \frac{P(u, k)(t_1)}{1 + t_1^{a-1}} - \frac{P(u, k)(t_2)}{1 + t_2^{a-1}} \right| \\ &\leq \frac{\phi_q(\tau)}{\Gamma(a)} \left[\int_0^{t_1} \left| \frac{(t_1-r)^{a-1}}{1 + t_1^{a-1}} - \frac{(t_2-r)^{a-1}}{1 + t_2^{a-1}} \right| dr + \frac{1}{a} \frac{(t_2 - t_1)^{a-1}}{1 + t_2^{a-1}} \right] \rightarrow 0 \text{ as } t_1 \rightarrow t_2, \end{aligned}$$

and

$$\begin{aligned}
 & |\phi_q(D_{0+}^a P(u, k)(t_1)) - \phi_q(D_{0+}^a P(u, k)(t_2))| \\
 &= \left| \frac{1}{\Gamma(a)} \int_{t_1}^{\infty} (t_1 - r)^{a-1} e^{-r} (I - F) N_k u(r) dr \right. \\
 &\quad \left. - \frac{1}{\Gamma(a)} \int_{t_2}^{\infty} (t_2 - r)^{a-1} e^{-r} (I - F) N_k u(r) dr \right| \\
 &\leq \frac{1}{\Gamma(a)} \int_{t_2}^{\infty} |e^{-r}| |(t_1 - r)^{a-1} - (t_2 - r)^{a-1}| |(I - F) N_k u(r)| dr \\
 &\quad + \frac{1}{\Gamma(a)} \int_{t_1}^{t_2} |e^{-r}| (t_1 - r)^{a-1} |(I - F) N_k u(r)| dr \rightarrow 0 \text{ as } t_1 \rightarrow t_2.
 \end{aligned}$$

By the uniform continuity of $\phi(r)$ on $[-H, H]$, $|\phi_q(D_{0+}^a P(u, k)(t_1)) - \phi_q(D_{0+}^a P(u, k)(t_2))| \rightarrow 0$ as $t_1 \rightarrow t_2$. Thus, $P(u, k)\bar{\Omega}$ is equicontinuous on $[0, G]$. We now establish equiconvergent property of $P(u, k)\bar{\Omega}$ at ∞ . By the uniform continuity property of $\phi_q(r)$ on $[-v - \phi_q(v), v + \phi_q(v)]$ for $v > 0$, there exists a constant A_1 such that for $\epsilon > 0$, $r \geq A_1$,

$$|g(r)| = \left| \phi_q \left(\int_t^{\infty} (t - r)^{a-1} e^{-r} (I - F) N_k u(r) dr \right) - D_{0+}^b u(0) \right| < \epsilon, \quad \forall u \in \bar{\Omega}.$$

Conversely, for $\epsilon > 0$ since $\lim_{t \rightarrow \infty} \frac{t^{a-1}}{1+t^{a-1}} = 1$, there exists $A > A_1 > 0$ such that for any $t_1, t_2 \geq A$ and $r \in [0, A_1]$, we have

$$\begin{aligned}
 \left| \frac{(t_1 - r)^{a-1}}{1 + t_1^{a-1}} - \frac{(t_2 - r)^{a-1}}{1 + t_2^{a-1}} \right| &= \left| -1 + \frac{(t_1 - r)^{a-1}}{1 + t_1^{a-1}} + 1 - \frac{(t_2 - r)^{a-1}}{1 + t_2^{a-1}} \right| \\
 &\leq \left| \frac{t_1^{a-1}}{1 + t_1^{a-1}} - 1 \right| + \left| 1 - \frac{t_2^{a-1}}{1 + t_2^{a-1}} \right| < \epsilon.
 \end{aligned}$$

