Abstract: In this article, we study the following noncoercive quasilinear parabolic problem

\[
\begin{aligned}
&\frac{\partial u}{\partial t} - \text{div} \varphi(x, t, u, \nabla u) + \nu |\nabla u|^{p-1} u = \lambda \frac{|u|^{p-2} u}{|x|^p} + f \quad \text{in } Q_T, \\
&u = 0 \quad \text{on } \Sigma_T, \\
&u(x, 0) = u_0 \quad \text{in } \Omega,
\end{aligned}
\]

with \( f \in L^1(Q_T) \) and \( u_0 \in L^1(\Omega) \) and show the existence of entropy solutions for this noncoercive parabolic problem with Hardy potential and \( L^1 \)-data.

Keywords: quasilinear parabolic equations, noncoercive equations, Hardy potential, entropy solutions

MSC 2020: 35K10, 35K59

1 Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) containing the origin. In the elliptic case, Abdellaoui et al. [3] studied the existence and nonexistence of positive solutions for the problem

\[
\begin{aligned}
&P_u \begin{cases}
-\Delta_p u \pm |\nabla u|^2 = \lambda \frac{|u|^{p-1}}{|x|^p} + f & \text{in } \Omega, \\
0 & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

(1.1)

They proved the existence of positive solutions in the absorption case \(+|\nabla u|^2\) for all \( \lambda > 0 \) and \( f \in L^1(\Omega) \). Furthermore, they showed the nonexistence of solution in the diffusion case \(-|\nabla u|^2\) and \( \lambda > 0 \) (even in a very weak sense) (see also [1] and [2]). These problems are related to the following classical Hardy inequality:

\[
\Lambda_{N,p} \int_{\mathbb{R}^N} \frac{|\phi|^p}{|x|^p} \, dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^p \, dx, \quad \text{for all } \phi \in W^{1,p}(\mathbb{R}^N),
\]

(1.2)

where \( \Lambda_{N,p} = \left( \frac{N-p}{p} \right)^p \) is optimal and is not attained; we refer to [7] for more details about Hardy inequality.

In [20], Porzio studied the quasilinear elliptic problem

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and proved that: for all $f \in L^1(\Omega)$, the problem (1.3) has a solution $u \in W_0^{1,q}(\Omega)$ such that $q < \frac{2s}{s+1}$ (see also [10,11,14,17,21,24]).

In [18], Porretta and Segura de León investigated the existence results of the problem

$$\begin{cases}
- \text{div} (a(x,u,\nabla u)) + g(x,u,\nabla u) = f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.4)

where $a(x,s,\xi)$ verifying a degenerate coercivity condition, and $g(x,s,\xi)$ is assumed to satisfy only some growth conditions. They proved the existence of solutions using rearrangement techniques, see also [22] and [12], for the unilateral case we refer the reader to [13]. In [6], Alvino et al. considered the nonlinear degenerated elliptic problem

$$\begin{cases}
- \text{div} (|\nabla u|^{p-2} \nabla u) = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.5)

and proved the existence of solutions and some regularity results in the case of $f \in L^m(\Omega)$, with $m \geq 1$ (see also [5]).

For $T > 0$, we denote by $Q_T$ the cylinder $\Omega \times (0, T)$ and by $\Sigma_T$ the lateral surface $\partial \Omega \times (0, T)$. Baras and Goldstein [8] studied the linear equation

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = \lambda \frac{u}{|x|^2} & \text{in } Q_T, \\
u = 0 & \text{on } \Sigma_T, \\
u(x,0) = u_0 & \text{in } \Omega.
\end{cases}$$

(1.6)

They proved the existence of solutions for the problem (1.6) in the case $\lambda \leq \Lambda_{N,2} = \left(\frac{N-2}{2}\right)^2$ and the initial datum $u_0$ in a convenient class. Moreover, if $\lambda > \Lambda_{N,2}$, the authors proved the nonexistence of a local solution for any $u_0 > 0$. Azorero and Alonso [7] studied the behavior of the nonlinear critical $p$-heat equation

$$\begin{cases}
\frac{\partial u}{\partial t} - \text{div}(|\nabla u|^{p-2} \nabla u) = \lambda \frac{u^{p-1}}{|x|^p} & \text{in } Q_T, \\
u = 0 & \text{on } \Sigma_T, \\
u(x,0) = f(x) \geq 0 & \text{in } \Omega.
\end{cases}$$

(1.7)

They showed the existence of a solution for $\lambda \leq \Lambda_N = \left(\frac{N-2}{p}\right)^p$. Moreover, they proved the nonexistence of a local solution for any $f(\cdot) > 0$ for $\lambda > \Lambda_N$, we refer the reader also to [19].

In this article, we study the existence of solutions for the problem associated with quasilinear parabolic equations involving a Leray-Lions-type operator with lower order terms and the so-called Hardy potential

$$\begin{cases}
\frac{\partial u}{\partial t} - \text{div} a(x,t,u,\nabla u) + \nu|u|^{p-1}u = \lambda \frac{|u|^{p-2}u}{|x|^p} + f & \text{in } Q_T, \\
u = 0 & \text{on } \Sigma_T, \\
u(x,0) = u_0 & \text{in } \Omega,
\end{cases}$$

(1.8)

where $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, and we consider the assumptions

$$\nu > 0, \quad \lambda \geq 0, \quad \frac{2N}{N+2} < p < N, \quad \text{and} \quad s > \max \left(\frac{N(p-1)}{N-p}, \frac{1}{p-1}\right)$$
Note that this article can be seen as a continuation of [7] and [8] for the nonlinear and noncoercive cases by adding the lower order term $|u|^{p-1}u$, which guarantees the existence of entropy solutions for any $\lambda \geq 0$.

This article is organized as follows: in Section 2, we present some assumptions on $a(x, t, s, \xi)$ for which our noncoercive parabolic problem (1.8) has at least one entropy solution. Section 3 contains some important lemmas useful to prove our main result. Section 4 will be devoted to the proof of the existence of entropy solutions for our quasilinear noncoercive parabolic problem (1.8).

## 2 Essential assumption

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ containing the origin, $T > 0$, and $p > \frac{2N}{N+2}$.

Consider a Leray-Lions operator $A$ that acts from $L^p(0, T; W_0^{1,p} (\Omega))$ into its dual $L^{q'} (0, T; W^{-1,q'} (\Omega))$, where $\frac{1}{p} + \frac{1}{q'} = 1$ defined by the formula

$$Au = -\text{div}a(x, t, u, \nabla u).$$

(2.1)

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function (measurable with respect to $(x, t)$ in $Q_T$ for every $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s, \xi)$ in $\mathbb{R} \times \mathbb{R}^N$ for almost every $(x, t)$ in $Q_T$), which satisfies the following conditions:

$$|a(x, t, s, \xi)| \leq \beta (K(x, t) + |s|^{p-1} + |\xi|^{p-1}),$$

(2.2)

$$(a(x, t, s, \xi) - a(x, t, s, \xi')) \cdot (\xi - \xi') > 0 \quad \text{for all } \xi \neq \xi' \quad \text{in } \mathbb{R}^N,$$

(2.3)

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K(x, t)$ is a nonnegative function lying in $L^p (Q_T)$ and $\beta > 0$.

We assume that the Carathéodory functions $a(x, t, s, \xi)$ verify the nonstandard coercivity condition

$$a(x, t, s, \xi) \cdot \xi \geq b(|s|) |\xi|^p$$

(2.4)

such that $b (\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a decreasing function, and there exists a positive constant $b_0$ that verifies

$$\frac{b_0}{(1 + |s|)^\delta} \leq b(|s|) \quad \text{for any } s \in \mathbb{R},$$

where $0 \leq \delta < p - 1$.

Let $\nu > 0$ and $\lambda \geq 0$, we consider the quasilinear noncoercive parabolic problem

$$\begin{cases}
    u_t - \text{div}a(x, t, u, \nabla u) + \nu |u|^{p-1}u = \lambda \frac{|u|^{p-2}u}{|x|^p} + f & \text{in } Q_T, \\
    u = 0 & \text{on } \Sigma_T, \\
    u(x, 0) = u_0 & \text{in } \Omega,
\end{cases}$$

(2.5)

with $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, and we assume that

$$s > \max \left( \frac{N(p-1)}{N-p}, \frac{1}{p-1} \right).$$

(2.6)

## 3 Some technical lemmas

**Lemma 3.1.** (cf. [23]) Let $B_0$, $B$, and $B_1$ be Banach spaces with $B_0 \subset B \subset B_1$. Let us set

$$Y = \{ u : u \in L^q(0, T; B_0) \text{ and } u' \in L^q(0, T; B_0) \},$$

where $q_0 > 1$ and $q_1 > 1$ are real numbers.

