Research article

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Optical Potts machine through networks of three-photon down-conversion oscillators

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Abstract: In recent years, there has been a growing interest in optical simulation of lattice spin models for applications in unconventional computing. Here, we propose optical implementation of a three-state Potts spin model by using networks of coupled parametric oscillators with phase tristability. We first show that the cubic nonlinear process of spontaneous three-photon down-conversion is accompanied by a tristability in the phase of the subharmonic signal between three states with $2\pi/3$ phase contrast. The phase of such a parametric oscillator behaves like a three-state spin system. Next, we show that a network of dissipatively coupled three-photon down-conversion oscillators emulates the three-state planar Potts model. We discuss potential applications of the proposed system for all-optical optimization of combinatorial problems such as graph 3-COL and MAX 3-CUT.

Keywords: graph coloring; optical computing; parametric three-photon down-conversion; Potts model.

1 Introduction

Combinatorial optimization deals with minimizing (maximizing) a cost function over a finite and discrete set of objects. These problems appear in a wide range of applications and generally involve large configuration spaces, which makes an exhaustive search impractical [1]. As a result, over the years, a number of heuristic and unconventional methods are developed aiming to speed up the search

process [1]. An interesting approach is to use physical systems as analog computing platforms that emulate spin lattice models, which allow for solving many computationally hard combinatorial problems [2]. Gain-dissipative optical systems have recently attracted much interest as a physical platform for simulating spin lattice models [3–14]. In particular, networks of coupled optical parametric oscillators have been used for optically implementing the Ising Hamiltonian [5]. This Ising solver has been also utilized in conjunction with digital processing to implement a Potts model solver [15]. The core to physical realization of the Ising model is the inherent phase bistability of the two-photon down-conversion process in an optical parametric oscillator, which emulates the two states of a classical spin.

In principle, many NP-complete combinatorial problems can be mapped to the two-state Ising spin model [16]. Nonetheless, the formulation of the problem as an Ising Hamiltonian usually involves adding extra spin degrees of freedom to embed penalty terms to favor the solutions within the required constraints of the original problem. Therefore, solving problems that do not have the exact same cost function as the Ising energy function may become challenging as finding the optimal embedding parameters that can rule out invalid solutions becomes harder for larger problems [17]. Furthermore, the extra spins add to the complexity and density of the network, which makes their physical implementation more difficult. Therefore, it is of great interest to develop novel physical systems that allow for compact embeddings of the cost function of certain optimization problems. In this regard, as a generalization of the Ising model, the Potts model is an interesting candidate [18, 19].

The planar Potts model is a lattice spin model, in which the spins are confined in a plane and oriented along any of the q discrete uniformly distributed angles of $\theta_m = 2\pi m/q$ ($m = 0, 1, \dots, q-1$), while the coupling of the interacting spins depends only on their relative angles [20]. The q -state planar Potts Hamiltonian is defined as [20]:

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \cos(\theta_i - \theta_j), \quad (1)$$

where, J_{ij} represents the interaction between the i th and j th lattice sites, with $J_{ij} > 0$ and $J_{ij} < 0$ corresponding to the

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ferromagnetic and antiferromagnetic cases, respectively. Alternatively, in the so-called standard Potts model, the Hamiltonian is defined as $H = -\sum_{\langle ij \rangle} J_{ij} \delta(\theta_i, \theta_j)$, where δ is the Kronecker delta function. Clearly, for $q = 2$, this Hamiltonian reduces to that of the Ising model.

In this letter, we propose an optical three-state Potts machine through networks of coupled parametric three-photon down-conversion (3PDC) oscillators. The nonlinear process of 3PDC is schematically depicted in Figure 1A. In these $\chi^{(3)}$ processes, a pump photon spontaneously breaks into three subharmonic photons of one-third of the pump frequency, that is, $3\omega \rightarrow \omega + \omega + \omega$. The subharmonic signal can be brought to self-sustained oscillations in a resonant structure (Figure 1B). Quite interestingly, the onset of 3PDC oscillations is accompanied with phase tristability of the subharmonic signal. In this case, depending on the initial conditions, the subharmonic signal takes either of three different stable phase states with $2\pi/3$ contrast as depicted in the in-phase and quadrature-phase coordinates in Figure 1C. This process thus emulates a three-state spin. Therefore, by coupling a network of such 3PDC oscillators, one can create a three-state Potts machine as depicted schematically in Figure 1D and E for a two-spin network. Given the gain thresholding nature of the oscillators, the coupling should be of dissipative form such that different phase states of the network are discriminated based on their level of dissipation. In this manner, the Potts Hamiltonian is mapped to the level of dissipation of different phase states of the optical system. In the following, first we derive a classical analytically solvable dynamical model for a 3PDC oscillator which allows us to calculate its oscillation threshold and to prove its phase tristability. Next, we introduce the optical Potts machine through networks of dissipatively coupled 3PDC oscillators. We numerically explore the performance of the proposed Potts machine for solving important combinatorial optimization problems such as maximum three-cut (MAX 3-CUT) and graph three-coloring (3-COL).

