Research article

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On the time evolution at a fluctuating exceptional point

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Abstract: We theoretically evaluate the impact of drift-free noise on the dynamics of $\mathcal{PT}$-symmetric non-Hermitian systems with an exceptional point, which have recently been proposed for sensors. Such systems are currently considered as promising templates for sensing applications, because of their intrinsically extremely sensitive response to external perturbations. However, this applies equally to the impact of fabrication imperfections and fluctuations in the system parameters. Here we focus on the influence of such fluctuations caused by inevitable (thermal) noise and show that the exceptional-point eigenstate is not stable in its presence. To this end, we derive an effective differential equation for the mean time evolution operator averaged over all realizations of the noise field, and via numerical analysis we find that the presence of noise leads to exponential divergence of any initial state after some characteristic period of time. We therefore show that it is rather demanding to design sensor systems based on continuous operation at an exceptional point.

Keywords: $\mathcal{PT}$-symmetric systems; exceptional point; thermal noise.

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1 Introduction

In the recent study of parity-time ($\mathcal{PT}$)-symmetric non-Hermitian dynamic systems [1–3], the notion of exceptional points [4–9] has attracted particular interest, e.g. for the realization of highly sensitive sensors [10]. A typical realization of such a system would consist of coupled oscillators (e.g. evanescently coupled optical resonators) where one oscillator is subject to gain and the other to an equal amount loss [11]. An exceptional point in the space formed by the parameters "gain" and "coupling strength" is characterized by the fact that not only two (or even more) eigenvalues are degenerate but that also their eigen*states* coalesce. In this sense it bears great similarity to the critically damped harmonic oscillator, which is the optimal operating point for various types of sensing equipment such as galvanometers [12]. Similarly, a coupled-oscillator system operated at an exceptional point is in itself particularly well suited for sensing applications, because any small perturbation Δ leads to a splitting of the eigenvalues that scales with the square root of Δ [11, 13]. In other words, exceptional points promise the design of extremely sensitive sensor configurations.

However, this extreme sensitivity is as much of a curse for practical purposes as it is a blessing, because even minuscule deviations of the operating point move the system away from the exceptional point and thus diminish the high sensitivity. It is clear that both imperfections during manufacturing as well as drift of the operating point are of major concern and that they must be compensated by the introduction of an active stabilization of the operation point via a feedback amplifier similar to chopper-stabilized operational amplifiers. Ideally, this would involve operating the sensor in its stationary state at the exceptional point. However, since the time evolution of such systems includes a linearly growing contribution [14–16], it is not clear from the outset how the system would react to inevitable noise. In this context, we define drift as any fluctuation that is eliminated by a stabilization circuit and refer to the remaining fluctuations as noise in a strict sense. It should be stressed that by this definition, the noise spectrum has a low-frequency gap around the value $\omega=0$, which will become crucial for the following. Furthermore, we remark that any stabilization circuit will itself contribute noise.

The problem of imperfections and noise on the performance of hypothetical exceptional-point sensors has been addressed before in recent literature. One major contribution to this was the realization that while systems at an exceptional point do exhibit a square root law for the eigenvalue splitting in response to small perturbations, this does not lead to an improvement in the ratio between the signal and the fundamental quantum noise level [17]. In other words, it has been shown that an exceptional point does not provide a benefit in the quantum-noise limited regime, although it might make it easier to reach this limit. A different angle was approached by our group in a recent paper on the effect of sample-to-sample variations [18], where we discuss (among other things) how drift in the system parameters leads to an exponentially growing error in the state of an exceptional point sensor. In this paper, we go beyond this preexisting work and show that eliminating drift does not eliminate this divergent behavior. We show that drift-free fluctuations in the system parameters (e.g. the site detuning due to inevitable thermal fluctuations of the resonator geometry or fluctuations in the gain) around the exceptional point lead to an exponential divergence of the state error. As a consequence, it is impossible to operate any real-world system at the exceptional point for a large period of time.

The paper is structured as follows: In Section 2, we introduce the problem and define our notation and conventions. In Section 3, we present the ordinary differential equation that describes the time evolution of the noisy system averaged over all realizations of the noise field. In Section 4, we present a brief summary of the analytical derivation that leads to this differential equation and compare its solutions to brute-force numerical calculations for some particular realizations of the noise spectrum. In Section 5, we discuss the consequences of our findings for the design and feasibility of optical sensors based on exceptional-point dynamics, and discuss the prospects of exceptional point-based sensing in view of our analysis. After Section 6, the Conclusion, the paper ends with two appendices with additional details about the solution and its numerical implementation.

