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Chebyshev Operational Matrix Method for Lane-Emden Problem

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Abstract: In this paper, a new modified method is proposed for solving linear and nonlinear Lane-Emden type equations using first kind Chebyshev operational matrix of differentiation. The properties of first kind Chebyshev polynomial and their shifted polynomial are first presented. These properties together with the operation matrix of differentiation of first kind Chebyshev polynomial are utilized to obtain numerical solutions of a class of linear and nonlinear Lane-Emden type singular initial value problems (IVPs). The absolute error of this method is graphically presented. The proposed framework is different from other numerical methods and can be used in differential equations of the same type. Several examples are illuminated to reveal the accuracy and validity of the proposed method.

Keywords: Lane-Emden equations, chebyshev first kind operational matrix of differentiation, Isothermal gas spheres equation, Nonlinear ODE

1 Introduction

In astrophysics, Lane-Emden equation model is one of the case of singular initial value problem in the form of second-order nonlinear ordinary differential equation (ODE), which describes several important phenomenon in mathematical science and astrophysics such as radiative cooling, behavior of spherical self-gravitating gas clouds, stellar structure [1], thermal explosions [2],

modeling of cluster of galaxies, treatment of a phase transition in critical absorption, thermionic currents [3]. The dimensionless Poisson's equation which describes the equilibrium density distribution in self-gravitating symmetric sphere of polytropic isothermal gas, was proposed by Jonathan H. & Lane [4]. The further equation described by Robert & Emden [5] which is known as Lane-Emden equation. This equation dictates the pressure and density variation with respect to each other, formulated as

$$y''(x) + \frac{\alpha}{x}y'(x) + f(x, y) = g(x) \quad \alpha, x \geq 0, \quad (1)$$

subject to supplementary conditions:

$$y(0) = A, \quad y'(0) = 0 \quad (2)$$

where A is constant, $f(x, y)$ is a continuous real valued function, and $g(x) \in C[0, 1]$. It is known that the analytical solution of lane-Emden equation exists for supplementary conditions (2) in the locality of the singular point at $x = 0$. Taking $\alpha = 2$, $g(x) = 0$, $f(x, y) = y^n$ and $A = 1$ in equation (1) and (2) respectively

$$y''(x) + \frac{2}{x}y'(x) + y^n = 0 \quad (3)$$

characteristically, Eq. (3), represents the standard Lane-Emden equation, which will be subjected to the supplementary conditions as given in Eq. (4).

$$y(0) = 1, \quad y'(0) = 0 \quad (4)$$

Similarly, Isothermal gas spheres are modeled by Davis [6]

$$y''(x) + \frac{2}{x}y'(x) + e^y = 0 \quad (5)$$

subject to supplementary conditions:

$$y(0) = 0, \quad y'(0) = 0 \quad (6)$$

Various methods have been presented to solve the Lane-Emden equation in unbound domain. Wazwaz et al. [7] presented the series solution of integral form of Lane-Emden equation by using Adomian Decomposition method. Bender et al. [8] proposed a new perturbative technique by replacing nonlinear terms in the Lagrangian and introduced a new parameter γ . Aminikhah et al. [9] presented the numerical solution of Lane-Emden equation

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based on cubic B-spline approximation. Krivec and Mandelzweig [10, 11] proposed quasilinearization approach to solve the Lane-Emden equation with singular potentials. Ramos [12] introduced the linearization methods for the piece-wise closed form solution of Lane-Emden equation, which depends on the Jacobian and derivative of independent variable. The other effective methods are variation iterative methods [13], non-perturbative approximation method [14] and the Pade-series method [15]. Russell and Shampine [16] explored the Lane-Emden equation for linear function $f(x, y) = ky + h(x)$ and proved that a unique solution exists if $h(x) \in C[0, 1]$ and $-\infty < k \leq \pi^2$.