Assuming that the embedding $B_0 \hookrightarrow \hookrightarrow B$ be compact. Then,
and this imbedding is compact.

**Remark 3.2.** Let $p > \frac{2N}{N+2}$, we set

$$B_0 = W^{1,p}_0(\Omega), \quad B = L^2(\Omega), \quad \text{and} \quad B_1 = W^{-1,p'}(\Omega),$$

In view of Lemma 3.1, we obtain

$$\{u : u \in L^p(0, T; W^{1,p}_0(\Omega)) \text{ and } u' \in L^p(0, T; W^{-1,p'}(\Omega))\} = Y \hookrightarrow L^2(Q_T).$$

(3.1)

Moreover, in view of [9], we have

$$\{u : u \in L^p(0, T; W^{1,p}_0(\Omega)) \text{ and } u' \in L^p(0, T; W^{-1,p'}(\Omega))\} \subseteq C([0, T]; L^1(\Omega)).$$

(3.2)

Now, let $\mu \geq 0$, and we introduce the time mollification $u_\mu$ of a function $u \in L^p(0, T; W^{1,p}_0(\Omega))$, by

$$u_\mu(x, t) = \mu \int_{-\infty}^{t} a(x, s) \exp(\mu(s - t))ds, \quad \text{where} \ a(x, s) = u(x, s) \cdot \chi_{(0,T)}(s).$$

**Proposition 3.3.** (see [4])

(i) If $u \in L^p(0, T; W^{1,p}_0(\Omega))$, then $u_\mu$ is a measurable in $Q_T$ such that $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and

$$\|u_\mu\|_{L^p(0,T;W^{0,\frac{p}{p-1}}_0(\Omega))} \leq \|u\|_{L^p(0,T;W^{0,\frac{p}{p-1}}_0(\Omega))}.$$

(ii) If $u \in L^p(0, T; W^{1,p}_0(\Omega))$, then $u_\mu \rightharpoonup u$ strongly in $L^p(0, T; W^{1,p}_0(\Omega))$ as $\mu \to \infty$.

(iii) If $u_n \rightharpoonup u$ strongly in $L^p(0, T; W^{1,p}_0(\Omega))$, then $(u_n)_\mu \rightharpoonup u_\mu$ strongly in $L^p(0, T; W^{1,p}_0(\Omega))$ as $n \to \infty$.

**4 Main results**

We set

$$\varphi_k(r) = \int_{0}^{r} T_k(s)ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ |k|r - \frac{k^2}{2} & \text{if } |r| > k. \end{cases}$$

**Definition 4.1.** A measurable function $u$ is an entropy solution for the parabolic problem (2.5) if

$$|u|^p \in L^1(Q_T) \quad \text{and} \quad T_k(u) \in L^p(0, T; W^{1,p}_0(\Omega)) \quad \forall k > 0,$$

such that $u$ verifying the inequality

$$\left\{ \begin{array}{l} \int_{Q_T} \varphi_k(u - \psi(T))dx - \int_{Q_T} \varphi_k(u_0 - \psi(0))dx + \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u - \psi)dxdt \\
+ \int_{Q_T} a(x, t, u, \nabla u) \cdot \nabla T_k(u - \psi)dxdt + \nu \int_{Q_T} |u|^{p-1}u \cdot T_k(u - \psi)dxdt \\
\leq \Lambda \int_{Q_T} \frac{|u|^{p-2}}{|x|^p} f_k(u - \psi)dxdt + \int_{Q_T} f_k(u - \psi)dxdt \end{array} \right. \quad (4.2)$$

for all $\psi \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q_T)$ with $\frac{\partial \psi}{\partial t} \in L^p(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)$. 

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Remark 4.1. In view of Young’s inequality, we have
\[ \int_{Q_r} |u|^{p+2} \frac{T_p(u - \psi)}{|x|^p} dx dt \leq k \left( \frac{p - 1}{s} \right) \int_{Q_r} |u|^s dx dt + kT_{\frac{s}{p+1}} \left( \frac{s - p + 1}{s} \right) \int_{\Omega} \frac{1}{|x|^\frac{p}{p-1}} dx. \]

Thanks to (4.1), we have \( |u|^s \in L^1(Q_r) \), and since \( s > \frac{N(p-1)}{N-p} \), then \( \frac{1}{|x|^p} \in L^{\frac{N}{N-p}}(\Omega) \). It follows that \( |u|^s \in L^1(Q_r) \) holds true.

Theorem 4.2. Under the assumptions (2.2)–(2.4) and (2.6), the noncoercive parabolic problem (2.5) has at least one entropy solution such that
\[ |u|^s \in L^1(Q_r) \quad \text{for all} \quad s > \frac{N(p-1)}{N-p}. \]

4.1 Proof of the Theorem 4.2

4.1.1 Step 1: Approximate problems

Let \((f_n)_n\) be a sequence in \(L^p(0, T; W^{1,p}_0(\Omega)) \cap L^1(Q_r)\) such that \(f_n \to f\) in \(L^1(Q_r)\) with \(|f_n| \leq |f|\), and let \((u_{0,n})_n\) be a sequence in \(C_0^\infty(\Omega)\) such that \(u_{0,n} \to u_0\) in \(L^1(\Omega)\) with \(|u_{0,n}| \leq |u_0|\). We consider the sequence of approximate problems
\[
\begin{aligned}
\begin{cases}
(u_n)_t - \text{div}\alpha(x, t, T_n(u_n), \nabla u_n) + v |T_n(u_n)|^{p-1} T_n(u_n) = \lambda \frac{|T_n(u_n)|^{p-2} T_n(u_n)}{|x|^p + \frac{1}{n}} + f_n & \text{in} \ Q_r, \\
u_n(x, t) = 0 & \text{on} \ \Sigma_r, \\
u_n(x, 0) = u_{0,n} & \text{in} \ \Omega.
\end{cases}
\end{aligned}
\]

There exists at least one weak solution \(u_n \in L^\infty(0, T; W^{1,p}_0(\Omega))\) of the problem (4.3), (cf. [16]).

4.1.2 Step 2: A priori estimates

Lemma 4.3. Under the assumptions of Theorem 4.2, there exists a constant \(C > 0\), not depending on \(n\), such that the following estimates hold true:
\[
\int_{Q_r} |T_n(u_n)|^s dx dt \leq C \quad \text{for all} \quad s > \frac{N(p-1)}{N-p},
\]
(4.4)
\[
\int_{Q_r} |\nabla u_n|^p dx dt \leq C \quad \text{for all} \quad \theta > 1,
\]
(4.5)
\[
\int_{Q_r} |\nabla T_n(u_n)|^p dx dt \leq C(1 + k)^{\theta - \delta} \quad \text{for all} \quad k > 0.
\]
(4.6)

Proof of the Lemma 4.3. Let \(\theta > 1\), by taking \(\psi(u_n) = \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right)\sign(u_n)\) as a test function in (4.3), we obtain
\[
\begin{align*}
\int_0^T & \langle (u_n)_t, \psi(u_n) \rangle dt + \int_{Q_r} \alpha(x, t, T_n(u_n), \nabla u_n) \cdot \nabla \psi(u_n) dx dt + v \int_{Q_r} |T_n(u_n)|^s \psi(u_n) dx dt \\
& = \lambda \int_{Q_r} \frac{|T_n(u_n)|^{p-1}}{|x|^p + \frac{1}{n}} \psi(u_n) dx dt + \int_{Q_r} f_n \psi(u_n) dx dt.
\end{align*}
\]
(4.7)
Using (2.4) and the fact that $|\psi(u_n)| \leq 1$, we arrive at

$$
\int_0^T \langle (u_n)_t, \psi(u_n) \rangle \, dt + b_0(\theta - 1) \int_{Q_T} \frac{|
abla u_n|^p}{(1 + |u_n|)^{p+\delta}} \, dx \, dt + v \int_{Q_T} |T_n(u_n)|^p \left(1 - \frac{1}{(1 + |u_n|)^{\theta \delta}}\right) \, dx \, dt
\leq \lambda \int_{Q_T} |T_n(u_n)|^{p-1} \, dx + \int_{Q_T} |f| \, dx \, dt. 
$$