2 Preliminaries

We study the time evolution of a two-site $\mathcal{PT}$-symmetric system at an exceptional point. Within a coupled-mode picture, the dynamics of any such system is described by the equation

$$\mathrm{i}\partial_t \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} \omega - \mathrm{i}g & \kappa \\ \kappa & \omega + \mathrm{i}g \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}, \tag{1}$$

where $a_1(t)$ and $a_2(t)$ are the complex amplitudes of the respective resonator modes, ω is their common eigenfrequency, g is the gain or loss (depending on the sign) that they are subjected to and κ is the coupling constant, which was chosen to be real by an appropriate choice for the relative phase between the modes. Next, we switch to a frame of reference that rotates with the phase $\exp(-\mathrm{i}\omega)$ and introduce a rescaled time variable $\tau=\kappa t$. The latter means that we measure time in units of the inverse coupling constant. Thus, we then find the equation of motion

$$\mathrm{i}\partial_\tau \psi(\tau) = \begin{pmatrix} -\mathrm{i}g/\kappa & 1 \\ 1 & \mathrm{i}g/\kappa \end{pmatrix} \psi(\tau), \tag{2}$$

where the state vector $\psi(\tau)=\exp(-\mathrm{i}\omega t/\kappa)[a_1(t), a_2(t)]^T$ comprises the mode amplitudes in the rotating frame. This system has an exceptional point for $g=\kappa$. The exceptional-point dynamics of every two-site $\mathcal{PT}$-symmetric system can be thus reduced to the prototypical Hamiltonian

$$\mathcal{H}_0 = \begin{pmatrix} -\mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix} = \sigma_x - \mathrm{i}\sigma_z, \tag{3}$$

where σ_i denotes the Pauli matrices, and the time evolution of a state $\psi(\tau)$ of this ideal system is given by a Schrödinger-type equation

$$\partial_\tau \psi(\tau) = -\mathrm{i}\mathcal{H}\psi(\tau), \tag{4}$$

with respect to the rescaled time variable τ of the transformed system.

We now assume that the operating point of the system is perturbed by some time-dependent real-valued fluctuation $\Delta(\tau)$, which can be represented as a Fourier integral

$$\Delta(\tau) = \int_{-\infty}^{\infty} \mathrm{d}\omega\; b(\omega)\exp(-\mathrm{i}\omega\tau). \tag{5}$$

The phase of the function $b(\omega)$ is assumed to fluctuate randomly and arbitrarily quickly in ω while its modulus is a smooth function of ω [19]. It is connected to the fluctuation power spectrum $P(\omega)$ (again in appropriately chosen dimensionless units):

$$P(\omega) = |b(\omega)|^2. \tag{6}$$

We assume the overall fluctuation power to be finite, so $P(\omega)$ must cut off at high frequencies. Furthermore, we distinguish between low-frequency and quasi-static fluctuations, which we call drift, and high-frequency fluctuations, which we call noise. The former are assumed to be eliminated by an active stabilization circuit, with only the latter remaining. In other words, we assume that the relevant fluctuation field $\Delta(\tau)$ vanishes in a neighborhood of $\omega=0$, if only to prevent the system from permanently drifting away from the exceptional point. We assume

$$P(\omega)=0 \quad \text{for} \quad |\omega|<\omega_{\min}. \tag{7}$$

The key characteristic of the noise function within this paper is the auto-correlation function $\Gamma(\tau)$. Since the auto-correlation of a white noise is a sharp peak at $\tau=0$, we approximate it as a Dirac distribution:

$$\Gamma(\tau)=\int_{-\infty}^{\infty} \mathrm{d}\tau'\Delta(\tau')\Delta(\tau'+\tau)\approx\gamma\delta(\tau), \tag{8}$$

with some constant γ, which is formally given by $\gamma=\iint_{-\infty}^{\infty}\mathrm{d}\tau'\mathrm{d}\tau\Delta(\tau')\Delta(\tau'+\tau)$.