Hermite, Laguerre, Legendre and Chebyshev polynomials are orthogonal over unbounded domain. These polynomial spectral methods are most common methods due to reveal the more accurate solution of a class of Lane-Emden equation type [17–20]. Parand et al. [21] proposed Hermite collocation method for Lane-Emden equation. Pandey et al. [22] developed efficient numerical method for solving Lane-Emden equation using Legendre operation matrix of differentiation. Doha et al. [23] presented a new algorithm using second kind chebyshev operational matrix of derivative to obtain numerical solutions of Lane-Emden type IVPs. Abd-Elhameed et al. [24, 25] proposed a new algorithm to describe Lane-Emden Singular type IVPs using new Galerkin operation matrix of differentiation.

Wavelet theory is a relatively new and comes into view of mathematical research area. It has been applied with the orthogonal polynomials to propose a new method for solving several engineering problems. Yousefi [26] has given the solution of Lane-Emden equation using Legendre wavelet method. Youssri et al. [27] introduced ultraspherical wavelets operational matrix of derivatives for approximation of a class of Lane-Emden type with high accuracy.

The aim of this paper is to develop a more accurate and fast computational method for solving linear and nonlinear Lane-Emden equation (1) using first kind Chebyshev operation matrix of differentiation. The manuscript is organized as follows: Section 2 introduces the definition of first kind Chebyshev polynomial and formulates differential operation matrix. Section 3 describes the application of differentiation matrix of Chebyshev polynomial for solving the Lane-Emden equation. Section 4 exhibits the approximate solution obtained by the proposed method for the case of physical significance of polytropic index range $0 < n \leq 5$, wherein the result shows high accuracy and efficiency than other existing numerical solutions.

2 First kind chebyshev polynomials and its operational matrix of Differentiation

The analytic form of the shifted chebyshev polynomial $P_m(x)$ of degree m are given by Atabakzadeh et al. [28]

$$P_m(x) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^k 2^{m-2k-1} \frac{m}{m-k} \binom{m-k}{k} (2x-1)^{m-2k} \quad (7)$$

Any function, $y(x) \in L^2[0, 1]$ can be approximated as a sum of shifted chebyshev polynomial (of the first kind) as:

$$y(x) = \sum_{i=0}^{\infty} c_i P_i(x) \quad (8)$$

where

$$c_i = \langle y(x), P_i(x) \rangle = (2i+1) \int_0^1 y(x) P_i(x) dx, \quad i = 0, 1, \dots \quad (9)$$

In general, the series in Equation(8) can be truncated with the first $(N+1)$ shifted chebyshev polynomial of first kind as

$$y_N(x) = \sum_{i=0}^N c_i P_i(x) = C^T \phi(x) \quad (10)$$

where

$$C^T = [c_0 \ c_1 \ \dots \ c_N], \quad \phi(x) = [P_0(x) \ P_1(x) \ \dots \ P_N(x)]^T$$

The operation matrix of derivative of the shifted chebyshev polynomial set $\phi(x)$ is defined as

$$\frac{d\phi(x)}{dx} = D^{(1)} \phi(x) \quad (11)$$

where $D^{(1)}$ in the $(N+1) \times (N+1)$ operation matrix of differentiation of chebyshev polynomial given as:

$$D^{(1)} = d_{ij} = \begin{cases} \frac{4i}{\eta_j} & , \quad j = i - k \begin{cases} k = 1, 3, \dots, N & \text{if } N \text{ even,} \\ k = 1, 3, \dots, N-1 & \text{if } N \text{ odd,} \end{cases} \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

where $\eta_0 = 2$ and $\eta_k = 1$ ($k \geq 1$).

For example $N = 6$, we have

$$D^{(1)} = 2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 6 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 \\ 5 & 0 & 10 & 0 & 10 & 0 \end{pmatrix} \quad (13)$$

where $m = 0, 1, 2 \dots N$.

From Eq. (10), it can be generalized for any $n \in N$ as:

$$\frac{d^n \phi(x)}{dx} = (D^{(1)})^n \phi(x) = D^n \phi(x), \quad \text{where } n = 1, 2, 3, \dots \quad (14)$$

3 Application of the method

This section describe the application of differentiation matrix of chebyshev polynomial for solving the Lane-Emden equation. The Lane-Emden equation is formulated as

$$y''(x) + \frac{\alpha}{x} y'(x) + f(x, y) = g(x) \quad \alpha, x \geq 0,$$

subject to supplementary conditions: $y(0) = A, y'(0) = 0$.