(4.8)

We have

$$
\left(1 - \frac{1}{(1 + |u_n|)^{\theta \delta}}\right) \geq \frac{1}{2} \quad \text{for} \quad |u_n| \geq R = 2^{\frac{1}{\theta \delta}} - 1,
$$

thus, we get

$$
\frac{1}{2} \int_{\{|u_n| \geq R\}} |T_n(u_n)|^p \, dx \, dt \leq \int_{\{|u_n| \geq R\}} |T_n(u_n)|^p \left(1 - \frac{1}{(1 + |u_n|)^{\theta \delta}}\right) \, dx \, dt
\leq \int_{Q_T} |T_n(u_n)|^p \left(1 - \frac{1}{(1 + |u_n|)^{\theta \delta}}\right) \, dx \, dt,
$$

it follows that

$$
\frac{1}{2} \int_{Q_T} |T_n(u_n)|^p \, dx \, dt \leq \frac{1}{2} TR^p \Omega + \int_{Q_T} |T_n(u_n)|^p \left(1 - \frac{1}{(1 + |u_n|)^{\theta \delta}}\right) \, dx \, dt.
$$

We conclude that

$$
\int_0^T \langle (u_n)_t, \psi(u_n) \rangle \, dt + b_0(\theta - 1) \int_{Q_T} \frac{|
abla u_n|^p}{(1 + |u_n|)^{p+\delta}} \, dx \, dt + v \int_{Q_T} |T_n(u_n)|^p \, dx \, dt
\leq \frac{v}{2} TR^p \Omega + \lambda \int_{Q_T} |T_n(u_n)|^{p-1} \, dx + \int_{Q_T} |f| \, dx \, dt.
$$

Since $s > p - 1$, Young’s inequality allows us to obtain

$$
\lambda \int_{Q_T} |T_n(u_n)|^{p-1} \, dx \, dt \leq \frac{v}{4} \int_{Q_T} |T_n(u_n)|^p \, dx \, dt + C_1 \int_{Q_T} \frac{1}{|x|^s} \, dx,
$$

with $C_1 = \frac{s-p+1}{s} \left(\frac{p-1}{s}\right)^{\frac{p-1}{s}}$. It follows that

$$
\int_0^T \langle (u_n)_t, \psi(u_n) \rangle \, dt + b_0(\theta - 1) \int_{Q_T} \frac{|
abla u_n|^p}{(1 + |u_n|)^{p+\delta}} \, dx \, dt + v \int_{Q_T} |T_n(u_n)|^p \, dx \, dt
\leq \frac{v}{2} TR^p \Omega + C_1 \int_{Q_T} \frac{1}{|x|^s} \, dx + \int_{Q_T} |f| \, dx \, dt.
$$

(4.9)

Concerning the first term on the right-hand side of (4.9), we set

$$
\omega(s) = \begin{cases} 
|s| + \frac{1}{\theta - 2 \left(1 + |s|\right)^{\theta - 2}} + \frac{1}{2 - \theta} & \text{for} \theta \in [1, 2] \cup [2, \infty], \\
|s| - \log(1 + |s|) & \text{for} \theta = 2,
\end{cases}
$$

then $\psi(s) = (\omega(s))'$. Taking $\theta \in [1, 2] \cup [2, \infty]$, we obtain
\[ \int_0^T \langle (u_n)_t, \psi(u_n) \rangle \, dt = \int_0^T \int_\Omega \frac{\partial \omega(u_n)}{\partial t} \, dx \, dt = \int_\Omega \omega(u_n(T)) \, dx - \int_\Omega \omega(u_0,n) \, dx, \]

it follows that

\[ \int_\Omega \omega(u_n(T)) \, dx + b_\theta(\theta - 1) \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta+\delta}} \, dx \, dt + \frac{\nu}{4} \int_\Omega |T_n(u_n)|^q \, dx \, dt \leq \frac{\nu}{2} TR^1|\Omega| + C_T \int_\Omega \frac{1}{|x|^{\frac{N(p-1)}{N-p}}} \, dx + \int_\Omega |f| \, dx + \int_\Omega \omega(u_0,n) \, dx. \]

(4.10)

Under the assumption \( s > \frac{N(p-1)}{N-p} \), we have \( \frac{1}{|x|^{\frac{N(p-1)}{N-p}}} \in L(\Omega) \). In the case of \( 1 < \theta < 2 \), since \( \omega(u_0,T) \geq 0 \) and \( \omega(u_{0,n}) \leq |u_{0,n}| \), we have

\[ b_\theta(\theta - 1) \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta+\delta}} \, dx \, dt + \frac{\nu}{4} \int_\Omega |T_n(u_n)|^q \, dx \, dt \leq C_2 + \int_\Omega |u_{0,n}| \, dx \leq C. \]

(4.11)

The same result can be obtained directly from (4.11) in the case of \( \theta > 2 \). In the case of \( \theta = 2 \), we have

\[ \int_0^T \langle (u_n)_t, \psi(u_n) \rangle \, dt = \int_\Omega |u_n(T)| - \log(1 + |u_n(T)|) \, dx - \int_\Omega |u_{0,n}| - \log(1 + |u_{0,n}|) \, dx \geq -\int_\Omega |u_0| \, dx, \]

and thanks to (4.9), we obtain

\[ b_\theta(\theta - 1) \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta+\delta}} \, dx \, dt + \frac{\nu}{4} \int_\Omega |T_n(u_n)|^q \, dx \, dt \leq C_2 + \int_\Omega |u_0| \, dx. \]

(4.12)

By combining (4.11) and (4.12), we deduce that for any \( \theta \in ]1, \infty[, \) there exists a constant \( C_\theta \) that does not depend on \( n \) such that

\[ b_\theta(\theta - 1) \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta+\delta}} \, dx \, dt + \frac{\nu}{4} \int_\Omega |T_n(u_n)|^q \, dx \, dt \leq C_\theta \quad \text{for all } \theta > 1. \]

(4.13)

Finally, we have

\[ \int_\Omega |\nabla T_n(u_n)|^p \, dx \, dt = \int_{\|u_n\| \leq k} |\nabla u_n|^p \, dx \, dt \leq (1 + k)^{\theta_1+\delta} \int_\Omega \frac{|\nabla u_n|^p}{(1 + |u_n|)^{\theta+\delta}} \, dx \, dt, \]

and by (4.5), we conclude (4.6). \( \square \)

### 4.1.3 Step 3: Weak convergence of truncations

Let \( n \geq k \), and thanks to (4.4), we have

\[ k^s \text{meas}\{|u_n| > k\} = \int_{\|u_n\| > k} |T_n(u_n)|^q \, dx \, dt \leq \int_{\Omega} |T_n(u_n)|^q \, dx \, dt \leq C. \]

It follows that

\[ \limsup_{n \to \infty} \text{meas}\{|u_n| > k\} \leq \frac{C}{k^s} \to 0 \quad \text{as } k \to +\infty. \]

(4.14)

For all \( \sigma > 0 \), we have
meas\{\|u_n - u_m\| > \sigma\} \leq \text{meas}\{\|u_n\| > k\} + \text{meas}\{\|u_m\| > k\} + \text{meas}\{\|T_k(u_n) - T_k(u_m)\| > \sigma\}.