In the following, we assume that the fluctuation field detunes the on-site energies of the two coupled sites; i.e. we introduce a perturbation operator $\mathcal{V}(\tau)=\Delta(\tau)\sigma_z$. A fluctuation in the gain and loss parameters of the two-site problem can be described by a second operator $\mathcal{V}'(\tau)=\mathrm{i}\Delta'(\tau)\sigma_z$ generated by a second fluctuation field $\Delta'(\tau)$. This leads to results that differ from the ones obtained for $\mathcal{V}(\tau)$ by only the imaginary unit, and (assuming no correlations between $\Delta(\tau)$ and $\Delta'(\tau)$) their respective corrections to the total time evolution can be simply added. Therefore, it is sufficient to study the problem of fluctuating on-site energy detuning:

$$\mathcal{H}(\tau)=\mathcal{H}_0+\Delta(\tau)\sigma_z. \tag{9}$$

This type of problem requires heavy use of nested time integrals for which we introduce a short-hand notation:

$$\int_0^\tau\cdots\mathrm{d}\tau_{\{n\}}=\int_0^\tau\mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\cdots\int_0^{\tau_{n-1}}\mathrm{d}\tau_n.$$

3 Main result

The topic of this study is the derivation of the mean time evolution of an exceptional-point system in the presence of a noisy perturbation to the system parameters, averaged over all possible realizations of the noise field. The natural description for the dynamics of such a system is the time evolution operator $\bar{\mathcal{U}}(\tau)$, which we find to satisfy the ordinary initial value problem

$$\bar{\mathcal{U}}''(\tau)=-\gamma[1+\mathrm{i}\mathcal{H}_0^\dagger\tau]\bar{\mathcal{U}}'(\tau)+2\gamma[\sigma_z-\tau]\bar{\mathcal{U}}(\tau), \tag{10a}$$

$$\bar{\mathcal{U}}'(0)=-\mathrm{i}\mathcal{H}_0, \tag{10b}$$

$$\bar{\mathcal{U}}(0)=1, \tag{10c}$$

where τ is the dimensionless time variable, $\mathcal{H}_0^\dagger$ is the adjoint of the (non-Hermitian) Hamiltonian at the unperturbed exceptional point and γ is the parameter of the noise autocorrelation function as introduced in Eq. (8).

The solution to Eqs. (10a–10c) at first follows the noiseless dynamics, which means that the norm of the exceptional-point eigenstate remains constant and the norm of non-eigenstates grows linearly. After some characteristic time the system enters a new regime, where the norm of any initial state grows exponentially, which can be expressed to very good accuracy by the equation

$$|\Delta\psi(\tau)|\approx\exp(\sqrt{2\gamma}\tau-1). \tag{11}$$

This means that a system with noisy system parameters can be operated at the exceptional point for a time no longer than $\tau_{\max}=1/\sqrt{2\gamma}$. At this point, any active stabilization circuit will kick in and try to stabilize the norm of the state, moving one system parameter away from the exceptional point. Depending on the characteristics of the feedback circuit, the system will either settle at this new equilibrium point or the feedback circuit will become unstable and enter oscillations. The growing sensitivity close to the exceptional point suggests that a reduction in noise (and hence in distance between the equilibrium operating point and the exceptional point) comes at the price of increased tendency for oscillation of the system parameters (detuning, gain or loss).

4 Sketch of the derivation

The time evolution of a state including a fluctuating perturbation of the exceptional point is described by the equation

$$\partial_\tau\psi(\tau)=-\mathrm{i}\mathcal{H}(\tau)\psi(\tau)=-\mathrm{i}[\mathcal{H}_0+\Delta(\tau)\sigma_z]\psi(\tau), \tag{12}$$

which, in analogy to the treatment within the Heisenberg representation in quantum mechanics [20, 21], is conventionally solved by the time evolution operator given as a Neumann series

$$\mathcal{U}(\tau)=1+\sum_{n=1}^{\infty}\mathcal{T}^{(n)}(\tau), \tag{13}$$

where each summand $\mathcal{T}^{(n)}(\tau)$ covers all terms of order τ^n:

$$\mathcal{T}^{(n)}(\tau)=\int_0^\tau\cdots\mathrm{d}\tau_{\{n\}}\prod_{j=1}^{n}(-\mathrm{i})[\mathcal{H}_0+\Delta(\tau_j)\sigma_z]. \tag{14}$$

With this, the time evolution operator satisfies the Dyson series-like [20] self-consistent equation

$$\begin{aligned}\mathcal{U}(\tau)=1-\mathrm{i}\mathcal{H}_0\tau-\mathrm{i}\int_0^\tau\cdots\mathrm{d}\tau_{\{2\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\\ \times\sigma_z\Delta(\tau_2)\mathcal{U}(\tau_2).\end{aligned} \tag{15}$$