Approximating $y(x), f(x, y)$ and $g(x)$ by the shifted chebyshev polynomial has

$$y(x) \approx \sum_{i=0}^n c_i P_i(x) = C^T \phi(x), \quad (15)$$

$$f(x, y) \approx f(x, C^T \phi(x)), \quad (16)$$

$$g(x) \approx \sum_{i=0}^n g_i P_i(x) = G^T \phi(x), \quad (17)$$

where the unknowns are, $C = [c_0 \ c_1 \ \dots \ c_N]^T$.

Using first kind chebyshev polynomial operation matrix of derivative, equation (15) can be expressed as

$$C^T D^2 \phi(x) + \frac{\alpha}{x} C^T D^1 \phi(x) + f(x, C^T \phi(x)) \approx G^T \phi(x) \quad (18)$$

The Residual $R_N(x)$ for equation(18) can be expressed as

$$R_N(x) \approx C^T D^2 \phi(x) + \frac{\alpha}{x} C^T D^1 \phi(x) + f(x, C^T \phi(x)) - G^T \phi(x) \quad (19)$$

Eq. (18) is converted in $N - 2$ nonlinear equations by applying Tau method [29], using following equation,

$$\langle R_N(x), P_i(x) \rangle = \int_0^1 R_N(x) P_i(x) dx = 0, \quad i = 0, 1, 2, \dots, N - 2. \quad (20)$$

The supplementary conditions of Eq. (17) are given by

$$y(0) = C^T \phi(0) = d_0, \quad y^{(1)}(0) = C^T D^{(1)} \phi(0) = d_1. \quad (21)$$

By substitute the shifted polynomials in Eq. (20), we have $(N - 2)$ nonlinear equations. These equation together with Eq. (24) generate N nonlinear equations. This nonlinear system can be solved using Newton iterative method to determine coefficients of vector C . We can get the numerical solution $y(x)$ of Lane-Emden equation by substituting C into Eq. (15).

4 Convergence and error analysis

In this section a convergence analysis is given for the method of solution discussed in last section for linear system of differential equations. It is well-known that shifted Chebyshev polynomials $\tau_i(x)$ forms a complete orthogonal set [30], $\Omega = [0, 1]$ and $L_\omega^2(\Omega)$ denotes the space of all functions $u : \Omega \rightarrow \mathbb{R}$ with weighted L^2 -norm and defined by

$$\|u\|_{L_\omega^2(\Omega)} = \int_0^1 u^2(t) \omega(t) dt$$

we recall that $H_\omega(\Omega)$ is the sobolev space of all function $u(t)$ on Ω such that $u(t)$ and all its weak derivative up to order m are in L_ω and define $\|\cdot\|_{H_\omega^l(\Omega)}$ as

$$\|u\|_{H_\omega^m(\Omega)} = \left(\sum_{k=0}^m \left\| \frac{\partial^k}{\partial t^k} u(t) \right\|_{L_\omega^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

The semi-norm also defined by

$$|u|_{H_\omega^{m,N}(\Omega)} = \sum_{i=\min(m,N)}^N N \|u^{(i)}(t)\|_{L_\omega^2(\Omega)}^2$$

Now, suppose $u_N = \sum_{k=0}^N \hat{U}_k \tau_k$ is the truncated Chebyshev approximations of a function $u \in H_\omega^m(\Omega)$. Then, as has been proved in [30], the truncation error is

$$\|u - u_N\|_{H_\omega^{l,N}(\Omega)} \leq CN^{2l-1/2-m} |u|_{H_\omega^{m,N}(\Omega)} \quad (22)$$