In view of (4.14), we obtain that for all \(\varepsilon > 0\), there exists \(k_0(\varepsilon) > 0\) such that
\[
\text{meas}\{\|u_n\| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{\|u_m\| > k\} \leq \frac{\varepsilon}{3} \quad \forall k \geq k_0(\varepsilon). \tag{4.15}
\]
In view of (4.6), the sequence \((T_k(u_n))_n\) is bounded in \(L^p(0, T; W^{1,p}_0(\Omega))\), then there exist \(\phi_k \in L^p(0, T; W^{1,p}_0(\Omega))\) and a subsequence still denoted \((T_k(u_n))_n\) such that
\[
\begin{cases}
T_k(u_n) \rightharpoonup \phi_k \quad \text{in} \quad L^p(0, T; W^{1,p}_0(\Omega)), \\
T_k(u_n) \to \phi_k \quad \text{in} \quad L^1(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T.
\end{cases}
\]
Thus, we can assume that \((T_k(u_n))_n\) is a Cauchy sequence in measure in \(Q_T\), then for all \(k > 0\) and \(\sigma > 0\), there exists \(n_0 = n_0(k, \sigma, \varepsilon)\) such that
\[
\text{meas}\{\|T_k(u_n) - T_k(u_m)\| > \sigma\} \leq \frac{\varepsilon}{3} \quad \forall n, m \geq n_0. \tag{4.16}
\]
By combining (4.15) and (4.16), we deduce that for all \(\varepsilon, \sigma > 0\), there exists \(n_0 = n_0(\sigma, \varepsilon)\) such that
\[
\text{meas}\{\|u_n - u_m\| > \sigma\} \leq \varepsilon \quad \forall n, m \geq n_0. \tag{4.17}
\]
It follows that \((u_n)_n\) is a Cauchy sequence in measure, then there exists a subsequence still denoted \((u_n)_n\) such that
\[u_n \to u \quad \text{a.e. in} \quad Q_T.\]
We deduce that
\[T_k(u_n) \to T_k(u) \quad \text{in} \quad L^p(0, T; W^{1,p}_0(\Omega)), \tag{4.18}\]
and in view of the Lebesgue dominated convergence theorem,
\[T_k(u_n) \to T_k(u) \quad \text{in} \quad L^p(Q_T). \tag{4.19}\]

### 4.1.4 Step 4 The weak convergence of \((u_n)_n\) in \(L^p(0, T; W^{-1,p}(\Omega)) + L^1(Q_T)\)

Let \(S_h(\cdot)\) be an increasing function in \(C^2(\mathbb{R})\) such that \(S_h(r) = r\) for \(|r| \leq h\) and \(\text{supp}(S_h) \subset [-h - 1, h + 1]\), then \(\text{supp}(S_h^\prime) \subset [-h - 1, -h] \cup [h, h + 1]\).

Let \(v \in L^p(0, T; W^{1,p}_0(\Omega)) \cap L^\infty(Q_T)\), by taking \(S_h(u_n)v\) as a test function in (4.3), we obtain
\[
\begin{aligned}
\int_0^T \left\{ \frac{\partial u_n}{\partial t}, S_h(u_n)v \right\} dt + \int_0^T \left( a(x, t, T_n(u_n), \nabla u_n) \cdot (S_h''(u_n)v + S_h'(u_n)v \nabla u_n) \right) \, dx \, dt &+ \int_{Q_T} \int |T_n(u_n)|^{p-1} |T_n(u_n)S_h''(u_n)v| \, dx \, dt \\
&= \lambda \int_{Q_T} \frac{|T_n(u_n)|^{p-2} |T_n(u_n)|}{|x|^p + \frac{1}{n}} S_h''(u_n)v \, dx \, dt + \int_{Q_T} f_n S_h'(u_n)v \, dx \, dt,
\end{aligned}
\]
then
\[
\begin{aligned}
\left| \int_0^T \left\{ \frac{\partial S_h(u_n)}{\partial t}, v \right\} dt \right| &\leq \int_0^T \left( a(x, t, T_n(u_n), \nabla u_n) \cdot (S_h''(u_n)v + S_h'(u_n)v \nabla u_n) \right) \, dx \, dt \\
&+ \int_{Q_T} |T_n(u_n)|^p \left| S_h''(u_n)v \right| \, dx \, dt + \lambda \int_{Q_T} \frac{|T_n(u_n)|^{p-1} |S_h''(u_n)v|}{|x|^p + \frac{1}{n}} \, dx \, dt \\
&+ \int_{Q_T} |f_n| \left| S_h'(u_n)v \right| \, dx \, dt. \tag{4.20}
\end{aligned}
\]
For the first term on the right-hand side of (4.20), we have
\[
\left| \int_{Q_T} a(x, t, T_n(u_n), \nabla u_n) \cdot (S_{h}''(u_n)\nabla v + S_{h}''(u_n)\nabla u_n)dxdt \right|
\leq \int_{[u_n \leq h + 1]} \beta(K(x, t) + |u_n|^p - 1 + |\nabla u_n|^p - 1) |S_{h}''(u_n)||\nabla v| + |S_{h}''(u_n)||\nabla u_n|dxdt
\leq \beta(K(x, t) + \|T_{h,1}(u_n)\|_{L^p(Q_T)} + \|T_{h,1}(u_n)\|_{L^p(Q_T)})
\times (\|S''_{h}\|_{L^\infty(R)}||\nabla v||_{L^\infty(Q_T)} + \|S''_{h}\|_{L^\infty(R)}||\nabla u_n||_{L^\infty(Q_T)})
\leq C_n(\|v\|_{L^p(0,T;W^1_0(\Omega))} + \|\nabla v\|_{L^\infty(Q_T)}).
\]

Concerning the last three terms on the right-hand side of (4.20), by using Young’s inequality, we have
\[
v \int_{Q_T} |T_n(u_n)|^p |S'(u_n)v| dxdt + \lambda \int_{Q_T} \frac{|T_n(u_n)|^p - 1}{|x|^{p+1}} |S'(u_n)v| dxdt + \int_{Q_T} |f_n| |S'(u_n)v| dxdt
\leq 2v \int_{Q_T} |T_n(u_n)|^p |S'(u_n)v| dxdt + C_9 \int_{Q_T} \frac{1}{|x|^{p+1}} |S'(u_n)v| dxdt + \int_{Q_T} |f_n| |S'(u_n)v| dxdt
\leq \left( 2v \|T_n(u_n)\|_{L^p(Q_T)} + C_9 T \right) \left( \frac{1}{|x|^{p+1}} \right) \left( \|S''_{h}\|_{L^\infty(R)} \|v\|_{L^\infty(Q_T)} \right)
\leq C_n(\|v\|_{L^p(0,T;W^1_0(\Omega))} + \|\nabla v\|_{L^\infty(Q_T)}).
\]

By combining (4.20)–(4.21), we deduce that
\[
\left| \int_{0}^{T} \left\langle \frac{\partial S_{h}(u_n)}{\partial t}, v \right\rangle dt \right| \leq C_n(\|v\|_{L^p(0,T;W^1_0(\Omega))} + \|\nabla v\|_{L^\infty(Q_T)}),
\]
with \(C_n\) being a constant that does not depend on \(n\), then the sequence \(\left\langle \frac{\partial S_{h}(u_n)}{\partial t} \right\rangle_n\) is bounded in norm for the space \(L^p(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)\), therefore \(\frac{\partial S_{h}(u_n)}{\partial t} \rightarrow \frac{\partial S_{h}(u)}{\partial t}\) weakly in \(L^p(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T)\).

### 4.1.5 Step 5: The equi-integrability of the sequences \(\left\{\frac{T_n(u_n)|^{p-2}T_n(u_n)\right\}_n\) and \(\left\{T_n(u_n)|^{1-p}T_n(u_n)\right\}_n\)

Now, we will show that
\[
|T_n(u_n)|^{p-1}T_n(u_n) \rightarrow |u|^{p-1}u \quad \text{and} \quad \frac{|T_n(u_n)|^{p-2}T_n(u_n)}{|x|^{p+1}} \rightarrow \frac{|u|^{p-2}u}{|x|^{p}} \quad \text{strongly in} \quad L^1(Q_T).
\]
We have
\[
|T_n(u_n)|^{p-1}T_n(u_n) \rightarrow |u|^{p-1}u \quad \text{and} \quad \frac{|T_n(u_n)|^{p-2}T_n(u_n)}{|x|^{p+1}} \rightarrow \frac{|u|^{p-2}u}{|x|^{p}} \quad \text{a.e. in} \quad Q_T.
\]
By using Vitali’s theorem, it is sufficient to prove that the two sequences \(\left\{|T_n(u_n)|^{p-1}T_n(u_n)\right\}_n\) and \(\left\{|T_n(u_n)|^{p-2}T_n(u_n)\right\}_n\) are uniformly equi-integrable.