2 Phase tristability of the 3PDC oscillator

The nonlinear optical process of spontaneous parametric 3PDC has been theoretically explored and experimentally demonstrated in the bulk [21], in optical fibers [22], and in resonant cavities [23–26]. However, the interesting phase tristability of the subharmonic signal seems to be overlooked in previous studies. Here, we derive a second-order nonlinear oscillator model for the subharmonic signal, which allows for analytical investigation of the 3PDC oscillations and its pertinent tristability.

Figure 1: (A) A schematic illustration of the 3PDC process with a virtual energy diagram. (B) A parametric 3PDC oscillator implemented in a Fabry-Pérot cavity. (C) The ternary phase of the 3PDC oscillator operating above the threshold depicted in the in-phase and quadrature-phase coordinates. (D) Dissipative coupling of two 3PDC oscillators. (E) A two-spin Potts system. 3PDC, three-photon down-conversion.

Considering a doubly resonant cavity driven with a pump laser as depicted in Figure 1B, coupled mode equations for the complex modal amplitudes of the pump (a_p) and signal (a_s) can be written as [26]:

$$\dot{a}_s = (-i\omega_s^0 - \kappa_s + i\gamma_{11}|a_s|^2 + i\gamma_{13}|a_p|^2)a_s + i3\mu a_p (a_s^*)^2, \quad (2a)$$

$$\dot{a}_p = (-i\omega_p^0 - \kappa_p + i\gamma_{31}|a_s|^2 + i\gamma_{33}|a_p|^2)a_p + i\mu^* a_s^3 + \sqrt{2\kappa_e} s_p. \quad (2b)$$

In these relations, $\omega_{s,p}^0$ are the cavity eigenfrequencies near the signal and pump frequencies ($\omega_p^0 \approx 3\omega_s^0$), $\kappa_{s,p}$ are the corresponding linewidths, γ_{ij} represent the self- and cross-phase modulation coefficients, μ describes the nonlinear interaction between the pump and the signal harmonics, μ_e is the out-coupling loss at the pump frequency, and s_p is the complex amplitude of the pump drive field. The fields are normalized such that $|a_{s,p}|^2$ represent the energy stored in the intracavity fields and $|s_p|^2$ shows the pump drive power.

The phase tristability can be investigated analytically based on a reduced model. Here, we neglect the frequency detunings and the self- and cross-phase modulation terms. In addition, we assume that the lifetime of the signal photons is much longer than that of the pump photons, that is, $\kappa_s \ll \kappa_p$, which allows for adiabatic elimination of the pump variable. Under these conditions, and using a transformation $a_s \rightarrow (1+i)a_s/\sqrt{2}$, the signal is found to be governed by the following equation:

$$\dot{a} = -\kappa a + g_0 (a^*)^2 - g_s |a|^4 a, \quad (3)$$

where the signal subscripts are removed for simplicity. In this relation, $g_0 = 3\mu\sqrt{2\kappa_e}s_p/\kappa_p$ is the small-signal gain and

$g_s = 3\mu^2/\kappa_p$ is the gain saturation coefficient. The latter equation admits a ternary set of fixed-point solutions $\bar{a} = Ae^{i\theta_{1,2,3}}$, where $\theta_{1,2,3} = -2\pi/3, 0, 2\pi/3$, and the amplitude A is a positive root of the quartic equation $g_s A^4 - g_0 A + \kappa = 0$. It is very important to note that Eq. (3) simultaneously admits a trivial stable fixed-point solution of $\bar{a} = 0$. Therefore, one should identify the parameter range for stabilization of the system in the ternary state rather than the trivial solution.

The three parameters involved are κ , g_0 , and g_s , which, respectively, represent linear signal loss, small-signal gain, and gain saturation coefficients. It is straightforward to show that the onset of parametric oscillations at the ternary states occurs at the threshold gain level g_{th} , where

$$g_{th}^4 = \frac{256}{27} g_s \kappa^3. \quad (4)$$

For gain levels above this threshold value, self-sustained oscillation of the signal becomes prominent. In this case, owing to the coexistence of the trivial solution, seeding is necessary to avoid death of the signal. However, by increasing the gain level, the seeding requirement becomes less critical and for large gain levels noise can trigger oscillations of the signal.

