Yalçın Öztürk*

An efficient numerical algorithm for solving system of Lane–Emden type equations arising in engineering

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Abstract: The purpose of this paper is to propose an efficient numerical method for solving system of Lane–Emden type equations using Chebyshev operational matrix method. This method transforms the system of Lane–Emden type equation into the system of algebraic equations with unknown Chebyshev coefficients. Some illustrative examples are given to demonstrate the efficiency and validity of the proposed algorithm.

Keywords: Lane–Emden system; engineering problem; Chebyshev polynomials; operational matrix method

1 Introduction

In this paper, we consider the system of the Lane–Emden type equations as:

$$\begin{aligned} P(t) \frac{d^2 y_1(t)}{dt^2} + \frac{\alpha}{t} \frac{dy_1(t)}{dt} + y_2^p(t) &= g_1(t) \\ H(t) \frac{d^2 y_2(t)}{dt^2} + \frac{\beta}{t} \frac{dy_2(t)}{dt} + y_1^q(t) &= g_2(t) \end{aligned} \quad (1)$$

with initial conditions

$$\begin{aligned} y_1(0) &= \lambda_0, \quad y_1'(0) = \lambda_1 \\ y_2(0) &= y_0, \quad y_2'(0) = y_1 \end{aligned} \quad (2)$$

where p, q are positive integers. Eq. (1) is the Lane–Emden equation of the first kind, or of index q , which is a basic equation in the theory of stellar structure, in the sequence of papers [1–28]. The Lane–Emden equation of the first kind describes the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of thermodynamics. Notice that the

Lane–Emden equation is linear for $q = 0, 1$ and nonlinear otherwise. In astrophysics, the Lane–Emden equation is Poisson’s equation for the gravitational potential of a self-gravitating spherically symmetric and polytropic fluid at hydrostatic equilibrium for [22]. Moreover, one of the important fields of application of this equation is the analysis of the diffusive transport and chemical reaction of species inside a porous catalyst particle [22].

The modelling of several physical phenomena such as pattern formation, population evolution, chemical reactions, and so on [7] gives rise to the systems of Lane–Emden equations, and have attracted much attention in recent years. Several authors have proved existence and uniqueness results for the Lane–Emden systems [8, 9].

Several methods for the solution of Lane–Emden equations and the system of Lane–Emden equations have been presented, which are sinc-collocation method [1], the variational iteration method [2], collocation method [10], the modified Homotopy analysis method [11], the variational iteration method [12], the Adomian decomposition method [13], Legendre operational matrix method [14], second kind Chebyshev operational algorithm [15], ultraspherical wavelets method [16], shifted Legendre operational matrix method [17], spectral second kind Chebyshev wavelets [18] other methods, in the sequence of papers [19–30].

Chebyshev polynomials are the most famous bases of polynomials space. It is well known that there are four kinds of Chebyshev polynomials, and all of them are special cases of the more widest class of Jacobi polynomials. These polynomials have reliable advantage such as easy to compute, rapid convergence and completeness, since Chebyshev polynomials (first kind) is the Fourier cosine series [31, 32]. From this reasons, we use the first kind shifted Chebyshev polynomials in approximation.

Spectral methods play prominent roles in numerical mathematics to solve many more important problems. In this article, we use the operational method which is the kind of spectral method. Operational method have become increasingly popular to solve various problems in applied mathematics. A large number of authors utilize

*Corresponding Author: Yalçın Öztürk, Ula Ali Koçman Vocational High School, Muğla Sıtkı Koçman University, Muğla, Turkey, E-mail: yozturk@mu.edu.tr

this method to solve various equations such as linear and nonlinear hyperbolic telegraph type equations [33], fractional order differential equation [34], higher-order ordinary differential equations [35, 36], Abel equation [37], two point boundary value problems [38], singularly perturbed boundary value problems of fractional multi-order [39], integro-differential equation [40].