For any $1 \leq l \leq m$ and C is a positive constant independent of m . Then, using proposed analytical approximation method, the approximation of y is $y_N = (y_{1N}, y_{1N}, \dots, y_{nN}), y_{iN}(x) = \sum_{k=0}^N c_{ik} \tau_k(x)$ and error is defined by $e(x) = (e_1, e_2, \dots, e_n) = y(x) - y_N(x)$. Therefore,

$$D^2 e_i + \sum_{j=1}^{k_i} a_{ij} e_j = e_{gi}, \quad i = 1, 2, \dots, n, \quad (23)$$

with $e_{gi} = g_i - g_{iN}$ and $g_{ik} \tau_k(x)$. Taking the norm $\|\cdot\|_{H_\omega^{l,N}(\Omega)}$ from both sides of (23) yields

$$\begin{aligned} \|e_{gi}\|_{H_\omega^{l,N}(\Omega)} &= \|D^2 e_i + \sum_{j=1}^{k_i} a_{i,j} e_j\|_{H_\omega^{l,N}(\Omega)} \leq \|D^2 e_i\| \\ &\quad + \sum_{j=1}^{k_i} |a_{i,j}| \|e_j\|_{H_\omega^{l,N}(\Omega)}. \end{aligned} \quad (24)$$

On the other hand, since operator D^2 on Chebyshev polynomial space is continuous and bounded, as has been proved in [31], there exists a constant $\tilde{C}$ such that $\|D^2 e_i\|_{H_\omega^{l,N}(\Omega)} \leq \tilde{C} \|e_i\|_{H_\omega^{l,N}(\Omega)}$. Now, using this and Eq. (24) we find

$$\|e_{gi}\|_{H_\omega^{l,N}(\Omega)} \leq \tilde{C} \|e_i\|_{H_\omega^{l,N}(\Omega)} + \sum_{j=1}^{k_i} |a_{i,j}| \|e_j\|_{H_\omega^{l,N}(\Omega)}. \quad (25)$$

Let the vector norm $\|e\|_{H_\omega^{l,N}(\Omega)} = \max_{1 \leq i \leq n} \|e_i\|_{H_\omega^{l,N}(\Omega)}$ and the vector semi-norm $|y|_{H_\omega^{l,N}(\Omega)} = \max_{1 \leq i \leq n} |y_i|_{H_\omega^{l,N}(\Omega)}$. Then Eq. (25) yields

$$\begin{aligned} \|e_{gi}\|_{H_\omega^{l,N}(\Omega)} &\leq \tilde{C} \|e\|_{H_\omega^{l,N}(\Omega)} + \sum_{j=1}^{k_i} |a_{i,j}| \|e\|_{H_\omega^{l,N}(\Omega)} \\ &\leq \|e\|_{H_\omega^{l,N}(\Omega)} (\tilde{C} + \sum_{j=1}^{k_i} |a_{i,j}|). \end{aligned} \quad (26)$$

Finally, combining (22) and (26) gives the following bound for the error in our presented analytical approximation method

$$\frac{\|e_{gi}\|_{H_\omega^{l,N}(\Omega)}}{\mathbb{A}} \leq \|e\|_{H_\omega^{l,N}(\Omega)} \leq CN^{2l-1/2-m} |y|_{H_\omega^{m,N}(\Omega)}$$

where $\mathbb{A} = \tilde{C} + \max_{1 \leq i \leq n} \sum_{j=1}^{k_i} |a_{i,j}|$. Note that single Ω is compact set, $|y|_{H_\omega^{m,N}(\Omega)}$ is bounded and this means that our approximation converges for sufficiently large N .

Remark 1. As we have stated above, the convergence analysis for the proposed method just applied into the linear system (20). However, convergence analysis cannot simply determine for the nonlinear case as above discussion and needs some more efforts. Indeed for each nonlinear case, we may need some additional conditions on nonlinear functions $y^n(x)$ in system (20).

5 Numerical results and discussions

In this section, the introduced first kind Chebyshev operational matrix of derivatives is employed for solving both

linear and nonlinear Lane-Emden IVPs. The singularity in the variable coefficient (α/x) is handled by reducing a class of Lane-Emden type singular initial value problems to a system of algebraic equations using Tau method and supplementary conditions. The Newton iterative method is used to determine the coefficients of system equations. Six examples are executed on MATLAB-2016 using the proposed numerical algorithm to show the accuracy and efficiency of Chebyshev operation matrix method.