Hence, for $t_1, t_2 \geq A$, we have

$$\begin{aligned}
 \left| \frac{P(u, k)(t_1)}{1 + t_1^{a-1}} - \frac{P(u, k)(t_2)}{1 + t_2^{a-1}} \right| &= \frac{1}{\Gamma(a)} \left| \int_0^{t_1} \frac{(t_1 - r)^{a-1}}{1 + t_1^{a-1}} g(r) dr - \int_0^{t_2} \frac{(t_2 - r)^{a-1}}{1 + t_2^{a-1}} g(r) dr \right| \\
 &\leq \frac{1}{\Gamma(a)} \int_0^{A_1} \left| \frac{(t_1 - r)^{a-1}}{1 + t_1^{a-1}} - \frac{(t_2 - r)^{a-1}}{1 + t_2^{a-1}} \right| |g(r)| dr + \frac{1}{\Gamma(a)} \int_{A_1}^{t_1} \frac{(t_1 - r)^{a-1}}{1 + t_1^{a-1}} |g(r)| dr \\
 &\quad + \frac{1}{\Gamma(a)} \int_{A_1}^{t_2} \frac{(t_2 - r)^{a-1}}{1 + t_2^{a-1}} |g(r)| dr \\
 &\leq \frac{A_1 \epsilon}{\Gamma(a)} + \frac{2A_1 \epsilon}{\Gamma(a+1)},
 \end{aligned}$$

and

$$\begin{aligned}
 |D_{0+}^a P(u, k)(t_1) - D_{0+}^a P(u, k)(t_2)| &= |g(t_1) - g(t_2)| \\
 &= |g(t_1)| + |g(t_2)| < 2\epsilon.
 \end{aligned}$$

□

Thus, $P(u, k)\overline{\Omega}$ is equiconvergent at ∞ . Therefore, by Lemma 2.3, $P : \overline{\Omega} \times [0, 1] \rightarrow U_2$ is relatively compact.

Let $u \in Q_k = \{u \in \overline{\Omega} : Mu = N_k u\}$, then, $e^t D_-^a \phi_p(D_{0+}^b u(t)) = N_k u(t) \in \text{Im } M = \ker F$. We will now prove (ρ_3) and (ρ_4) of Definition 2.2 hold. Clearly, (ρ_3) holds since $P(u, 0) = 0$ and

$$\begin{aligned} P(u, k)(t) &= I_{0+}^b \left[\phi_q \left(I_-^a e^{-r} N_k u(r) dr \right) - D_{0+}^b u(0) \right] \\ &= I_{0+}^b \left[\phi_q \left(I_-^a D_-^a \phi_p(D_{0+}^b u(t)) \right) - D_{0+}^b u(0) \right]. \end{aligned}$$

For $u \in \text{dom } M$, since $I_{0+}^{1-b} u(0) = 0$, the $u(0) = \phi_p(D_{0+}^b u(0)) = \phi_p(D_{0+}^b u(\infty)) = 0$, then $d_1 = 0$ and

$$P(u, k)(t) = I_{0+}^b (D_{0+}^b u(t) - D_{0+}^b u(0)).$$

By applying boundary conditions, we get

$$P(u, k)(t) = u(t) - \frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b = [(I - F)u](t).$$

Also, for $u \in \overline{\Omega}$

$$\begin{aligned} M[Eu + P(u, k)](t) &= M \left[\frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b + I_{0+}^b \left[\phi_q \left(I_-^a e^{-r} (I - F) N_k u(r) dr \right) - D_{0+}^b u(0) \right] \right] \\ &= e^t D_-^a \phi_p \left(D_{0+}^b \left[\frac{D_{0+}^b u(0)}{\Gamma(b+1)} t^b + I_{0+}^b \left[\phi_q \left(I_-^a e^{-r} (I - F) N_k u(r) dr \right) - D_{0+}^b u(0) \right] \right] \right) \\ &= (I - F) N_k u(t). \end{aligned}$$

Hence, N_k is M -compact on $\overline{\Omega}$.

3 Main Results

In this section, we will establish the existence of solution for a fractional-order p -Laplacian boundary value problem involving both the left Caputo and right Riemann-Liouville fractional derivatives on the half-line. We start with this theorem.

Theorem 3.1. Suppose $w : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and we assume the following to be true (C_1) there exist functions $n_1(t) > 0$, $n_2(t) > 0$, $n_3(t) > 0$ in Z such that

$$|w(t, u, v)| \leq n_1(t) + n_2(t) \frac{|u|^{p-1}}{(1+t^a)^{p-1}} + n_3(t) |v|^{p-1}, \quad \forall t \in [0, \infty), (u, v) \in \mathbb{R}^2,$$

(C_2) there exists constant $Q_1 > 0$ such that for every $t_0 \in [0, \infty)$, $u \in \mathbb{R}$, $|D_{0+}^a u(t)| < Q_2$ either

$$\frac{e^{-t} \int_0^\infty z(r) dr}{\Gamma(a)} > 0,$$

or

$$\frac{e^{-t} \int_0^\infty z(r) dr}{\Gamma(a)} < 0,$$

(C_3) there exists $Q_2 > 0$ such that, if $|d| > Q_2$ or for all $t \in [0, \infty)$, then either

$$d \int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dr < 0, \quad (9)$$

or

$$d \int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dr > 0. \quad (10)$$

Then, the boundary value problem (1)-(2) has at least one solution in U if

$$\|n_2\|_1 + \|n_3\|_1 < 1.$$

In order to prove Theorem 3.1, two lemmas are required.