Indeed, by taking \(T_{h,1}(u_n) - T_n(u_n)\) as a test function in (4.3), and since \(T_{h,1}(u_n) - T_n(u_n)\) have the same sign as \(u_n\), we obtain
We have
\[
\left\{ \frac{\partial u_n}{\partial t} + \nabla \cdot \nabla u_n \right\} + \nu \int |T_n(u_n)|^p |T_{n+1}(u_n) - T_n(u_n)| dx dt = 
\int |T_n(u_n)|^{p-1} |T_{n+1}(u_n) - T_n(u_n)| dx dt + 
\int f_n \cdot (T_{n+1}(u_n) - T_n(u_n)) dx.
\]

We conclude that
\[
\int a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx dt + \nu \int |T_n(u_n)|^p |T_{n+1}(u_n) - T_n(u_n)| dx dt 
\leq \lambda \int |T_n(u_n)|^{p-1} |T_{n+1}(u_n) - T_n(u_n)| dx dt 
+ \nu \sum_{[h < |u_n|]} \int |T_n(u_n)|^p |T_{n+1}(u_n) - T_n(u_n)| dx dt 
+ C_{12} \int \frac{|T_n(u_n) - T_n(u_n)|}{|x|^{\frac{p-1}{2}}} dx dt.
\]

We have
\[
\lim_{h \to \infty} \int_{[h < |u_n|]} \frac{1}{|x|^{\frac{p-1}{2}}} dx dt = 0 \quad \text{and} \quad \lim_{h \to \infty} \int |f| dx dt = 0,
\]
and since \( u_0 \in L^1(\Omega) \), then
\[
\int_{\Omega} \varphi_{h,n}(u_{0,n}) \, dx - \int_{\Omega} \varphi_h(u_{0,n}) \, dx = \int_{\{h < |u_{0,n}| \cap [h+1] \}} \frac{|u_{0,n}|^2}{2} - h|u_{0,n}| + \frac{h^2}{2} \, dx + \int_{\{h |u_{0,n}| \} \cap [h+1]} \frac{|u_{0,n}| - h - \frac{1}{2}}{2} \, dx
\]
\[
\leq \frac{1}{2} \int_{\{h < |u_{0,n}| \cap [h+1] \}} dx + \int_{\{h |u_{0,n}| \} \cap [h+1]} \frac{|u_{0} - h - \frac{1}{2}|}{2} \, dx \to 0 \quad \text{as} \quad h \to \infty,
\]
we deduce that
\[
\lim_{h \to \infty} \limsup_{n \to \infty} \int_{\{h < |u_{0,n}| \cap [h+1] \}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dxdt = 0 \quad (4.26)
\]
and
\[
\int_{\{h+1 < |u_{0,n}| \}} |T_n(u_{0,n})|^p \, dxdt + \int_{\{h+1 < |u_{0,n}| \}} \frac{|T_n(u_{0,n})|^{p-1}}{|x|^p} \, dxdt \to 0 \quad \text{as} \quad h \to \infty. \quad (4.27)
\]
Thus, for all \( \eta > 0 \), there exists \( h(\eta) > 0 \) such that
\[
\int_{\{h(\eta) < |u_{0,n}| \}} |T_n(u_{0,n})|^p \, dxdt + \int_{\{h(\eta) < |u_{0,n}| \}} \frac{|T_n(u_{0,n})|^{p-1}}{|x|^p} \, dxdt \leq \frac{\eta}{2}. \quad (4.28)
\]
On the other hand, for any measurable subset \( E \subset Q_T \), we have
\[
\int_{E} |T_n(u_{0,n})|^p \, dxdt + \int_{E} \frac{|T_n(u_{0,n})|^{p-1}}{|x|^p} \, dxdt
\]
\[
\leq \int_{E} |T_n(u_{0,n})|^p \, dxdt + \int_{E} \frac{|T_{h(\eta)}(u_{0,n})|^{p-1}}{|x|^p} \, dxdt + \int_{\{h(\eta) < |u_{0,n}| \}} |T_n(u_{0,n})|^p \, dxdt + \int_{\{h(\eta) < |u_{0,n}| \}} \frac{|T_n(u_{0,n})|^{p-1}}{|x|^p} \, dxdt, \quad (4.29)
\]
and there exists \( \beta(\eta) > 0 \) such that
\[
\int_{E} |T_{h(\eta)}(u_{0,n})|^p \, dxdt + \int_{E} \frac{|T_{h(\eta)}(u_{0,n})|^{p-1}}{|x|^p} \, dxdt \leq \frac{\eta}{2} \quad \text{for} \quad \text{meas}(E) \leq \beta(\eta). \quad (4.30)
\]
Finally, by combining (4.28)–(4.30), we conclude that
\[
\int_{E} |T_n(u_{0,n})|^p \, dxdt + \int_{E} \frac{|T_n(u_{0,n})|^{p-1}}{|x|^p} \, dxdt \leq \eta, \quad \text{with} \quad \text{meas}(E) \leq \beta(\eta), \quad (4.31)
\]
then the sequences \( (|T_n(u_{0,n})|^{p-1}T_n(u_{0,n}))_n \) and \( \left( \frac{T_n(u_{0,n})|^{p-2}}{|x|^p} \right)_n \) are uniformly equi-integrable, and in view of Vitali’s Theorem, we deduce that
\[
|T_n(u_{0,n})|^{s-1}T_n(u_{0,n}) \to |u|^{s-1}u \quad \text{and} \quad \frac{|T_n(u_{0,n})|^{p-2}}{|x|^p} \to \frac{|u|^{p-2}u}{|x|^p} \quad \text{strongly in} \quad L^1(Q_T). \quad (4.32)
\]

### 4.1.6 Step 6 Convergence of the gradient

In the sequel, we denote by \( \varepsilon_{j}(n) \), \( j = 1, 2, \ldots \) some various functions of real numbers, which converge to 0 as \( n \) tends to infinity. Similarly, we define \( \varepsilon_{j}(h) \), \( \varepsilon_{j}(n, h) \) and \( \varepsilon_{j}(n, \mu, h) \).
Let $0 < k \leq h$ and $n$ large enough, and taking $S_h(u_n)T_k(u_n)$ as a test function in (4.3), we obtain

\[
\int_0^T \int_\Omega \partial_t S_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt + \int_\Omega S_h(u_n)\alpha(x, t, T_k(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt \\
+ \int_\Omega (T_k(u_n) - T_k(u))S''_h(u_n)\alpha(x, t, T_k(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt \\
+ \nu \int_\Omega |T_k(u_n)|^{p-1}T_k(u_n)S_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt \\
= \lambda \int_\Omega \frac{|T_k(u_n)|^{p-2}T_k(u_n)}{|x|^p + \frac{1}{n}} S_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt + \int_\Omega f_nS_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt.
\]

(4.33)

Since $S_h(u_n)(T_k(u_n) - T_k(u))$ has the same sign as $u_n$ on $\{|u_n| > k\}$, and in view of Young's inequality, we have

\[
\int_0^T \int_\Omega \partial_t S_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt + \int_\Omega S_h(u_n)\alpha(x, t, T_k(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt \\
+ \int_\Omega (T_k(u_n) - T_k(u))S''_h(u_n)\alpha(x, t, T_k(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt \\
+ \nu \int_\{|u_n| \leq k\} |T_k(u_n)|^{p-1}T_k(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt \\
\leq \lambda \int_\{|u_n| \leq k\} \frac{|T_k(u_n)|^{p-2}T_k(u_n)}{|x|^p + \frac{1}{n}} (T_k(u_n) - T_k(u)) \, dx \, dt + C_{13} \int_\{|u_n| > k\} \frac{|S_h(u_n)(T_k(u_n) - T_k(u))|}{|x|^p + \frac{1}{n}} \, dx \, dt \\
+ \int_\Omega f_nS_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt.
\]

(4.34)

For the first term on the right-hand side of (4.34), we have $\frac{\partial S_h(u_n)}{\partial t} - \frac{\partial S_h(u)}{\partial t}$ in $L^p(0, T; W^{-1,p}(\Omega)) + L^1(Q_T)$, then

\[
\liminf_{n \to \infty} \int_0^T \int_\Omega \partial_t S_h(u_n)(T_k(u_n) - T_k(u)) \, dx \, dt \\
= \liminf_{n \to \infty} \int_\Omega \frac{\partial S_h(u_n)}{\partial t} T_k(u_n) \, dx \, dt - \lim_{n \to \infty} \int_\Omega \frac{\partial S_h(u_n)}{\partial t} T_k(u) \, dx \, dt \\
= \liminf_{n \to \infty} \int_\Omega \frac{\partial S_h(u_n)}{\partial t} T_k(S_h(u_n)) \, dx - \int_\Omega \frac{\partial S_h(u)}{\partial t} T_k(S_h(u)) \, dx \\
= \int_\Omega \varphi_h(S_h(u_n(T))) \, dx - \int_\Omega \varphi_h(S_h(u(T))) \, dx \\
= \int_\Omega \varphi_h(S_h(u_n(T))) \, dx - \int_\Omega \varphi_h(S_h(u(T))) \, dx.
\]