This expansion is ill suited for our goal to derive the mean time evolution operator $\bar{\mathcal{U}}(\tau)$ averaged over all realizations of the noise function $\Delta(\tau)$. This is because it does not separate the odd powers of the perturbation $\mathcal{V}(\tau)$, which all average to 0, from the even powers, whose averages have finite values (see Appendix A). Therefore, we reorder the series to find the equivalent equation

$$\begin{aligned}\mathcal{U}(\tau)=1-\mathrm{i}\mathcal{H}_0\tau-\mathrm{i}\int_0^\tau\cdots\mathrm{d}\tau_{\{2\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\\ \times\sigma_z\Delta(\tau_2)[1-\mathrm{i}\mathcal{H}_0\tau_2]\\ -\int_0^\tau\cdots\mathrm{d}\tau_{\{4\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\sigma_z\Delta(\tau_2)\\ \times[\delta(\tau_3-\tau_4)-\mathrm{i}\mathcal{H}_0]\sigma_z\Delta(\tau_4)\mathcal{U}(\tau_4).\end{aligned} \tag{16}$$

The solution to this equation depends critically on the precise function $\Delta(\tau)$, which is of course unknown. Instead, we only know statistical properties such as the power spectrum moments and the autocorrelation function [Eq. (8)]. Therefore, the natural quantity to investigate is the average $\bar{\mathcal{U}}(\tau)$ of the evolution operator $\mathcal{U}(\tau)$ over all realizations of $\Delta(\tau)$. Next, we assume that the different realizations of $\Delta(\tau)$ are in fact just an explicit dependence on the absolute start point τ_0 of the time evolution. In other words, a change of the realization of the noise function $\Delta(\tau)$ is equivalent to a shift of the temporal origin $\Delta(\tau)\to\Delta(\tau+\tau_0)$. The evolution operator associated with start time τ_0 satisfies the following:

$$\begin{aligned}\mathcal{U}_{\tau_0}(\tau)=1-\mathrm{i}\mathcal{H}_0\tau-\overbrace{\mathrm{i}\int_0^\tau\cdots\mathrm{d}\tau_{\{2\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\sigma_z\Delta(\tau_2-\tau_0)[1-\mathrm{i}\mathcal{H}_0\tau_2]}^{\text{averages to zero when integrated over }\tau_0}\\ -\int_0^\tau\cdots\mathrm{d}\tau_{\{4\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\sigma_z\Delta(\tau_2-\tau_0)[\delta(\tau_3-\tau_4)-\mathrm{i}\mathcal{H}_0]\sigma_z\Delta(\tau_4-\tau_0)\mathcal{U}_{\tau_0}(\tau_4),\end{aligned} \tag{17}$$

and the realization-averaged evolution operator $\bar{\mathcal{U}}(\tau)$ is given as an integral over τ_0:

$$\bar{\mathcal{U}}(\tau)=\int_{-\infty}^{\infty}\mathrm{d}\tau_0\mathcal{U}_{\tau_0}(\tau). \tag{18}$$

To interpret the realization dependence as an explicit time dependence may seem a bit unconventional at first, but it is just the reverse of the usual strategy in statistical physics to replace time averages with ensemble averages assuming quasi-ergodicity. Assuming that the fluctuation function $\Delta(\tau)$ is the result of a (e.g. thermodynamic) process that satisfies quasi-ergodicity justifies our choice.

Next, we substitute Eq. (18) in Eq. (17) and replace $\mathcal{U}_{\tau_0}(\tau)$ with $\bar{\mathcal{U}}(\tau)$ under the integral. This essentially constitutes a random-phase approximation and should be a decent approximation if the noise spectrum is negligible in the frequency range of the coupling parameter (i.e. frequencies comparable with the Rabi frequency away from the exceptional point). Finally, we drop the first integral from the recursion equations as annotated in Eq. (17). This assumption holds for evolution times τ that are large compared to the lower cut-off frequency of the noise spectrum, because the moments of $\langle\tau^n\Delta(\tau)\rangle=0$ (see Appendix A). We find for the realization-averaged evolution operator