3 The optical Potts machine

By trapping into the triple attractors, the continuous phase of the 3PDC oscillator evolves into one of the three stable discrete values of $\theta = -2\pi/3, 0$ and $2\pi/3$. Therefore, a coupled network of such oscillators can emulate the three-state Potts model. Here, we consider a network of N identical oscillators with pairwise dissipative coupling among the oscillators. The time evolution of the complex field amplitude of the m th oscillator is governed by

$$\dot{a}_m = -\kappa_l a_m + g_0 (a_m^*)^2 - g_s |a_m|^4 a_m - \sum_n \kappa_{mn} (a_n + a_m). \quad (5)$$

Here, κ_l represents the intrinsic loss of each oscillator, and κ_{mn} is a real non-negative constant representing the coupling coefficient between the m th and n th oscillators. The presence of the diagonal term in the summation is due to energy conservation. According to this term, the total loss of the m th cavity is $\kappa_m = \kappa_l + \sum_n \kappa_{mn}$. The sign of the coupling coefficient corresponds to equal addition of the decaying fields in the dissipation channels, such that two in-phase cavity fields add up and decay maximally in the channel. This choice of the sign of the coupling becomes equivalent to an antiferromagnetic interaction in spin systems which is of relevance for computing applications.

By defining $\mathbf{a} = [a_1, \dots, a_N]^T$, Eq. (5) can be cast in a matrix form as $\dot{\mathbf{a}} = \mathbf{f}(\mathbf{a}, \mathbf{a}^*) - Q\mathbf{a}$, where the first and second terms describe the individual and interaction dynamics of the oscillators. Here, $\mathbf{f} = [f_1, \dots, f_N]^T$, where $f_m = -\kappa_l a_m + g_0 (a_m^*)^2 - g_s |a_m|^4 a_m$, and Q is the coupling matrix, with off-diagonal elements $q_{mn} = \kappa_{mn}$ and diagonal elements $q_{mm} = \sum_n \kappa_{mn}$. Numerical simulations of the system of dynamical Eq. (5) show that for a proper choice of the parameters involved, the network reaches an equilibrium state where the oscillators maintain their characteristics of stabilizing in a constant amplitude $|a_m| = |\bar{a}|$, and exhibiting ternary discrete phases $\theta_m = \{-2\pi/3, 0, +2\pi/3\}$. In addition, the orientation of individual phases tends to create a phase pattern that minimizes the energy of the corresponding Potts Hamiltonian.

To reach a nontrivial equilibrium, the gain injected to the Potts machine should compensate the total of internal and interaction losses. Therefore, the threshold gain can be found in terms of the smallest modal loss of the coupled cavity network. Taking α_{min} as the smallest eigenvalue of the coupling matrix Q , the minimum network loss is $\kappa_l + \alpha_{min}$. Therefore, in accordance with Eq. (4), the threshold gain of the network is obtained from $g_{th}^4 = (256/27)g_s(\kappa_l + \alpha_{min})^3$. Operating above this threshold gain level does not guarantee stabilization to a nontrivial state as simultaneous death of all oscillators is possible because of the very nature of the 3PDC oscillator, which allows coexistence of the trivial solution above the threshold.

The equilibrium properties of the Potts machine can be best discussed in terms of a potential landscape function $F(a_1, a_1^*, \dots, a_N, a_N^*)$ such that

$$\dot{a}_m = -\frac{\partial F}{\partial a_m^*}, \quad (6a)$$

$$\dot{a}_m^* = -\frac{\partial F}{\partial a_m}. \quad (6b)$$

By directly integrating Eq. (5), F is found to be

$$F = \sum_m \left[\kappa_l |a_m|^2 - \frac{g_0}{3} (a_m^3 + (a_m^*)^3) + \frac{g_s}{3} |a_m|^6 \right] + \frac{1}{2} \sum_{m,n} \kappa_{mn} |a_m + a_n|^2 \quad (7)$$

The functional F can be viewed as a potential defined over a $2N$ -dimensional space that involves all possible trajectories of the set of dynamical Eq. (5) for various initial conditions, while the fixed-points are located at the extrema (local or global) of this potential. It is straightforward to show that the total derivative of this function along the trajectories of Eq. (5) is negative semidefinite: $\dot{F} = -2\sum_m |\dot{a}_m|^2$. In addition,

this potential landscape is radially unbounded. These conditions ensure that the evolution of the network from given initial conditions is toward the minima (local or global) of function F . The existence of the potential function F guarantees local stability of the network at equilibrium points.