Let $f(x, y) = y^n(x)$, $g(x) = 0$ and $A = 1$, Eq.(1) can be written as

$$y''(x) + \frac{2}{x} y'(x) + y^n(x) = 0 \quad (27)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (28)$$

Example 1. for $n = 0$, the linear Lane-Emden equation from Eq. (27) has the form

$$y''(x) + \frac{2}{x} y'(x) + 1 = 0 \quad (29)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (30)$$

Applying the method, proposed in section 3 for $N = 2$, we have

$$y(x) = C^T \phi(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \quad (31)$$

From Eq.(18), $D^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}$ and $D^{(2)} =$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \end{pmatrix}.$$

By using Eq. (25), we obtain

$$48c_2 + 1 = 0, \quad (32)$$

By applying the supplementary conditions from Eq. (27) we have

$$c_0 - c_1 + c_2 = 1, \quad (33)$$

$$2c_1 - 8c_2 = 0, \quad (34)$$

Solving Eqs. (32)-(34) we get,

$$c_0 = \frac{15}{16}, \quad c_1 = \frac{-1}{12}, \quad c_2 = \frac{-1}{48}.$$

Substituting the values of constant vector in Eq. (31), the solution is obtained as:

$$\begin{aligned} y(x) &= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \\ &= \begin{pmatrix} \frac{15}{16} & \frac{-1}{12} & \frac{-1}{48} \end{pmatrix} \begin{pmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \end{pmatrix} = 1 - \frac{x^2}{6}, \end{aligned}$$

which is the exact solution.

Example 2. For $n = 1$, the nonlinear Lane-Emden equation from Eq. (27) has the form

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 0 \quad (35)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (36)$$

Solving Eq. (35), the exact solution is $y(x) = \frac{\sin x}{x}$.

Here,

$$f(x, y) = y(x) = C^T \phi(x).$$

The proposed method has been applied into the Eq. (35) with supplementary conditions Eq. (36) and the unknown matrix C^T has evaluated by solving the nonlinear system for $N = 8$ and $N = 10$. The values of coefficients are are shown in Table 4 and 5 respectively. The exact solution and obtained solution using shifted first kind chebyshev polynomial in larger range $[0, 8]$ are plotted in Fig. 1 and absolute error is graphically presented in Fig. 2.

Fig. 1: Plot between x & obtained solution $y(x)$ for $N = 8, N = 10$ and exact solution.

Example 3. For $n = 3$, the nonlinear Lane-Emden equation from Eq. (27) has the form

$$y''(x) + \frac{2}{x}y'(x) + y^3(x) = 0, \quad (37)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (38)$$

Here,

$$y(x) = C^T \phi(x) \quad \text{and} \quad f(x, y) = y^3(x) = (C^T \phi(x))^3.$$

The values of unknown coefficients C^T for $N = 8$ and $N = 10$ are shown in table 4 and 5 respectively. The analytic series solution [7] and obtained result using first kind

Fig. 2: Absolute errors at $N = 8$ and $N = 10$ for example 2.

chebyshev polynomial are graphically presented in Fig. 3 in range $[0, 5]$.

The Table 1 shows the comparison of results and absolute errors with Wazwaz et al. [7] and Pandey et al. [22]. From Table 1 and Fig. 3, we observe that the proposed method providing better results compare to the recently developed methods.

Fig. 3: Plot of obtained solution $y(x)$ for $N = 8, N = 10$ and series solution presented by Wazwaz [7] for example 3.

Example 4. For $n = 5$, the nonlinear Lane-Emden equation from Eq. (27) has the form

$$y''(x) + \frac{2}{x}y'(x) + y^5(x) = 0, \quad (39)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (40)$$

Table 1: Comparison of approximate solutions and error obtained by modified method with series solution wazwaz [7] and newly advanced method Pandey [22] for Example 3.

x	Present method	Wazwaz [7]	Error at $N = 10$	Error at Pandey [22]
0	0.9999999999	1	2.23E-12	1.14E-13
0.1	0.9983358295	0.9983358295	2.12E-11	1.86E-11
0.2	0.9933730935	0.9933730935	1.38E-11	1.65E-11
0.5	0.9598390699	0.9598391144	4.45E-08	4.45E-08
1	0.8550575685	0.8550950543	3.74E-05	3.74E-05

having exact solution $y(x) = (1 + \frac{x^2}{3})^{-\frac{1}{2}}$.