$$\begin{aligned}\bar{\mathcal{U}}(\tau)\approx1-\mathrm{i}\mathcal{H}_0\tau-\int_0^\tau\cdots\mathrm{d}\tau_{\{4\}}\,[\delta(\tau_1-\tau_2)-\mathrm{i}\mathcal{H}_0]\sigma_z\\ \times[\delta(\tau_3-\tau_4)-\mathrm{i}\mathcal{H}_0]\sigma_z\Gamma(\tau_2-\tau_4)\bar{\mathcal{U}}(\tau_4),\end{aligned} \tag{19}$$

where $\Gamma(\tau)$ is the auto-correlation function as introduced in Eq. (8). We can now simplify Eq. (19):

$$\begin{aligned}\bar{\mathcal{U}}(\tau)=1-\mathrm{i}\mathcal{H}_0\tau-\gamma\int_0^\tau\mathrm{d}\tau_1\sigma_z[1-\mathrm{i}\mathcal{H}_0\tau_1]\sigma_z\bar{\mathcal{U}}(\tau_1)\\ +\mathrm{i}\gamma\int_0^\tau\cdots\mathrm{d}\tau_{\{2\}}\mathcal{H}_0\sigma_z[1-\mathrm{i}\mathcal{H}_0\tau_2]\sigma_z\mathcal{U}(\tau_2).\end{aligned} \tag{20}$$

This can be transformed to the ordinary initial value problem that we have already stated in Eq. (10).

5 Results and discussion

The full analytical solution for equations like Eq. (10) with scalar coefficients is a product between the exponential and the hypergeometric function. A matrix generalization of this with non-commuting arguments would be beyond the scope of this paper. Nonetheless, a numerical solution is straightforward and compared in Figure 1 to brute-force solutions of the full problem for two quite different realizations of the noise function $\Delta(\tau)$. Further details on the numerics employed can be found in Appendix B.

Both the fully numerical examples and our effective description show that the time evolution of an initial state at first follows the behavior expected for the noiseless Hamiltonian. This is a stationary evolution for the eigenstate $\psi(0)=(1,\ \mathrm{i})^T/\sqrt{2}$ and linearly growing norm for any non-eigenstate, e.g. $\psi(0)=(0,\ 1)^T$. After a characteristic time τ_0 that depends on the noise amplitude γ, the system enters an exponentially divergent regime. This is more clearly seen in the semi-logarithmic plot in the left-hand panel of Figure 2. Here, we compare the effective numerical solution of the time evolution of the exceptional-point eigenstate to the simple Ansatz

$$f(\tau)=\exp[\alpha(\tau-\tau_0)],\qquad (21)$$

where α describes how quickly the state diverges from the expected behavior and τ_0 after which time the divergent regime sets in. As we show in the right-hand panel of Figure 2, the γ dependence of both quantities is reasonably well described by simple square root laws: $\alpha(\gamma)=\sqrt{2\gamma}$ and $\alpha\tau_0(\gamma)=1$. This is the result summarized in Eq. (11).

One might wonder what values for the characteristic time can be expected in an actual experiment. This is fairly hard to answer without knowledge of the possible origins of noise and the respective amplitudes in a given experiment. However, we are able to make some very rough estimates based on the inevitable noise of the pump laser in optically pumped $\mathcal{PT}$-symmetric microring dimers with a few nanometer resonance splitting of the symmetrically pumped resonators such as those presented in Ref. [9]. The natural time unit is the inverse coupling parameter κ, which in this example is of the order $\kappa\approx 10^{12}\mathrm{s}^{-1}$; i.e. $\tau=1$ is on the order of picoseconds. The fluctuations of the gain parameter relative to the mean gain can be roughly estimated as identical to the relative intensity fluctuations of the pump. According to the well-known Wiener-Khinchine theorem [19], this number is given by the relative spectral power density of the wide band intensity fluctuations. A realistic intensity noise figure for a small laser is around –120 dBc. Therefore, we find the rough order of magnitude $\gamma\approx 10^{-12}$, which is well within the range of validity for our perturbative expansion; we would also like to point out that this range of γ comes close to double-precision machine accuracy and that brute-force simulations of Eq. (12) would thus not be trustworthy for this value. From Figure 2, we can then estimate $\tau\approx 10^{6}$, which in real time corresponds to the order of microseconds. By reducing the noise figure, this can be of course increased with a square root law.