Now, using $S_h(u_n)T_k(u_n)$ as a test function in (4.3) and from Young's inequality, we obtain
\[
\int_{0}^{r} \left( \frac{\partial S_{0}(u_{n})}{\partial t}, T_{0}(u_{n}) \right) dt + \int_{Q_{r}} S_{h}^{i}(u_{n}) T_{i}(u_{n}) \cdot a(x, t, T_{h}(u_{n}), \nabla u_{n}) \nabla u_{n} dx dt \\
+ \int_{Q_{r}} S_{h}^{i}(u_{n}) a(x, t, T_{h}(u_{n}), \nabla u_{n}) \cdot \nabla T_{h}(u_{n}) dx dt + \frac{\nu}{2} \int_{Q_{r}} |T_{h}(u_{n})|^{\nu-1} T_{h}(u_{n}) S_{h}^{i}(u_{n}) T_{h}(u_{n}) dx dt (4.36)
\]

\[
\leq C_{14} \int_{Q_{r}} \frac{1}{|x|^{\frac{n}{n-1}}} S_{h}^{i}(u_{n}) T_{i}(u_{n}) dx dt + \int_{Q_{r}} f_{n} S_{h}^{i}(u_{n}) T_{h}(u_{n}) dx dt,
\]

and since \( S_{h}^{i}(u_{n}) \geq 0 \), the third and fourth terms on the left-hand side of (4.36) are nonnegative. Thanks to (4.26), we have

\[
\int_{\Omega} \varphi_{h}(S_{h}(u_{n}(T))) dx - \int_{\Omega} \varphi_{h}(S_{h}(u_{0,n})) dx \leq k \| f \|_{L^{\nu}(Q_{T})} \left( \| f \|_{L^{\nu}(Q_{T})} + C_{14} T \left( \frac{1}{|x|^{\frac{n}{n-1}}} \right) \right) \\
+ k \| S_{h}^{i} \|_{L^{\nu}(R)} \int_{\{ |h| \leq |u_{n}| \leq h+1 \}} |a(x, t, T_{h+1}(u_{n}), \nabla T_{h+1}(u_{n}))| \varnothing \nabla T_{h+1}(u_{n})| dx dt \leq k C_{15},
\]

with \( C_{15} \) being a constant that does not depend on \( n \), then

\[
\int_{\Omega} \varphi_{h}(S_{h}(u_{n}(T))) dx \leq k C_{15} + \int_{\Omega} \varphi_{h}(S_{h}(u_{0})) dx.
\]

We have \( \varphi_{h}(S_{h}(u_{n}(T))) \geq 0 \) and \( \varphi_{h}(S_{h}(u_{n}(T))) \to \varphi_{h}(S_{h}(u(T))) \) a.e. in \( \Omega \) thanks to Fatou’s Lemma, we have

\[
\int_{\Omega} \varphi_{h}(S_{h}(u(T))) dx \leq \liminf_{n \to \infty} \int_{\Omega} \varphi_{h}(S_{h}(u_{n}(T))) dx,
\]

and in view of (4.35), we deduce that

\[
\liminf_{n \to \infty} \int_{Q_{r}} \partial_{t} S_{h}(u_{n})(T_{h}(u_{n}) - T_{h}(u)) dx dt = \liminf_{n \to \infty} \int_{\Omega} \varphi_{h}(S_{h}(u_{n}(T))) dx - \int_{\Omega} \varphi_{h}(S_{h}(u(T))) dx \geq 0,
\]

then

\[
\int_{Q_{r}} \partial_{t} S_{h}(u_{n})(T_{h}(u_{n}) - T_{h}(u)) dx dt \geq \varepsilon_{3}(n). (4.37)
\]

Concerning the second term on the left-hand side of (4.34), we have \( S_{h}^{i}(s) \geq 0 \) and \( S_{h}^{i}(s) = 1 \) for \( |s| \leq k \), with \( \text{supp}(S_{h}^{i}) \subset [-h-1, h+1] \), then

\[
\int_{Q_{r}} S_{h}^{i}(u_{n}) a(x, t, u_{n}, \nabla u_{n}) \cdot (\nabla T_{h}(u_{n}) - \nabla T_{h}(u)) dx dt \\
= \int_{Q_{r}} a(x, t, T_{h}(u_{n}), \nabla T_{h}(u_{n})) \cdot (\nabla T_{h}(u_{n}) - \nabla T_{h}(u)) dx dt \\
- \int_{\{|k| \leq |u_{n}| \leq h+1\}} S_{h}^{i}(u_{n}) a(x, t, T_{h+1}(u_{n}), \nabla T_{h+1}(u_{n})) \cdot \nabla T_{h}(u) dx dt \\
\geq \int_{Q_{r}} (a(x, t, T_{h}(u_{n}), \nabla T_{h}(u_{n})) - a(x, t, T_{h}(u_{n}), \nabla T_{h}(u))) \cdot (\nabla T_{h}(u_{n}) - \nabla T_{h}(u)) dx dt \\
+ \int_{Q_{r}} a(x, t, T_{h}(u_{n}), \nabla T_{h}(u)) \cdot (\nabla T_{h}(u_{n}) - \nabla T_{h}(u)) dx dt \\
- \| S_{h}^{i} \|_{L^{\nu}(R)} \int_{\{|k| \leq |u_{n}| \leq h+1\}} a(x, t, T_{h+1}(u_{n}), \nabla T_{h+1}(u_{n})) \cdot \nabla T_{h}(u) dx dt. (4.38)
\]
We have \( a(x, t, T_k(u_n), \nabla T_k(u_n)) \to a(x, t, T_k(u), \nabla T_k(u)) \) in \((L^p(Q_T))^N\), and since \( \nabla T_k(u_n) \to \nabla T_k(u) \) in \((L^q(Q_T))^N\), then
\[
\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{4.39}
\]

For the last term on the right-hand side of (4.38), we have \( a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \) being bounded in \((L^p(Q_T))^N\), then there exists \( \xi_h \in (L^p(Q_T))^N \) such that \( a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \to \xi_h \) weakly in \((L^p(Q_T))^N\). Therefore,
\[
\int_{Q_T} a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \cdot \nabla T_k(u) \, dx \, dt \to \int_{\{ |u| > k \}} \xi_h \cdot \nabla T_k(u) \, dx \, dt = 0. \tag{4.40}
\]

By combining (4.38)–(4.40), we deduce that
\[
\int_{Q_T} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt \leq \int_{Q_T} \mathbb{S}_h(u_n) a(x, t, u_n, \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt + \varepsilon_2(n). \tag{4.41}
\]

For the third and fourth terms on the left-hand side of (4.34), thanks to (4.26), we have \( \text{supp}(\mathbb{S}_h^n) \subset [-h - 1, -h] \cup [h, h + 1] \) and
\[
\varepsilon_3(n, h) \leq \| \mathbb{S}_h^n \|_{L^\infty(\mathbb{R})} \int_{\{ |h| - |u_n| \leq h + 1 \}} |T_k(u_n) - T_k(u)| a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \cdot \nabla T_{h+1}(u_n) \, dx \, dt \leq 2k \| \mathbb{S}_h^n \|_{L^\infty(\mathbb{R})} \int_{\{ |h| - |u_n| \leq h + 1 \}} a(x, t, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) \cdot \nabla T_{h+1}(u_n) \, dx \, dt \to 0 \quad \text{as} \quad h \to \infty. \tag{4.42}
\]

Moreover, it is clear that
\[
\varepsilon_4(n) = \int_{\{ |u_n| \leq k \}} |T_k(u_n)|^{p-1} |T_k(u_n) - T_k(u)| \, dx \, dt \leq k^{p-1} \int_{\{ |u_n| \leq k \}} |T_k(u_n) - T_k(u)| \, dx \, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{4.43}
\]