Lemma 3.1. The set $\Omega_1 = \{u \in \text{dom } M : \ker M : Mu = N_k u, k \in [0, 1]\}$ is bounded in U if (C_1) and (C_2) of Theorem 3.1 hold.

Proof. Let $u \in \Omega_1$, then $Mu = N_k u$ and $FN_k u = 0$. Hence, from Lemma 2.1, we have

$$u(t) = d_1 t^{b-1} + I_{0+}^b D_{0+}^b u(t), \quad d_1 \in \mathbb{R}.$$

Since $I_{0+}^{1-b} u(0) = 0$, then $d_1 = 0$, and

$$|u(t)| = \left| \frac{1}{\Gamma(b)} \int_0^t (t-r)^{b-1} D_{0+}^b u(r) dr \right|.$$

Therefore,

$$\|u\|_0 \leq \sup_{t \in [0, \infty)} \frac{t^{b-1}}{1+t^{b-1}} \frac{1}{\Gamma(b+1)} \|D_{0+}^b u\|_\infty \leq \frac{1}{\Gamma(b+1)} \|D_{0+}^b u\|_\infty.$$

Thus, from (C_2) there exist constants $t_0 \in [0, \infty)$ such that $|D_{0+}^a u(t_0)| < Q_2$. Also, by (C_1) of Theorem 3.1 and from $e^t D_-^a \phi_p(D_{0+}^u(t))' = -w(t, u(t), D_{0+}^b u(t))$, we have

$$\begin{aligned} |\phi_p(D_{0+}^b u(t))| &= \left| \phi_p(D_{0+}^a u(t_0)) - \frac{k}{\Gamma(a)} \int_{t_0}^\infty (r-t_0)^{a-1} e^{-r} w(r, u(r), D_{0+}^b u(r)) dr \right| \\ &\leq |\phi_p(D_{0+}^a u(t_0))| \\ &\quad + \frac{k}{\Gamma(a)} \int_{t_0}^\infty (r-t_0)^{a-1} e^{-r} \left[n_1(r) + n_2(r) \frac{|u|^{p-1}}{(1+r^{a-1})^{p-1}} + n_3(r) |v|^{p-1} \right] dr \\ &\leq \phi_p(Q_2) + [\|n_1\|_1 + \|n_2\|_1 \phi_p(\|u\|_0) + [\|n_3\|_1 \phi_p(\|D_{0+}^b u\|_\infty)] \frac{1}{\Gamma(a)} \int_0^\infty r^{a-1} e^{-r} dr] \\ &\leq \phi_p(Q_2) + [\|n_1\|_1 + \|n_2\|_1 \phi_p(\|u\|_0) + [\|n_3\|_1 \phi_p(\|D_{0+}^b u\|_\infty)]]. \end{aligned}$$

Then

$$\begin{aligned} \phi_p(\|D_{0+}^a u\|_\infty) &\leq \phi_p(Q_2) + \|n_1\|_1 + [\|n_2\|_1 + \|n_3\|_1] \phi_p(\|D_{0+}^b u\|_\infty) \\ &\leq \frac{\phi_p(Q_2) + \|n_1\|_1}{1 - [\|n_2\|_1 + \|n_3\|_1]}. \end{aligned}$$

Therefore

$$(\|D_{0+}^a u\|_\infty) \leq \phi_q(\Theta)$$

where $\Theta = \frac{\phi_p(Q_2) + \|n_1\|_1}{1 - [\|n_2\|_1 + \|n_3\|_1]}$. Hence,

$$\|u\| = \max\{\|u\|_0, \|D_{0+}^a u\|_\infty\} = \max\left\{\frac{1}{\Gamma(b+1)} \|D_{0+}^a u\|_\infty, \|D_{0+}^a u\|_\infty\right\} \leq \phi_q(\Theta).$$

Therefore Ω_1 is bounded in U_1 . □

Lemma 3.2. $\Omega_2 = \{u \in \ker M : Nu \in \text{Im } M\}$ is bounded if (C_3) holds.