The aim of this study is to get solution as truncated Chebyshev series defined by

$$y_1^N(t) = \sum_{n=0}^N ' a_n T_n^*(t), y_2^N(t) = \sum_{n=0}^N ' b_n T_n^*(t) \quad (3)$$

where $T_n^*(t) = \cos(n\theta)$, $2t - 1 = \cos \theta$, $0 \leq t \leq 1$, denotes the shifted Chebyshev polynomials of the first kind; $\sum'$ denotes a sum whose first term is halved; a_n , b_n ($0 \leq n \leq N$) are unknown Chebyshev coefficients, and N is chosen any positive integer such that $N \geq m$.

2 The first kind shifted Chebyshev polynomials

The Chebyshev polynomials of the first kind $T_n(x)$ is a polynomials in x of degree n , defined by relation [31, 32]

$$T_n(x) = \cos n\theta, \text{ when } x = \cos \theta$$

If the range of the variable x is the interval $[-1, 1]$, the range the corresponding variables θ can be taken $[0, \pi]$. We map the independent variable t in $[0, 1]$ to the variable s in $[-1, 1]$ by transformation

$$s = 2x - 1 \text{ or } x = \frac{1}{2}(s + 1)$$

and this lead to the shifted Chebyshev polynomial of the first kind $T_n^*(x)$ of degree n in x on $[0, 1]$ given by [31]

$$T_n^*(x) = T_n(s) = T_n(2x - 1).$$

These polynomials may be generated by using recurrence relation

$$T_n^*(x) = 2(2x - 1)T_{n-1}^*(x) - T_{n-2}^*(x)$$

with initial conditions

$$T_0^*(x) = 1, T_1^*(x) = 2x - 1,$$

Any given function, $y(x) \in L^2[0, 1]$, can be approximated as a sum of shifted Chebyshev polynomials as:

$$y(x) = \sum_{n=0}^{\infty} a_n T_n^*(x)$$

where

$$c_n = \left\langle y(x), T_n^*(x) \right\rangle = \int_0^1 y(x) T_n^*(x) dx, n = 0, 1, \dots$$

3 Fundamental relations

Let us consider Eq. (1) and find the matrix forms of the equation. First we can convert the solution $y_{1,2}^N(t)$ defined by a truncated shifted Chebyshev series (3) and its derivative $(y_{1,2}^N(t))^{(k)}$ to matrix forms

$$y_1^N(t) = \mathbf{T}^*(t)\mathbf{A}, y_1^{N(k)}(t) = \mathbf{T}^{*(k)}(t)\mathbf{A}, k = 0, 1, 2 \quad (4)$$

$$y_2^N(t) = \mathbf{T}^*(t)\mathbf{B}, y_2^{N(k)}(t) = \mathbf{T}^{*(k)}(t)\mathbf{B}, k = 0, 1, 2 \quad (5)$$

where

$$\begin{aligned} \mathbf{T}^*(t) &= [T_0^*(t) T_1^*(t) \dots T_N^*(t)] \mathbf{T}^{*(k)}(t) \\ &= [T_0^{*(k)}(t) T_1^{*(k)}(t) \dots T_N^{*(k)}(t)] \end{aligned}$$

$$\mathbf{A} = \left[\frac{1}{2} a_0 \ a_1 \dots \ a_N \right]^T$$

$$\mathbf{B} = \left[\frac{1}{2} b_0 \ b_1 \dots \ b_N \right]^T$$

It is well known that the relation between the powers x^n and the shifted Chebyshev polynomials $T_n^*(x)$ is

$$x^n = 2^{-2n+1} \sum_{k=0}^n ' \binom{2n}{k} T_{n-k}^*(x), 0 \leq x \leq 1 \quad (6)$$

where $\sum'$ denotes a sum whose first term is halved.