Here,

$$y(x) = C^T \phi(x)$$

and

$$f(x, y) = y^5(x) = (C^T \phi(x))^5.$$

The unknown matrix C^T has evaluated by solving the non-linear system for $N = 8$ and $N = 10$. The values of coefficients are shown in Table 4 and 5 respectively. The exact solution and obtained solution using shifted first kind chebyshev polynomial in larger range [0,5] are plotted in Fig. 4 and absolute error is graphically presented in Fig. 5.

Fig. 4: Plot of obtained solution $y(x)$ for $N = 8, N = 10$ and exact solution for example 4.

Example 5. Isothermal gas spheres equation from Eq.(6) has the form

$$y''(x) + \frac{2}{x}y'(x) + \exp(y) = 0, \quad x \geq 0, \quad (41)$$

with supplementary conditions

$$y(0) = 0, \quad y'(0) = 0 \quad (42)$$

Here, we have

Fig. 5: Absolute errors at $N = 8$ and $N = 10$ for example 4.

$$y(x) = C^T \phi(x) \quad \text{and} \quad f(x, y) = e^y.$$

Expansion of $f(x, y)$ according to Taylor series is

$$f(x, y) = e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \frac{y^5}{120} + \dots \quad (43)$$

By considering first five terms, $f(x, y)$ can be expressed as

$$f(x, y) = 1 + C^T \phi(x) + \frac{(C^T \phi(x))^2}{2} + \frac{(C^T \phi(x))^3}{6} + \frac{(C^T \phi(x))^4}{24} \quad (44)$$

The unknown matrix C^T is shown in Tables 4 and 5 respectively. The Wazwaz [7] analytic series solution and evaluated results by solving nonlinear system using developed method for $N = 8$ and $N = 10$, are shown in Fig. 6 in range [0,1].

The Table 2 shows the superiority of obtained approximate solutions than Wazwaz et al. [7], Pandey et al. [22] and Parand et al. [21].

Example 6. Consider the ordinary differential equation of Lane-Emden type of shape factor 2,

$$y''(x) + \frac{2}{x}y'(x) + \sin(y) = 0, \quad (45)$$

with supplementary conditions

$$y(0) = 1, \quad y'(0) = 0 \quad (46)$$

Table 2: Comparison of approximate solutions and error obtained by modified method with series solution wazwaz [7], newly advanced method Pandey [22] and Parand [21] for Example 5.

x	Present method	Wazwaz [7]	Error for $N = 10$	Error at [22]	Error at [21]
0	9.411877E-11	0.0000000000	9.41E-11	9.24E-18	0.00E+00
0.1	-0.0016658337	-0.0016658338	9.93E-11	5.28 E-10	5.85E-07
0.2	-0.0066533669	-0.0066533671	1.06E-10	3.37E-08	6.04E-07
0.5	-0.0411539572	-0.0411539567	3.42E-10	8.12E-06	5.58E-07
1	-0.1588276829	-0.1588272857	3.29E-07	4.93E-04	8.20E-07

Table 3: Comparison of approximate solutions and error obtained by modified method with series solution wazwaz [7], and newly advanced method Pandey [22] & Saadtmandi and Dehgan [29] for example 6.

x	Present method	Wazwaz [7]	Error at $N=10$	Error at [22]	Error at [29]
0	0.9999999999	1.0000000000	3.78E-14	1.11E-16	0.00E+00
0.1	0.9985976022	0.9985979273	5.83E-08	3.25E-07	7.21E-06
0.2	0.9943949769	0.9943962648	5.42E-07	1.28E-06	1.00E-05
0.5	0.9651702487	0.9651777797	1.67E-07	7.53E-06	1.04E-05
1	0.8636571676	0.8636807315	3.27E-06	2.35E-05	7.03E-06

Table 4: The value of the unknowns vector C^T for discussed examples for $N = 8$.