For the terms on the right-hand side of (4.34), since \( T_k(u_n) - T_k(u) \to 0 \) weak-* in \( L^\infty(Q_T) \), then
\[
\varepsilon_5(n) = \int_{\{ |u_n| \leq k \}} |T_k(u_n)|^{p-1} |T_k(u_n) - T_k(u)| \, dx \, dt \leq k^{p-1} \int_{Q_T} |T_k(u_n) - T_k(u)| \, dx \, dt \to 0 \quad \text{as} \quad n \to \infty, \tag{4.44}
\]

and
\[
\varepsilon_6(n) = \int_{\{ |u_n| > k \}} |\mathbb{S}_h^n(u_n)(T_k(u_n) - T_k(u))| \, dx \, dt \leq \| \mathbb{S}_h^n \|_{L^\infty(\mathbb{R})} \int_{\{ |u_n| > k \}} |T_k(u_n) - T_k(u)| \, dx \, dt \to 0 \quad \text{as} \quad n \to \infty. \tag{4.45}
\]
\[
\varepsilon(n) = \left| \int_{Q_T} f_n S_n(u_n)(T_h(u_n) - T_h(u)) \, dx \, dt \right| \\
\leq \|S_n\|_{L^\infty(R)} \int_{Q_T} |f_n| |T_h(u_n) - T_h(u)| \, dx \, dt 
\longrightarrow 0 \quad \text{as } n \to \infty. 
\] (4.46)

Combining (4.37) and (4.41)–(4.46), we deduce that
\[
\int_{Q_T} (a(x, t, T_h(u_n)), \nabla T_h(u_n)) - a(x, t, T_h(u_n), \nabla u_n) \cdot (\nabla T_h(u_n) - \nabla T_h(u)) \, dx \, dt \\
\leq \int_{Q_T} S_n(u_n) a(x, t, u_n, \nabla u_n) \cdot (\nabla T_h(u_n) - \nabla T_h(u)) \, dx \, dt + \varepsilon_s(n, h) \\
\leq \varepsilon_s(n, h). 
\] (4.47)

By passing \( n \) and \( h \) to infinity, we obtain
\[
\lim_{n \to \infty} \int_{Q_T} (a(x, t, T_h(u_n)), \nabla T_h(u_n)) - a(x, t, T_h(u_n), \nabla u_n) \cdot (\nabla T_h(u_n) - \nabla T_h(u)) \, dx \, dt = 0. 
\] (4.48)

In view of Lemma 3.3 in [15], we deduce that
\[
T_h(u_n) \to T_h(u) \quad \text{in } L^p(0, T; W^{1, p}_0(\Omega)), \quad \text{then } \nabla u_n \to \nabla u \quad \text{a.e. in } Q_T. 
\] (4.49)

### 4.1.7 Step 7 The convergence of \( u_n \) in \( C([0, T]; L^1(\Omega)) \)

Let \( h \geq 1 \). For \( 0 < s \leq T \), by taking \( T_h(u_n) - (T_h(u))_\mu \chi_{[0,s]} \) as a test function in (4.3), we obtain
\[
\int_0^s \int_\Omega \frac{\partial u_n}{\partial t} (T_h(u_n) - (T_h(u))_\mu) \, dx \, dt + \int_0^s \int_\Omega a(x, t, T_h(u_n), \nabla u_n) \cdot \nabla T_h(u_n) - (T_h(u))_\mu \, dx \, dt \\
+ \nu \int_0^s \int_\Omega |T_h(u_n)|^{p-2} T_h(u_n) T_h(u_n) - (T_h(u))_\mu \, dx \, dt \\
= \lambda \int_0^s \int_\Omega |T_h(u_n)|^{p-2} T_h(u_n) T_h(u_n) - (T_h(u))_\mu \, dx \, dt + \int_0^s \int_\Omega f_n (T_h(u_n) - (T_h(u))_\mu) \, dx \, dt. 
\] (4.50)

We have
\[
\frac{\partial u_n}{\partial t} = \frac{\partial}{\partial t} (u_n - (T_h(u))_\mu) + \frac{\partial (T_h(u))_\mu}{\partial t} = \frac{\partial}{\partial t} (u_n - (T_h(u))_\mu) + \mu (T_h(u) - (T_h(u))_\mu),
\]

it follows that
\[
\int_0^s \int_\Omega \frac{\partial u_n}{\partial t} (T_h(u_n) - (T_h(u))_\mu) \, dx \, dt = \int_0^s \int_\Omega \frac{\partial}{\partial t} (u_n - (T_h(u))_\mu) T_h(u_n) - (T_h(u))_\mu \, dx \, dt \\
+ \mu \int_0^s \int_\Omega (T_h(u) - (T_h(u))_\mu) T_h(u_n) - (T_h(u))_\mu \, dx \, dt. 
\] (4.51)

Observe that, for every \( s \in [0, T] \), by letting \( n \) tends to infinity, we obtain
Concerning the second term on the left-hand side of (4.50), we have
\[
\int_{\Omega} \int_{0}^{s} \sigma(x, t, T\theta n(u_n), \nabla u_n) \cdot \nabla T\theta(u_n) - (T\theta(u))_\mu) dx dt = \int_{\Omega} \int_{0}^{s} \sigma(x, t, T\theta n(u_n), \nabla u_n) \cdot \nabla T\theta(u_n) - (T\theta(u))_\mu) dx dt
\]
\[
= \int_{\Omega} \int_{0}^{s} \sigma(x, t, T\theta n(u_n), \nabla u_n) \cdot \nabla T\theta(u_n) - (T\theta(u))_\mu) dx dt \quad \text{as } n, \mu \to \infty
\]
\[
= \int_{\{b \leq |u| \leq b+1\}} \sigma(x, t, T\theta n(u_n), \nabla u_n) \cdot \nabla u dx dt = 0.
\]
For the third term on the left-hand side of (4.50), thanks to (4.32), and since \( T\theta(u_n) - (T\theta(u))_\mu \to T\theta(u) - T\theta(u) \) weakly in \( L^\infty(Q_T) \), we have
\[
\int_{\Omega} \int_{0}^{s} |T\theta(u_n)|^{s-1} T\theta(u_n) T\theta(u) - (T\theta(u))_\mu) dx dt \quad \text{as } n, \mu \to \infty.
\]
Moreover, we have
\[
\int_{\Omega} \int_{0}^{s} \frac{|T\theta(u_n)|^{p-1}}{|x|^p + \frac{1}{n}} |T\theta(u_n) - (T\theta(u))_\mu| dx dt \quad \text{as } n, \mu \to \infty.
\]
On the other hand, since \( f_n \to f \) strongly in \( L^1(Q_T) \), we have
\[
\int_{\Omega} \int_{0}^{s} |f_n| T\theta(u_n) - (T\theta(u))_\mu) dx dt \quad \text{as } n, \mu \to \infty.
\]
By combining (4.50)–(4.57), we have
\[
\int \varphi(u_0(s) - (T\theta(u(s)))_\mu) dx + \int_{0}^{s} |u|^{p-1} u T\theta(u - T\theta(u)) dx dt
\]
\[
\leq \varepsilon(n, \mu) + \int_{0}^{s} \frac{|u|^{p-1}}{|x|^p} |T\theta(u - T\theta(u))| dx dt + \int_{0}^{s} |f| |T\theta(u - T\theta(u))| dx dt + \int \varphi(u_0 - T\theta(u_0)) dx.
\]
Concerning the second term on the left-hand side of (4.58), it is clear that
\[ \int_0^s \int_{\Omega} |u|^p u T_h(u - T_h(u)) \, dx \, dt \to 0 \quad \text{as} \quad h \to \infty. \tag{4.59} \]

On the other hand, for the three last terms on the right-hand side of (4.58), we have
\[ \int_0^s \int_{\Omega} \frac{|u|^p - |T_h(u - T_h(u))|}{|x|^p} \, dx \, dt + \int_0^s \int_{\Omega} |f| |T_h(u - T_h(u))| \, dx \, dt + \int_{\Omega} \varphi_1(u_0 - T_h(u_0)) \, dx \to 0 \quad \text{as} \quad h \to \infty. \tag{4.60} \]

By combining (4.58)–(4.60), we conclude that
\[ \int_{\Omega} \varphi_1(u_0(s) - (T_h(u(s)))_n) \, dx \leq \epsilon(n, \mu, h). \tag{4.61} \]