By using the expression (6) and taking $n=0,1,\dots,N$ we find the corresponding matrix relation as follows:

$$(\mathbf{X}(t))^T = \mathbf{D}(\mathbf{T}^*(t))^T \text{ and } \mathbf{X}(t) = \mathbf{T}^*(t)\mathbf{D}^T \quad (7)$$

where

$$\mathbf{X}(t) = [1 \ t \ \dots \ t^N]$$

$$\mathbf{D} = \begin{bmatrix} 2^0 \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & 0 & 0 & \dots & 0 \\ 2^{-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} & 2^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & 0 & 0 & \dots & 0 \\ 2^{-4} \begin{pmatrix} 4 \\ 2 \end{pmatrix} & 2^{-3} \begin{pmatrix} 4 \\ 1 \end{pmatrix} & 2^{-3} \begin{pmatrix} 4 \\ 0 \end{pmatrix} & 0 & \dots & 0 \\ 2^{-6} \begin{pmatrix} 6 \\ 3 \end{pmatrix} & 2^{-5} \begin{pmatrix} 6 \\ 2 \end{pmatrix} & 2^{-5} \begin{pmatrix} 6 \\ 1 \end{pmatrix} & 2^{-5} \begin{pmatrix} 6 \\ 0 \end{pmatrix} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2^{-2N} \begin{pmatrix} 2N \\ N \end{pmatrix} & 2^{-2N+1} \begin{pmatrix} 2N \\ N-1 \end{pmatrix} & 2^{-2N+1} \begin{pmatrix} 2N \\ N-2 \end{pmatrix} & 2^{-2N+1} \begin{pmatrix} 2N \\ N-3 \end{pmatrix} & \dots & 2^{-2N+1} \begin{pmatrix} 2N \\ 0 \end{pmatrix} \end{bmatrix}$$

Then, by taking into account (7) we obtain

$$\mathbf{T}^*(t) = \mathbf{X}(t)(\mathbf{D}^{-1})^T \quad (8)$$

and

$$(\mathbf{T}^*(t))^{(k)} = \mathbf{X}^{(k)}(t)(\mathbf{D}^{-1})^T, \quad k = 0, 1, 2,$$

To obtain the matrix $\mathbf{X}^{(k)}(t)$ in terms of the matrix $\mathbf{X}(t)$, we can use the following relation:

$$\mathbf{X}^{(1)}(t) = \mathbf{X}(t)\mathbf{C}$$

$$\mathbf{X}^{(2)}(t) = \mathbf{X}^{(1)}(t)\mathbf{C} = \mathbf{X}(t)(\mathbf{C})^2 \quad (9)$$

where

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (8) and (9) into (4)-(5) we have the matrix relation

$$y_1^N(t)^{(k)} = \mathbf{X}(t)\mathbf{C}^k(\mathbf{D}^T)^{-1}\mathbf{A}, \quad k = 0, 1, 2 \quad (10)$$

$$y_2^N(t)^{(k)} = \mathbf{X}(t)\mathbf{C}^k(\mathbf{D}^T)^{-1}\mathbf{B}, \quad k = 0, 1, 2 \quad (11)$$

4 Method of Solution

Firstly, we deal with singularity in Eq. (1). If they set in Eq. (1) by [16]

$$g_1(t) = tf_1(t) \quad g_2(t) = tf_2(t),$$

Then Eq. (1) is turned into

$$\begin{aligned} tP(t)\frac{d^2y_1(t)}{dt^2} + \alpha\frac{dy_1(t)}{dt} + ty_2^p(t) &= f_1(t) \\ tH(t)\frac{d^2y_2(t)}{dt^2} + \beta\frac{dy_2(t)}{dt} + ty_1^q(t) &= f_2(t) \end{aligned} \quad (12)$$

$$\begin{aligned} y_1(0) &= \lambda_0, \quad y_1'(0) = \lambda_1 \\ y_2(0) &= y_0, \quad y_2'(0) = y_1 \end{aligned} \quad (13)$$

Now, we present the numerical solution method for Eq. (12) with initial conditions Eq. (13). We can give the expansion of $f_i(x)$ by the shifted Chebyshev polynomials as:

$$f_i(t) \approx \mathbf{F}_i^T \mathbf{X}(t) (\mathbf{D}^T)^{-1}, \quad i = 1, 2 \quad (14)$$

Using matrix representation of approximate solution and its derivatives, Eq. (12) can be written as:

$$tP(t)\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{A} + \alpha\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{A} + t(\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{B})^p = f_1(t) \quad (15)$$

$$tH(t)\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{B} + \beta\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{B} + t(\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A})^q = f_2(t) \quad (16)$$