Proof. Let $u \in \Omega_2$, then $u = dt^b$ where $d \in \mathbb{R}$. Since, $Nu \in \text{Im } M$ then

$$\int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dr = 0. \quad (11)$$

(C₃) and (11) implies $d \leq Q_2$, thus,

$$\|u\| \leq Q_2.$$

Therefore, Ω_2 is bounded. $\square$

Proof of Theorem 3.1. We have previously proved that M is quasi-linear and N_k is M -compact on $\overline{\Omega}$. Also, from Lemma 3.1 and Lemma 3.2 we proved that (τ_1) and (τ_1) of Theorem 2.1 holds. Finally, we will prove that (τ_3) of Theorem 2.1 also hold. Let $J : \text{Im } F \rightarrow \ker M$ be defined as

$$J(de^{-t}) = dt^b.$$

If (9) holds for any $u \in \text{dom } \partial\Omega \cap \ker M$, where $u = dt^b \neq 0$, we define the homeomorphism by

$$H(u, k) = -ku + (1-k)JFNU, \quad k \in [0, 1].$$

Then $H(u, 1) = -u \neq 0$ and $H(u, 0) = JFNU \neq 0$ since $Nu \notin \text{Im } M$. For $k \in (0, 1)$ and by way of contradiction, we assume $H(u, k) = 0$, then

$$H(u, k) = -kdt^b + (1-k) \frac{\int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dt}{\Gamma(a)} t^b,$$

which implies that

$$kd^2\Gamma(a) = (1-k)d \int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dt < 0$$

which contradicts $kd^2\Gamma(a) \geq 0$. If (10) holds, then we define

$$H(u, k) = -ku - (1-k)JFNU, \quad k \in [0, 1].$$

Then

$$-kd^2\Gamma(a) = (1-k)d \int_0^\infty r^{a-1} w(r, dr^b, \Gamma(b+1)d) dt > 0$$

which also contradicts $-kd^2\Gamma(a) \leq 0$. Hence, by the homotopy property of Brouwer degree, we have

$$\begin{aligned} \deg(JFN, \Omega \cap M, 0) &= \deg(H(\cdot, 1), \Omega \cap \ker M, 0) \\ &= \deg(H(\cdot, 0), \Omega \cap \ker M, 0) \\ &= \deg(I, \Omega \cap \ker M, 0) \neq 0. \end{aligned}$$

Therefore, at least one solution of (1)-(2) exists in $\overline{\Omega}$. $\square$

4 Example

Example 4.1. Consider the following boundary value problem

$$D_-^{4/3} \phi_{4/3}(D_{0+}^{3/2} u(t)) = e^{-t} \left(\frac{1}{6} e^{-2t} + \frac{e^{-4t} \cos \sqrt[3]{u(t)}}{5\sqrt[3]{1+t^{1/3}}} + \frac{e^{-4t}}{4} \phi_{4/3}(D_{0+}^{3/2} u(t)) \right), \quad t \in (0, \infty), \quad (12)$$

$$I_{0+}^{-1/2} u(0) = 0, \quad \phi_{4/3}(D_{0+}^{3/2} u(\infty)) = \phi_{4/3}(D_{0+}^{3/2} u(0)), \quad (13)$$

$\|n_2\|_1 = \int_0^\infty \frac{e^{-4t}}{5} dt = \frac{1}{20}$, $\|n_1\|_1 = \int_0^\infty \frac{e^{-2t}}{4} dt = \frac{1}{16}$, then

$$\|n_1\|_1 + \|n_2\|_1 = \frac{1}{20} + \frac{1}{16} < 1.$$

Conditions (C₁), (C₂) and (C₃) can also be shown to hold. Hence, there exists at least one solution of (12)-(13).

5 Conclusion

This work considered existence of solution for a fractional-order p -Laplacian boundary value at resonance involving the left-sided Caputo and right-sided Riemann-Liouville fractional derivatives on the half-line. The main tool that was used is the Ge and Ren Coincidence degree theory. The solutions obtained here are new and are validated using an example.

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