The functional of Eq. (7) can be considered as a cost function for the Potts machine, that is, the evolution of the system is toward minimizing F . Therefore, it is of interest to show its connection with the Potts Hamiltonian. This cost function is composed of two terms associated with self-oscillations of individual oscillators and the interaction between oscillators. For a given gain level, the self-oscillation term is minimized when each oscillator stabilizes to its ternary state discussed in the previous section. The interaction term, on the other hand, is minimized when each pair of coupled oscillators are out of phase. The relative contribution of these two terms in the total cost function can be evaluated with the gain parameter g_0 and the coupling coefficients κ_{mn} . For large gain levels, the self-oscillation term carries a large weight. Thus, minimizing this term in isolation is necessary for the minimization of the total cost function. This ensures the stabilization of all oscillators into equal amplitude states while the phases admit discrete ternary values. The minimization of the total cost function can then be considered as minimizing the interaction term subject to the constraint imposed by the first term. This behavior is confirmed by our numerical simulations of the dynamical Eq. (5). Assuming uniform amplitudes for all oscillators, the cost function reduces to

$$F = F_0 + |\bar{a}|^2 \sum_{m,n} \kappa_{mn} \cos(\phi_n - \phi_m) \quad (8)$$

where, $F_0 = N[\kappa_l |\bar{a}|^2 - (2g_0/3)|\bar{a}|^3 + (g_s/3)|\bar{a}|^6] + |\bar{a}|^2 \sum_{m,n} \kappa_{mn}$, is a constant that depends on the parameters and the network structure. According to relation 8, apart from a constant, the cost function governing the network of coupled 3PDC oscillators is mathematically identical with the Potts Hamiltonian.

To numerically investigate the Potts machine, we consider the example of a simple network with three equally coupled oscillators as shown in Figure 2. Here, we consider three different drive conditions and evaluate the dynamics of the amplitudes and phases of the oscillators. First, the gain is abruptly turned on and kept constant at a small value below the network threshold, which results in the death of all oscillators to zero amplitudes (Figure 2A). Next, for a gain higher than the total network loss (Figure 2B), the network reaches a nontrivial equilibrium, where all oscillators stabilize to constant amplitudes and phases. In this case, the equilibrium phase state of the

Figure 2: Dynamics of a three-spin system (inset) for different gain levels.

(A) For small gain (top), the network does not reach its oscillation threshold, thus the amplitudes (middle) approach zero and the phases (bottom) end up with arbitrary values. (B) When the gain (top) is abruptly turned on and kept at a constant large value, the amplitudes (middle) become equal and phases (bottom) converge to a valid Potts state, which in this case is not the ground state of the Potts Hamiltonian. (C) By adiabatically increasing the gain (top) from zero until the amplitudes (middle) converge, the phases (bottom) end up in the ground state of the Potts Hamiltonian. The simulation parameters are $\kappa_l = 1$, $\kappa_{mn} = \kappa = 0.1$, and $g_s = 0.1$, and initial conditions and gain are set as shown in each panel.

network $(\phi_1, \phi_2, \phi_3) = (-2\pi/3, 0, 0)$ is a valid state of the associated Potts model, although it is not the ground state. In fact, because the gain is considerably higher compared with the coupling across the network, the individual behavior of the oscillators dominate their collective behavior as elements of a larger system. On the other hand, when the gain is adiabatically increased (Figure 2C), the system ends up in the state $(\phi_1, \phi_2, \phi_3) = (-2\pi/3, 0, 2\pi/3)$, which is equivalent with the ground state of the corresponding Potts Hamiltonian.

4 Application to combinatorial optimization

Lattice spin models have been widely utilized as heuristics to solve computationally hard combinatorial optimization problems [16]. For this purpose, the cost function of the optimization problem is generally cast as the Hamiltonian of a spin model, which will then be solved for its ground state. The Potts model, in particular, is ideally suited for maximum q -cut (MAX q -CUT) and graph q -coloring (q -COL) problems

[20], which belong to the class of NP-hard problems and arise in applications such as register allocation and timetable/examination scheduling [27, 28]. The MAX 3-CUT problem can be stated as partitioning a graph into three disjoint sets such that the number of edges between disjoint parts is maximized. This is also equivalent to finding a three-coloring of the graph vertices such that the number of connected vertices of the same color is minimized.