The residual $R_i(t)$ for Eq. (15) can be written as

$$R_1(t) \approx tP(t)\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{A} + \alpha\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{A} + t(\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{B})^p - f_1(t) \quad (17)$$

$$R_2(t) \approx tH(t)\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{B} + \beta\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{B} + t(\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A})^q - f_2(t) \quad (18)$$

and if it is applied typical tau method which is used in the sense of a particular form of the Petrov-Galerkin method [33–40], Eqs. (16) – (17) can be converted in $2 \times (N - 1)$ nonlinear equations by calculating

$$\left\langle R_i(t), T_n^*(t) \right\rangle = \int_0^1 R_i(t) T_n^*(t) dt = 0, \quad n = 0, 1, \dots, N - 1 \quad (19)$$

The initial conditions (13) lead to the four equations

$$\begin{aligned} y_1^N(0) &= \mathbf{X}(0)(\mathbf{D}^T)^{-1}\mathbf{A} = \lambda_0, \\ (y_1^N(x))' &= \mathbf{X}(0)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{A} = \lambda_1 \end{aligned} \quad (20)$$

$$\begin{aligned} y_2^N(0) &= \mathbf{X}(0)(\mathbf{D}^T)^{-1}\mathbf{B} = y_0, \\ (y_2^N(x))' &= \mathbf{X}(0)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{B} = y_1 \end{aligned} \quad (21)$$

Therefore, we get the $2 \times (N + 1)$ sets of equations with $2 \times (N + 1)$ unknowns by Eq. (18) and Eqs. (19) – (20). We write the program in the Maple 13 and solve the $2 \times (N + 1)$ sets of equations with $2 \times (N + 1)$ unknowns, and so approximate solution $y_i^N(x)$, $i = 1, 2$ can be calculated.

4.1 Error estimation and convergence analysis

Assume that $H = L^2[0, 1]$, where $P_N = \{T_0^*(t), T_1^*(t), \dots, T_N^*(t)\} \subset H$ be the set of polynomials of n -th degree and $W = \text{Span}(P_N)$. Clearly, W is a finite dimensional vector space. Let $f \in H$, then f has the unique best approximation out of W such that $g_0 \in W$, that is [41]

$$\|f - g_0\|_2 \leq \|f - g\|_2, \quad \forall g \in W \quad (22)$$

where $\|f\|_2^2 = \langle f, f \rangle$. There exist unique coefficients $\mathbf{A} = [a_0 \ a_1 \ \dots \ a_N]$ for g_0 such that

$$f \approx g_0 = \sum_{k=0}^N a_k T_k^*(t) = \mathbf{T}(t)\mathbf{A} \quad (23)$$

where $\mathbf{T}(x) = [T_0^*(t), T_1^*(t), \dots, T_N^*(t)]$ and coefficient matrix $\mathbf{A}$ can be given by the following equation

$$\mathbf{A} \langle \mathbf{T}(t), \mathbf{T}(t) \rangle = \langle f, \mathbf{T}(t) \rangle$$

where

$$\begin{aligned} \langle f, \mathbf{T}(t) \rangle &= \int_0^1 f(t) \mathbf{T}(t)^T dx = \\ &= \left[\langle f, T_0^*(t) \rangle \ \langle f, T_1^*(t) \rangle \ \dots \ \langle f, T_N^*(t) \rangle \right] \end{aligned}$$

and $\langle \mathbf{T}(t), \mathbf{T}(t) \rangle$ is an $(N + 1) \times (N + 1)$ matrix and

$$\phi = \langle \mathbf{T}(t), \mathbf{T}(t) \rangle = \int_0^1 \mathbf{T}(t) \mathbf{T}(t)^T dt$$

and so

$$\mathbf{A} = \left(\int_0^1 f(t) \mathbf{T}(t)^T dt \right) \phi^{-1} \quad (24)$$