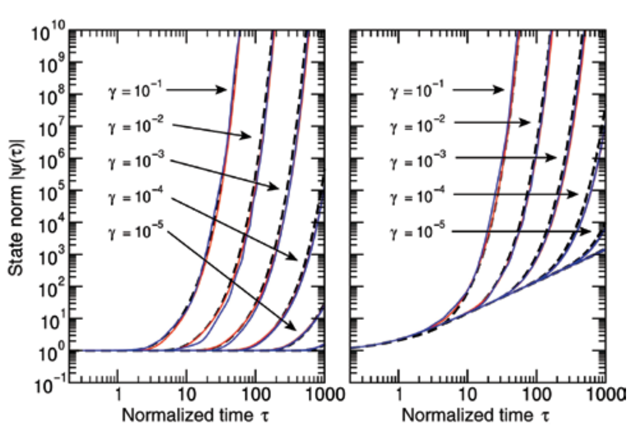

Figure 1: Left panel: Double-logarithmic plot of the norm $|\psi(\tau)|$ of the state starting with the exceptional-point eigenstate $\psi(0)=(1,\ \mathrm{i})^T/\sqrt{2}$ for different values of γ (see graph annotations). The black dashed curves are the numerical solutions to the effective differential equation (10a–10c) using fourth-order Runge-Kutta integration. The red and blue solid lines are brute-force calculations of Eq. (12) modeling the noise as two quite different ensembles consisting of 1000 harmonic oscillators each (red curve, high-frequency noise; blue curve, low-frequency noise; see Appendix B for details).
Right panel: Same as left panel starting with the non-eigenstate $\psi(0)=(0,\ 1)^T$.

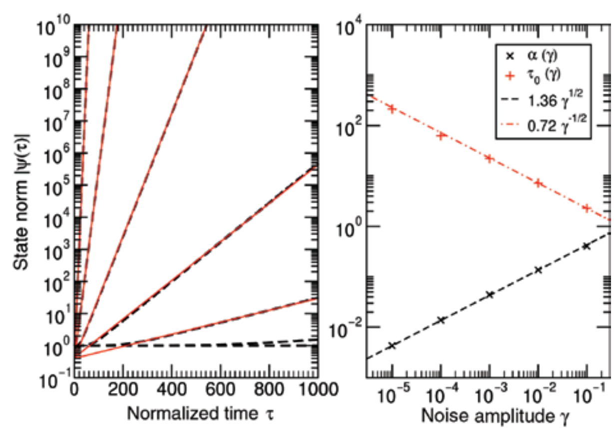

Figure 2: Left panel: Time evolution of the effective initial value problem [Eqs. (10a–10c), dashed black lines for values of γ annotated in Figure 1] starting from the exceptional-point eigenstate $\psi(0)=(1,\ \mathrm{i})^T/\sqrt{2}$ compared to expressions of the form $f(\tau)=\exp[\alpha(\tau-\tau_0)]$ (red solid lines) with parameters $\alpha(\gamma)$ and $\tau_0(\gamma)$ determined by least squares fitting. Clearly, the long-term behavior is an exponential growth; the parameter τ_0 is a measure for the time over which the time evolution mostly follows the unperturbed dynamics.
Right panel: Behavior of the functions $\alpha(\gamma)$ and $\tau_0(\gamma)$ as extracted from the left panel graphs. Clearly, both quantities are proportional to $\gamma^{1/2}$ and $\gamma^{-1/2}$, respectively, over a large dynamic range.

In view of our present analysis, one would naturally inquire if it is possible to design systems that fully take advantage of the fine sensitivity of exceptional points while at the same time eliminating the instabilities discussed here. While we cannot at the moment foresee a practical solution, topologically protected structures with exceptional points have been attracting increasing interest in recent years [22, 23], and in [23], in particular, it was shown that they can be very efficient in dealing with imperfections and loss in the case of waveguiding. It is not unreasonable therefore to speculate that they could potentially offer a route towards a solution in the case of sensing as well.