coefficients	Example 2	Example 3	Example 4	Example 5	Example 6
c_0	0.9397344	0.9435839	0.9460855	-0.0603338	0.9484656
c_1	-0.0797633	-0.0736793	-0.0686545	-0.0798799	-0.0684238
c_2	-0.0190577	-0.0161548	-0.0138788	-0.0191298	-0.0166709
c_3	0.0004999	0.0012224	0.0016977	0.0004686	0.0002518
c_4	0.0000588	0.0001001	0.0000853	0.0000499	0.0000338
c_5	-0.0000012	-0.0000145	-0.0000309	-2.536E-07	6.782E-07
c_6	8.702E-08	-3.823E-07	7.421E-07	-1.444E-07	3.324E-08
c_7	1.289E-09	1.398E-07	4.318E-07	1.293E-08	-5.028E-09
c_8	7.557E-11	9.082E-10	3.783E-08	3.846E-10	-2.986E-10

Table 5: The value of the unknowns vector C^T for discussed examples for $N = 10$.

coefficients	Example 2	Example 3	Example 4	Example 5	Example 6
c_0	0.9397344	0.9435839	0.9468055	-0.0603338	0.9484656
c_1	-0.0797633	-0.0736793	-0.0686545	-0.0798799	-0.0684238
c_2	-0.0190577	-0.0161548	-0.0138788	-0.0191298	-0.0166709
c_3	0.0004999	0.0012224	0.0016977	0.0004686	0.0002518
c_4	0.0000588	0.0001001	0.0000853	0.0000499	0.0000338
c_5	-0.0000012	-0.0000145	-0.0000309	-2.536E-07	6.782E-07
c_6	8.702E-08	-3.812E-07	4.794E-07	-1.444E-07	3.324E-08
c_7	1.291E-09	1.426E-07	4.426E-07	1.309E-08	-5.04E-09
c_8	7.522E-11	1.642E-09	4.250E-08	3.532E-10	-2.979E-10
c_9	9.165E-13	1.196E-09	4.373E-09	-6.520E-11	4.790E-12
c_{10}	4.275E-14	5.245E-11	9.754E-10	-5.20E-13	6.51E-13

Fig. 6: Plot of obtained solution $y(x)$ for $N = 8$, $N = 10$ and series solution presented by Wazwaz [7] for example 5.

Here,

$$y(x) = C^T \phi(x) \text{ and } f(x, y) = \sin(y).$$

Expansion of $f(x, y)$ according to Taylor series is

$$f(x, y) = \sin(y) = y - \frac{y^3}{6} + \frac{y^5}{120} + \dots \quad (47)$$

By considering only first three terms, $f(x, y)$ can be expressed as

$$f(x, y) = C^T \phi(x) - \frac{(C^T \phi(x))^3}{6} + \frac{(C^T \phi(x))^5}{120} \quad (48)$$

The coefficient vector C^T is solved for $N = 8$ & $N = 10$ and shown in Tables 4 and 5 respectively. The obtained results are presented in Fig. 7 in range $[0, 1]$. Table 3 presents the comparison of solution $y(x)$ and absolute error with Wazwaz et al. [7], Pandey et al. [22] and Saadatmandi and Dehghan [29].

Fig. 7: Plot of obtained solution $y(x)$ for $N = 8$, $N = 10$ and series solution presented by Wazwaz [7] for example 6.

6 Conclusions

In this manuscript, we have proposed a modified approach to solve nonlinear Lane-Emden equation using first kind chebyshev operational matrix of differentiation. The benefit of this approach over other existing methods is that only operation matrix of differentiation have need of present the solution more precisely. This differentiation matrix contains mostly number of zeros, which reduces the run time and provides simplicity in computation. This algorithm generates nonlinear system of equations, which can be performed on computer using a program run in MATLAB2016. N must be selected large sufficient to solve the Lane-Emden equation most precisely.

Consequently, the numerical solution concurs with the exact and series solution even with a small number of first kind of chebyshev polynomial used in estimation. Several examples have been worked out to reveal the accuracy and performance of modified algorithm than other recently developed methods.

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