Theorem 1. [41] Let assume that H is an Hilbert space, W is a closed subspace of H such that $\dim W$ is finite and $\{y_1, y_2, y_3, \dots, y_N\}$ is an y basis for W . Let f be an arbitrary element in H and g_0 be the unique best approximation to f out of W . Then we have

$$\|f - g_0\|_2^2 = \frac{D(f, y_1, y_2, \dots, y_N)}{D(y_1, y_2, \dots, y_N)} \quad (25)$$

where

$$D(f, y_1, y_2, \dots, y_N) = \begin{vmatrix} \langle f, f \rangle & \langle f, y_1 \rangle & \dots & \langle f, y_N \rangle \\ \langle y_1, f \rangle & \langle y_1, y_1 \rangle & \dots & \langle y_1, y_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle y_N, f \rangle & \langle y_N, y_1 \rangle & \dots & \langle y_N, y_N \rangle \end{vmatrix}$$

It is defined the inner product in H by $\langle f, g \rangle = \int_a^b f(t)g(t) dt$ and the subspace $W = \text{Span}(P_N)$, so the presented absolute error in Theorem 1 can be written

$$\|f - g_0\| = \frac{\det\left(\int_0^1 \Psi(t)\Psi(t)^T dt\right)}{\det\left(\int_0^1 \Phi(t)\Phi(t)^T dt\right)} \quad (26)$$

for which $\Phi(t)^T = [T_0^*(t) T_1^*(t) \dots T_N^*(t)]$ and $\Psi(x)^T = [f T_0^*(t) T_1^*(t) \dots T_N^*(t)]$.

Theorem 2. Assume that the function $g : [0, 1] \rightarrow R$ is $(N + 1)$ times continuously differentiable, $g \in C^{N+1}[0, 1]$ and $W = \text{Span}\{T_0^*(t), T_1^*(t), \dots, T_N^*(t)\}$. If $\mathbf{A}\Phi$ is the best approximation to g out of W , then a bound for absolute error is presented by

$$\|g - \mathbf{A}\Phi\| \leq \frac{M^2}{2^{2N}((N+1)!)^2}$$

where $M = \max_{x \in [0, 1]} (g_i(t)^{(N+1)})$.

Proof. We consider the interpolation polynomial $g^*(t)$ is the interpolating polynomial to g at t_i , where t_i , $i = 0, 1, \dots, N$ are the Chebyshev-Gauss grid points, then we have

$$g(t) - g^*(t) = \frac{g^{(N+1)}(\lambda)}{(N+1)!} \prod_{i=0}^N (t - t_i), \quad \lambda \in [0, 1] \quad (27)$$

Since, $\|T_N^*(t)\|_\infty = 1$, we conclude that if we choose the grid nodes $(t_i)_{0 \leq i \leq N}$ to be zero the $(N+1)$ zeroes of the Chebyshev polynomials $T_N^*(t)$, we have [25, 26]

$$\left\| \prod_{i=0}^N (t - t_i) \right\| = \frac{1}{2^N} \quad (28)$$

this is the smallest possible value. From (25), we obtain

$$\|g(t) - g^*(t)\|_\infty \leq \frac{1}{2^N(N+1)!} \|g(t)^{(N+1)}\|_\infty. \quad (29)$$

Since $\mathbf{A}\Phi$ is the best approximation to g out of W , considering $g^* \in W$ and using (28), we have

$$\begin{aligned} \|g - \mathbf{A}\Phi\| &\leq \|g - g^*\| = \int_0^1 |g(t) - g^*(t)|^2 dt \\ &\leq \int_0^1 \left(\frac{M}{2^N(N+1)!} \right)^2 dx \leq \frac{M^2}{2^{2N}((N+1)!)^2} \end{aligned}$$

Another comparison can be given for quality of the approximation by well known algorithm [32]. Since the approximate solution $y_{1,2}^N(t)$ is the approximate solution of