On the other hand, we have
\[ \int_{\Omega} \varphi_1\left( \frac{u_0(s) - u_m(s)}{2} \right) \, dx \leq \frac{1}{2} \left( \int_{\Omega} \varphi_1\left( (u_0(s) - (T_h(u(s)))_n) \right) \, dx + \int_{\Omega} \varphi_1\left( (u_m(s) - (T_h(u(s)))_n) \right) \, dx \right) \to 0 \quad \text{as} \quad n, m \to \infty. \tag{4.62} \]

We have
\[ \int_{\{|u_0(s) - u_m(s)| \leq 2\}} \left| \frac{u_0(s) - u_m(s)}{2} \right|^2 \, dx + \int_{\{|u_0(s) - u_m(s)| > 2\}} \left| \frac{u_0(s) - u_m(s)}{2} \right| \, dx \leq 2 \int_{\Omega} \varphi_1\left( \frac{u_0(s) - u_m(s)}{2} \right) \, dx, \tag{4.63} \]
and
\[ \int_{\Omega} |u_0(s) - u_m(s)| \, dx = \int_{\{|u_0(s) - u_m(s)| \leq 2\}} |u_0(s) - u_m(s)| \, dx + \int_{\{|u_0(s) - u_m(s)| > 2\}} |u_0(s) - u_m(s)| \, dx \leq \left( \int_{\{|u_0(s) - u_m(s)| \leq 2\}} |u_0(s) - u_m(s)|^2 \, dx \right)^{\frac{1}{2}} \text{meas}(\Omega)^{\frac{1}{2}} \tag{4.64} \]
\[ + \int_{\{|u_0(s) - u_m(s)| > 2\}} |u_0(s) - u_m(s)| \, dx. \]

According to (4.61)–(4.64), we deduce that
\[ \int_{\Omega} |u_0(s) - u_m(s)| \, dx \to 0 \quad \text{as} \quad m, n \to \infty. \tag{4.65} \]

Hence, \((u_n)_n\) is a Cauchy sequence in \(C([0, T]; L^1(\Omega))\), thus \(u \in C([0, T]; L^1(\Omega))\), and for \(0 \leq s \leq T\), we have \(u_0(s) \to u(s)\) in \(L^1(\Omega)\).

### 4.1.8 Step 8 Passage to the limit

Let \( \psi \in L^p(0, T; W^k_0(\Omega)) \cap L^\infty(Q_T) \) with \( \frac{\partial \psi}{\partial t} \in L^p(0, T; W^{-1,p'}(\Omega)) + L^1(Q_T) \), and \( M = k + \| \psi \|_{L^1(Q_T)} \) with \( k > 0 \).

By taking \( T_k(u_n - \psi) \) as a test function in (4.3), we obtain
\[
\int_0^T \left( \frac{\partial u_n}{\partial t}, T_k(u_n - \psi) \right) \, dt + \int \mathcal{Q}_T a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \psi) \, dx \, dt \\
+ \nu \int_0^T |T_k(u_n)|^{p-1} T_k(u_n - \psi) \, dx \\
= \lambda \int \frac{|T_k(u_n)|^{p-2} T_k(u_n)}{|x|^p + \frac{1}{n}} T_k(u_n - \psi) \, dx \, dt + \int \mathcal{Q}_T T_k(u_n - \psi) \, dx. 
\]

(4.66)

If \(|u_n| > M\), then \(|u_n - \psi| \geq |u_n| - \|\psi\|_\infty > k\), therefore \(|u_n - \psi| \leq k\) \(\subseteq \{|u_n| \leq M\}\), which implies that

\[
\int \mathcal{Q}_T a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n - \psi) \, dx \, dt \\
\geq \int \mathcal{Q}_T a(x, t, T_n(u_n), \nabla T_k(u_n)) - a(x, t, T_n(u_n), \nabla \psi) (\nabla T_k(u_n) - \nabla \psi) \, dx \, dt \\
+ \int \mathcal{Q}_T a(x, t, T_n(u_n), \nabla \psi) (\nabla T_k(u_n) - \nabla \psi) \, dx \, dt \\
\geq \int \mathcal{Q}_T a(x, t, u, \nabla u) \cdot \nabla T_k(u - \psi) \, dx \, dt. 
\]

(4.67)

On the other hand, for the first term on the left-hand side of (4.66), we have \(\frac{\partial u_n}{\partial t} = \frac{\partial (u_n - \psi)}{\partial t} + \frac{\partial \psi}{\partial t}\), then

\[
\int_0^T \left( \frac{\partial u_n}{\partial t}, T_k(u_n - \psi) \right) \, dt = \int_0^T \left( \frac{\partial (u_n - \psi)}{\partial t}, T_k(u_n - \psi) \right) \, dt + \int_0^T \left( \frac{\partial \psi}{\partial t}, T_k(u_n - \psi) \right) \, dt \\
= \int_\Omega \varphi_k(u_n(T) - \psi(T)) \, dx - \int_\Omega \varphi_k(u_{0,n} - \psi(0)) \, dx + \int_\Omega \frac{\partial \psi}{\partial t} T_k(u_n - \psi) \, dx \, dt, 
\]

and since \(u_n \to u\) in \(C([0, T]; L^4(\Omega))\), then \(u_n(T) \to u(T)\) in \(L^4(\Omega)\), it follows that

\[
\int_\Omega \varphi_k(u_{0,n} - \psi(0)) \, dx \longrightarrow \int_\Omega \varphi_k(u_0 - \psi(0)) \, dx, 
\]

(4.68)

and

\[
\int_\Omega \varphi_k(u_n(T) - \psi(T)) \, dx \longrightarrow \int_\Omega \varphi_k(u(T) - \psi(T)) \, dx. 
\]

(4.69)

Moreover, we have \(\frac{\partial \psi}{\partial t} \in L^p(0, T; W^{-1,p'}(\Omega)) \cup L^q(\mathcal{Q}_T)\), and \(T_k(u_n - \psi) \rightharpoonup T_k(u - \psi)\) weakly in \(L^p(0, T; W_0^{1,p}(\Omega))\) and weak-\(\ast\) in \(L^{\infty}(\mathcal{Q}_T)\)
\[
\int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u_n - \psi) \, dx \, dt \to \int_{Q_T} \frac{\partial \psi}{\partial t} T_k(u - \psi) \, dx \, dt,
\]
(4.70)

and
\[
\int_{Q_T} f_n T_k(u_n - \psi) \, dx \, dt \to \int_{Q_T} f T_k(u - \psi) \, dx \, dt.
\]
(4.71)

Having in mind (4.32), we deduce that
\[
\int_{Q_T} |T_n(u_n)|^{p-1} T_n(u_n - \psi) \, dx \, dt \to \int_{Q_T} |u|^{p-1} u T_k(u - \psi) \, dx \, dt,
\]
(4.72)

and
\[
\int_{Q_T} \frac{|T_n(u_n)|^{p-2} T_n(u_n)}{|x|^p + \frac{1}{n}} T_k(u_n - \psi) \, dx \, dt \to \int_{Q_T} \frac{|u|^{p-2} u}{|x|^p} T_k(u - \psi) \, dx \, dt.
\]
(4.73)

By combining (4.66)–(4.73), we deduce that
\[
\int_{\Omega} \phi_k(T(x) - \psi(x)) \, dx - \int_{\Omega} \phi_k(u_0 - \psi(0)) \, dx + \int_{Q_T} \frac{\partial \phi}{\partial t} T_k(u - \psi) \, dx \, dt
\]
\[
+ \int_{Q_T} a(x, t, u, \nabla u) \cdot \nabla T_k(u - \psi) \, dx \, dt + \lambda \int_{Q_T} |u|^{p-2} u T_k(u - \psi) \, dx \, dt
\]
\[
\leq \lambda \int_{Q_T} \frac{|u|^{p-2} u}{|x|^p} T_k(u - \psi) \, dx \, dt + \int_{Q_T} f T_k(u - \psi) \, dx \, dt,
\]

which concludes the proof Theorem 4.2.

Acknowledgement: The authors would like to thank the anonymous referees for their constructive comments and valuable suggestions, which are helpful to improve the quality of this article.

Funding information: The authors declare that they have no financial interest.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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