6 Conclusions

We have provided a detailed analytical study of the dynamics of a $\mathcal{PT}$-symmetric two-site coupled-mode system at the exceptional point, subject to drift-free fluctuations in its system parameters. To this end, we have analytically derived an effective differential equation that describes the mean time-evolution operator of this type of system. The fluctuations are assumed to be due to (e.g. thermal) noise, where the quasi-static contributions (drift) have been eliminated by means of an external stabilization system. The numerical solution of the effective differential equation shows that the presence of noise leads inevitably to the exponential divergence of both the noiseless system's eigenstate and non-eigenstates. As we find, the divergence occurs on a time scale that depends on the noise amplitude. The numerical solutions of the effective model are in excellent agreement with brute-force simulations that we performed by modeling the noise as the result of a bath of incoherent harmonic oscillators. This implies that harnessing the characteristic dynamics at an exceptional point for the design of highly sensitive sensors in practical applications faces not only the challenge that the quantum-noise limit cannot be overcome and that the delicate balance of system parameters is extremely sensitive to drift, but also that stabilization measures to keep the system at the exceptional point are exceedingly prone to amplifier noise and might suffer from regulation instabilities. Maintaining operation at the exceptional point for long enough times to detect minute resonance splittings seems to require very careful design of the feedback system. We believe that our effective differential equation for the time evolution at noisy exceptional points provides a valuable tool in this engineering feat as it can be extended to an analytical model for the complete system including the active stabilization system. This in turn would provide insight into the underlying processes, could be analyzed analytically, e.g. for overall stability, and could be used to optimize a system for the maximally sensitive equilibrium operating point without the need for a large number of brute-force simulations involving different realizations of the noise field.

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Appendix A Fundamental properties of the fluctuation field

We assume that the phases of $b(\omega)$ fluctuate arbitrarily quickly in frequency and the modulus of $b(\omega)$ decays for high ω. As a result, all moments of $b(\omega)$ with respect to ω vanish

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^n b(\omega) = 0, \tag{22}$$

for any positive exponent $n > 0$. We can generalize this to a wider class of functions

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, f(\omega) b(\omega) = 0, \tag{23}$$

for any $f(\omega)$ that is holomorphic on the union of a finite number of intervals that cover the support of $b(\omega)$. The reason is that under these conditions the integral can be decomposed into a finite number of integrals each covering an interval on which $f(\omega)$ can be represented by a Taylor series to whose terms Eq. (22) applies. Thus, assuming that $b(\omega)$ vanishes in a neighborhood of $\omega = 0$, we can extend Eq. (22) to negative exponents n and also allow that the integrand be multiplied with an arbitrary entire function

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, g(\omega) \omega^n b(\omega) = 0, \tag{24}$$

for $g(\omega)$ entire, and any integer $n \in \mathbb{Z}$. The assumption that $b(\omega)$ vanishes around $\omega = 0$ is intimately connected to the distinction between high-frequency fluctuations (noise) and low-frequency fluctuations (drift).

With this we can now show that all moments of the fluctuation field $\Delta(\tau)$ with respect to time vanish. The nth moment of $\Delta(\tau)$ is given as

$$\langle \tau^n \Delta(\tau)\rangle = \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\omega\, b(\omega)(\tau')^n \exp(\mathrm{i}\omega\tau') \tag{25}$$

$$= \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\omega\, b(\omega)\mathrm{i}^n \partial_\omega^n \exp(-\mathrm{i}\omega\tau'). \tag{26}$$

Next, we perform the temporal integral to find

$$\langle \tau^n \Delta(\tau)\rangle = \mathrm{i}^{n-1} \int \mathrm{d}\omega\, b(\omega)\partial_\omega^n \left[\frac{1-\exp(-\mathrm{i}\omega\tau)}{\omega}\right]. \tag{27}$$

This expression is of the type presented in Eq. (24) and therefore vanishes:

$$\langle \tau^n \Delta(\tau)\rangle = 0. \tag{28}$$

Appendix B Numerical methods

The comparison in Figure 1 was computed numerically in the following way. First, the differential operator of the effective differential equation (10a) was brought to a first-order form

$$\left[\partial_\tau + \begin{pmatrix} 0 & 1 \\ 2\gamma(\tau-\sigma_z) & \gamma(1+\mathrm{i}\mathcal{H}_0^{\dagger}\tau) \end{pmatrix}\right]$$

Then, the problem was integrated numerically using a standard fourth-order Runge-Kutta for the two initial column vectors $(1, 0, -1, 1)^T$ and $(0, 1, 1, 1)^T$ equivalent to applying the conditions Eqs. (10b) and (10c) to the physical states $(1, 0)^T$ and $(0, 1)^T$. This provides the columns of $\bar{\mathcal{U}}(\tau)$ [and as a byproduct those of $\bar{\mathcal{U}}'(\tau)$].