Eq. (1), Eq. (1) must be approximately satisfied by the function $y_N(x)$, that is;

$$\left| P(t) \frac{d^2 y_1^N(t)}{dt^2} + \frac{\alpha}{t} \frac{dy_1^N(t)}{dt} + (y_2^N(t))^p - g_1(t) \right| \approx 0 \quad (30)$$

$$\left| H(t) \frac{d^2 y_2^N(t)}{dt^2} + \frac{\beta}{t} \frac{dy_2^N(t)}{dt} + (y_1^N(t))^q - g_2(t) \right| \approx 0 \quad (31)$$

This comparison should be advised strongly by [32]. Then the error can be estimated by the error function [32]

$$E_1^N = \left| P(t) \frac{d^2 y_1^N(t)}{dt^2} + \frac{\alpha}{t} \frac{dy_1^N(t)}{dt} + (y_2^N(t))^p - g_1(t) \right| \quad (32)$$

$$E_2^N = \left| H(t) \frac{d^2 y_2^N(t)}{dt^2} + \frac{\beta}{t} \frac{dy_2^N(t)}{dt} + (y_1^N(t))^q - g_2(t) \right| \quad (33)$$

□

5 Examples

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple 13. Let us define

$$E_i^{N,L} = \left(\int_0^1 (y(t) - y_N(t))^2 dt \right)^{1/2}$$

where $y_i(x)$ and $y_i^N(x)$, $i = 1, 2$ denote the approximate solution obtained by the present method and the exact solution, respectively. In Tables, $N_e = |y_i(t) - y_i^N(t)|$ are absolute error for selected points.

Example 1. Let us consider the following linear systems of Lane-Emden equations [26]

$$\begin{aligned} y_1''(t) + \frac{3}{t} y_1'(t) - 4(y_1(t) + y_2(t)) &= 0 \\ y_2''(t) + \frac{2}{t} y_2'(t) + 3(y_1(t) + y_2(t)) &= 0 \end{aligned} \quad (34)$$

subject to initial conditions

$$y_1(0) = 1, \quad y_2(0) = 1, \quad y_1'(0) = 0, \quad y_2'(0) = 0.$$

For $N = 2$, We have the residual $R_i^N(t)$ for this problem

$$\begin{aligned} \mathbf{R}_1 &\approx t\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{A} + 3\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{A} - 4t\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A} \\ &\quad - 4t\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{B} - \mathbf{F}_1^T\mathbf{X}(t)(\mathbf{D}^T)^{-1} \end{aligned} \quad (35)$$

$$\begin{aligned} \mathbf{R}_2 &\approx t\mathbf{X}(t)\mathbf{C}^2(\mathbf{D}^T)^{-1}\mathbf{B} + 2\mathbf{X}(t)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{B} + 3t\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{A} \\ &\quad + 3t\mathbf{X}(t)(\mathbf{D}^T)^{-1}\mathbf{B} - \mathbf{F}_2^T\mathbf{X}(t)(\mathbf{D}^T)^{-1} \end{aligned} \quad (36)$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 3/8 & 1/2 & 1/8 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{X}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \quad \mathbf{F}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{F}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

Then, using Eqs.(34)-(35) we obtain the linear algebraic system

$$-2a_0 + \frac{16}{3}a_1 + \frac{26}{3}a_2 - 2b_0 - \frac{2}{3}b_1 + \frac{2}{3}b_2 = 0 \quad (37)$$

$$\frac{3}{2}a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 + \frac{3}{2}b_0 + \frac{9}{2}b_1 + \frac{15}{2}b_2 = 0 \quad (38)$$

with conditions

$$y_1^2(0) = \mathbf{X}(0)(\mathbf{D}^T)^{-1}\mathbf{A} = a_0 - a_1 + a_2 = 1 \quad (39)$$

$$y_2^2(0) = \mathbf{X}(0)(\mathbf{D}^T)^{-1}\mathbf{B} = 2a_1 - 8a_2 = 0 \quad (40)$$

$$(y_1^2(x))'(0) = \mathbf{X}(0)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{A} = b_0 - b_1 + b_2 = 1 \quad (41)$$

$$(y_2^2(x))'(0) = \mathbf{X}(0)\mathbf{C}(\mathbf{D}^T)^{-1}\mathbf{B} = 2b_1 - 8b_2 = 0 \quad (42)$$

Solving Eqs. (36) – (41), we obtain

$$a_0 = 1.375, \quad a_1 = 0.5, \quad a_2 = 0.125$$

$$b_0 = 0.625, \quad b_1 = -0.5, \quad b_2 = -0.125$$

Substituting these coefficients in Eqs. (4) – (5), then we get

$$y_1^2(x) = 1 + t^2$$

$$y_2^2(x) = 1 - t^2$$

These solutions are exact solution of this problem.