This problem can be readily formulated as finding the ground state of the Potts Hamiltonian (Eq. 1). For the antiferromagnetic case with $\kappa_{mn} < 0$, the Potts Hamiltonian is minimized when $\phi_n \neq \phi_m$ for the maximum possible number of interacting $\{m, n\}$ pairs. The equivalence of these problems with the Potts Hamiltonian is particularly clear using the standard Potts Hamiltonian form $H = -\sum_{\langle ij \rangle} J_{ij} \delta(\theta_i, \theta_j)$. The ground state of this Hamiltonian is obtained when the number of connected same-phase oscillators is minimized. Therefore, using a system which implements the Potts Hamiltonian, we are able to find solutions to the corresponding optimization problems. To numerically explore the performance of the Potts machine as an optimizer, we solved the MAX 3-CUT problem with three different graphs shown in Figure 3. These three graphs, named Petersen (Figure 3A), Doyle (Figure 3B), and Meringer (Figure 3C), respectively, are known to be three-colorable. Thus, the ground state energy of their

corresponding (standard) Potts Hamiltonian should be zero. Here, we consider uniform coupling across the network of rate κ , while the gain g_0 is linearly increased from zero until the network stabilizes into an equilibrium state, where all intensities are equal and the phases remain constant. The oscillators are initialized with small amplitudes and random phases. For each network, the simulations are repeated for an ensemble of 1000 random initial conditions and a histogram of the equilibrium-state energy is plotted.

In all cases, the Potts machine shows good performance as an optimizer. However, in many incidents the systems is trapped into local minima and the global minimum is not found. This problem can be largely circumvented by injecting random fluctuations into the oscillators, which assist the network to escape from shallow local minima. The performance of the Potts machine for the two scenarios of without and with noise is compared through their associated histograms in Figure 3A–C. The time domain evolution of the intensities and phases of the Peterson network is depicted for these two scenarios in Figure 3D and E. As these figures clearly indicate, the phases undergo a complex dynamics in the scenario when noise is involved until the network settles into an equilibrium state that cannot be escaped with random fluctuations. Our simulations indicate that the performance of the Potts machine as an

Figure 3: Examples of graph three-coloring using the proposed Potts machine without and with noise.

(A–C): Distribution of the steady-state energy $H_s = \sum_{\langle ij \rangle} \delta(\theta_i, \theta_j)$ calculated for 1000 random initializations, without noise (blue) and with noise (red), for (A) a three-regular graph with 10 vertices (Petersen graph), (B) a four-regular graph with 27 vertices (Doyle graph), and (C) a five-regular graph with 30 vertices (Meringer graph). These graphs are all three-colorable. (D, E) Time evolution of the amplitudes and phases of the Peterson network without and with noise, respectively. In these simulations, $\kappa_{mn} = \kappa = 1$, $g_s = 0.1$ and $\kappa_l = 1$, while the gain is increased linearly with the slope of $20g_{th}\kappa_l$ for (D), and $0.1g_{th}\kappa_l$ for (E). The noise is taken to be uncorrelated for different oscillators and with uniform amplitude and phase distributions in the ranges of $[0, 0.1]$ and $[0, 2\pi]$, respectively.

optimizer depends strongly on the pump and the coupling parameters. The pace of increasing the gain is found to play an important role in finding the optimal solution. In fact, depending on the network threshold and noise level, the gain should be increased fast enough to prevent death of oscillators. On the other hand, a rapid increase in the gain level to a large value may force the system into a local minimum without giving it enough time to search for states with lower energies.

5 Conclusion and discussion

In summary, we proposed optical implementation of the three-state Potts model by using a network of coupled 3PDC oscillators. The state of the proposed Potts machine is in general described by continuous intensities and phases of the governing oscillators. However, when reaching an equilibrium, the oscillators reach an equal intensity state and the phases take ternary discrete values. Thus, the equilibrium phase pattern of the Potts machine represents a valid spin configuration of the Potts model. In this case, the energy of the Potts spin model is mapped into the cost function of the oscillator network. By adiabatically increasing the gain, the oscillator network tends to evolve toward an equilibrium phase pattern associated with the ground state of the Potts Hamiltonian.

Our numerical simulations suggest that the Potts machine suffers from trapping into local minima, which is a challenge in many heuristic optimization techniques. In this regard, an important future direction is to find the optimal initialization and parameter tuning which allows for finding the global minimum for an arbitrary network. The parametric 3PDC oscillations which forms the basis of the proposed Potts machine is within experimental reach [26]. In addition, arbitrary network coupling topologies can be implemented through a time domain multiplexing technique similar to the scheme utilized in Ref. [5]. In principle, the proposed machine can be generalized to the q -state Potts model (for $q > 3$) by harnessing the q -photon down-conversion processes. Such high-order nonlinear effects suffer from poor conversion efficiencies. However, an alternative approach is to implement subharmonic generation at more than one stage and through cascading of lower-order down-conversion processes.

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