This is compared to a brute-force calculation. Random noise, being an intrinsically non-smooth signal, is not very well suited for numerical integration, especially because higher-order Runge-Kutta methods require the evaluation at different intermediate times. Therefore, we took some inspiration from Eq. (5) and modeled it as an ensemble of 1000 harmonic oscillators (the bath) with eigenfrequencies roughly equidistantly spaced in a spectral window and with random initial phases. The (real-valued) amplitudes of the oscillators were added up to give a consistent and smooth approximation to the noise function $\Delta(\tau)$, which was then fed into the Hamiltonian. With this, Eq. (12) was integrated in time alongside the ensemble of harmonic oscillators. We show results for two frequency bands: a high-frequency noise band with bath eigenfrequencies between 3.0 and 30.0, i.e. satisfying the assumption that underlies the approximation $\mathcal{U}(\tau) \approx \bar{\mathcal{U}}(\tau)$ in Eq. (19). The second example is for a low-frequency noise band spanning from 0.3 to 3.0, i.e. not satisfying said assumption. Still, our effective description seems to remain remarkably accurate.

References

[1] Bender CM, Boettcher S. Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. Phys Rev Lett 1998;80:5243–6.
[2] El-Ganainy R, Makris KG, Khajavikhan M, Musslimani ZH, Rotter S, Christodoulides DN. Non-Hermitian physics and PT symmetry. Nature Phys 2018;14:11–9.
[3] Longhi S. Parity-time symmetry meets photonics: a new twist in non-Hermitian optics. EPL 2017;120:64001.
[4] Guo A, Salamo GJ, Duchesne D, et al. Observation of $\mathcal{PT}$-symmetry breaking in complex optical potentials. Phys Rev Lett 2009;103:093902.
[5] Rüter CE, Makris KG, El-Ganainy R, Christodoulides DN, Segev M, Kip D. Observation of parity-time symmetry in optics. Nature Phys 2010;6:192–5.
[6] Regensburger A, Bersch C, Miri M-A, Onishchukov G, Christodoulides DN, Peschel U. Parity-time synthetic photonic lattices. Nature 2012;488:167–71.
[7] Brandstetter M, Liertzer M, Deutsch C, et al. Reversing the pump dependence of a laser at an exceptional point. Nature Commun 2014;5:4034.
[8] Peng B, Özdemir SK, Lei F, et al. Parity-time-symmetric whispering-gallery microcavities. Nature Phys 2014;10:394–8.
[9] Hodaei H, Miri M-A, Heinrich M, Christodoulides DN, Khajavikhan M. Parity-time-symmetric microring lasers. Science 2014;346:975–8.
[10] Miller J. Exceptional points make for exceptional sensors. Phys Today 2017;70:23–6.
[11] Hodaei H, Hassan AU, Wittek S, et al. Enhanced sensitivity at higher-order exceptional points. Nature 2017;548:187–91.
[12] Bajaj NK. The physics of waves and oscillations. New York, USA: McGraw-Hill, 2004.
[13] Wiersig J. Enhancing the sensitivity of frequency and energy splitting detection by using exceptional points: application to microcavity sensors for single-particle detection. Phys Rev Lett 2014;112:203901.
[14] Wiersig J, Kim SW, Hentschel M. Asymmetric scattering and nonorthogonal mode patterns in optical microspirals. Phys Rev A 2008;78:053809.
[15] Heiss WD. Time behaviour near to spectral singularities. Eur Phys J D 2010;60:257–61.
[16] Heiss WD. The physics of exceptional points. J Phys A: Math Theor 2012;45:444016.
[17] Langbein W. No exceptional precision of exceptional-point sensors. Phys Rev A 2018;98:023805.

[18] Mortensen NA, Gonçalves PAD, Khajavikhan M, Christodoulides DN, Tserkezis C, Wolff C. Fluctuations and noise-limited sensing near the exceptional point of parity-time-symmetric resonator systems. Optica 2018;5:1342–36.
[19] Born M, Wolf E. Principles of optics. Cambridge, UK: Cambridge University Press, 1999.
[20] Sakurai JJ. Modern quantum mechanics, Revised Edition. Reading, Massachussets, USA: Addison-Wesley, 1994.
[21] Baym G. Lectures on quantum mechanics. New York, USA: Westview Press, 1990.
[22] Shen H, Zhen B, Fu L. Topological band theory for non-Hermitian Hamiltonians. Phys Rev Lett 2018;120:146402.
[23] Gangaraj SAH, Monticone F. Topological waveguiding near an exceptional point: defect-immune, slow-light, and loss-immune propagation. Phys Rev Lett 2018;121:093901.