Example 2. Let us consider the linear, nonhomogeneous systems of Lane–Emden equations [26]

$$\begin{aligned} y_1''(t) + \frac{2}{t}y_1'(t) - (4t^2 + 6)y_1(t) + y_2(t) &= t^4 - t^3 \\ y_2''(t) + \frac{8}{t}y_2'(t) + ty_2(t) + y_1(t) &= e^{t^2} + t^5 - t^4 + 44t^2 - 30t \end{aligned}$$

subject to conditions

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 0$$

The comparison among present method and exact solution are shown in Table 1 and Table 2. In Table 3, the computational results of the L^2 -norm error between the approximate solutions and exact solution for different N and truncated errors are summarized. Figs. 1 – 2 represent the error between exact and approximate solutions for $N = 6, 8$. These figures show that errors are decreasing when N increasing.

Fig. 1: Comparison of absolute errors function for $y_1(t)$

Fig. 2: Comparison of absolute errors function for $y_2(t)$

Example 3. Let us consider the nonlinear equation [26]

$$\begin{aligned} y_1''(t) + \frac{5}{t}y_1'(t) + 8(e^{y_1(x)} + 2e^{-y_2/2}) &= 0 \\ y_2''(t) + \frac{3}{t}y_2'(t) - 8(e^{-y_2} + e^{y_1/2}) &= 0 \end{aligned}$$

with initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 0$$

Table 1: Numerical result for approximate solution of $y_1(t)$ in Example 2.

t	Exact Solution	N=5	$N_e=5$	N=6	$N_e=6$	N=8	$N_e=8$
0.0	1.000000	0.999999	0.800E-8	0.999999	0.500E-9	1.000000	0.000E-0
0.2	1.040810	1.040834	0.238E-4	1.040810	0.135E-6	1.040810	0.102E-6
0.4	1.173510	1.173384	0.126E-3	1.173517	0.690E-5	1.173510	0.261E-6
0.6	1.433329	1.433539	0.209E-3	1.433298	0.305E-4	1.433329	0.471E-6
0.8	1.896480	1.895792	0.688E-2	1.896583	0.102E-3	1.896481	0.909E-6
1.0	2.718281	2.686791	0.314E-1	2.712165	0.611E-3	2.718083	0.197E-3

Table 2: Numerical result for approximate solution of $y_2(t)$ in Example 2.

t	Exact Solution	N=5	$N_e=5$	N=6	$N_e=6$	N=8	$N_e=8$
0.0	0.0000	0.000E-0	0.000E-0	0.000E-0	0.000E-0	0.00000	0.000E-0
0.2	-0.0064	-0.006400	0.436E-7	-0.006400	0.189E-9	-0.00640	0.122E-9
0.4	-0.0384	-0.038399	0.343E-6	-0.038400	0.358E-7	-0.03840	0.399E-9
0.6	-0.0864	-0.086399	0.770E-5	-0.086399	0.102E-6	-0.08640	0.138E-8
0.8	-0.1024	-0.102400	0.620E-5	-0.102400	0.259E-6	-0.10240	0.455E-8
1.0	0.0000	-0.419E-4	0.419E-4	-0.722E-5	0.722E-5	0.168E-6	0.168E-6

Table 3: Numerical result for Example 2.

Present method	E_N^L		E_N^T	
	$y_1(t)$	$y_2(t)$	$y_1(t)$	$y_2(t)$
$N = 5$	0.615773×10^{-2}	101474×10^{-4}	10^{-2}	10^{-6}
$N = 6$	0.103500×10^{-2}	0.148370×10^{-5}	10^{-3}	10^{-8}
$N = 8$	0.263404×10^{-4}	0.256832×10^{-7}	10^{-5}	10^{-10}

Above algorithm is applied to this problem, we have $y_1(x) = x^2 - 2 \ln 1$, $y_2(x) = x^2 + 2 \ln 1$ which is the exact solution of this problem.

Example 4. Let us consider the following nonlinear problem [26]

$$\begin{aligned} y_1''(t) + \frac{1}{t} y_1'(t) - y_2^3(t)(y_1^2(t) + 1) &= 0 \\ y_2''(t) + \frac{3}{t} y_2'(t) + y_2^5(t)(y_1^2(t) + 3) &= 0 \end{aligned}$$

subject to conditions

$$y_1(0) = 1, y_1'(0) = 1, y_2(0) = 0, y_2'(0) = 0$$

with exact solutions

$$y_1 = \sqrt{1+t^2}, \quad y_2 = \frac{1}{\sqrt{1+t^2}},$$

Applying our method for $N = 4, 5, 6$, obtained numerical results are displayed in Table 4 and Table 5. Also, absolute errors are displayed in Figures 3 and 4. The tables and the figures show that proposed method is in good agreement with the analytical solution.

6 Conclusion

An operational matrix method for the solution of systems of Lane–Emden equations has been proposed and investi-

Fig. 3: Comparison of absolute errors function for $y_1(t)$ in Ex. 4

gated. Moreover, the convergence and error analysis have been determined. The presented numerical examples have exhibited the high accuracy, applicability, and efficiency of the proposed algorithms. The proposed method has been given to find the analytical solutions if the problem has exact solutions that are polynomial functions. This method has some considerable advantage that only small size operational matrix is required to provide the solution

Table 4: Numerical result for approximate solution of $y_1(t)$ in Example 5.

t	Exact Solution	N=4	$N_e=4$	N=5	$N_e=5$	N=6	$N_e=6$
0.0	1.00000	1.000000	0.000E-0	1.00000	0.000E-0	1.00000	0.000E-0
0.2	1.019803	1.020313	0.509E-3	1.019803	0.565E-4	1.019803	0.642E-5
0.4	1.077032	1.077661	0.628E-3	1.077032	0.216E-4	1.077032	0.220E-5
0.6	1.166190	1.166468	0.277E-3	1.166190	0.557E-5	1.166190	0.611E-5
0.8	1.280624	1.280897	0.272E-3	1.280624	0.738E-4	1.280624	0.471E-5
1.0	1.414213	1.414858	0.644E-3	1.414213	0.746E-4	1.414213	0.556E-5

Table 5: Numerical result for approximate solution of $y_2(t)$ in Example 5.

t	Exact Solution	N=4	$N_e=4$	N=5	$N_e=5$	N=6	$N_e=6$
0.0	1.000000	1.000000	0.000E-0	1.000000	0.000E-0	1.000000	0.000E-0
0.2	0.980580	0.979703	0.887E-3	0.980580	0.165E-4	0.980580	0.467E-5
0.4	0.928476	0.928249	0.227E-3	0.928476	0.151E-4	0.928476	0.405E-5
0.6	0.857492	0.858494	0.100E-3	0.857492	0.166E-4	0.857492	0.185E-5
0.8	0.780868	0.781561	0.692E-3	0.780868	0.167E-4	0.780868	0.327E-5
1.0	0.707106	0.706844	0.262E-3	0.707106	0.608E-4	0.707106	0.134E-6

Fig. 4: Comparison of absolute errors for $y_2(t)$ in Ex. 4

of high accuracy because most of matrix involves more numbers of zeroes and thus, reduces the run time such as 1.2 sn for Example 3 and lower operation count results in reduction of cumulative truncation errors and improvement of overall accuracy. It follows from the numerical results that the accuracy of the solutions obtained using the operational method is quite good in comparison with the exact solution. Moreover, in Tables 3, we give the L^2 -norm errors for Example 2.

Afterwards, the method can also be extended to the system of integro-differential equations, but some modifications are required.

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