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Influence of temperature-dependent properties on a gravity-driven thin film along inclined plate

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Abstract: A numerical investigation into the effect of temperature-dependent fluid properties on the thin film flow along an inclined heated plate is presented. The equations governing the coupled flow and heat transfer are formulated based on couple stress non-Newtonian model. Solutions of the coupled nonlinear differential equations are tackled numerically by using the combination of shooting method and the Fehlberg-Runge-Kutta method. Findings are presented graphically and discussed precisely.

Keywords: variable properties; thin film; heated plate; free and adiabatic conditions

1 Introduction

The importance of thin fluid flow along an inclined plate in many physical and engineering environment has accounted for the increasing number of studies in recent times. These studies are necessary to provide better understanding and improvement in the theory and applications of many real-life scenarios that are based on thin films. Some of the examples includes photography, heat exchangers, lubrications, electroplating of surfaces, and synovial movements. In view of this, Kay *et al.* [1] presented the thin film flow analysis over heated inclined plane using environmental model approach. Alharbi and Naire [2] introduced the R-adaptive mesh method for solv-

ing spreading thin fluid by taking the surface tension at the free surface into consideration. In a study by Shuaib *et al.* [3] a numerical study of a thin fluid flow with a leakage was presented along an inclined surface. Yu and Cheng [4] presented experimental findings of the thin fluid flow behavior on the inclined plate. Du *et al.* [5] performed an experiment on the gravity driven thin film of water under constantly changing inclination along an ellipsoidal heated plate. More interesting results can be found in ref. [6–10] and references cited within.

Motivated by several findings of many thermal and mechanical engineering applications, the present article is devoted to temperature dependent properties of thin film flow. Since findings [11–16] documented in the literature have shown over the last few years that formulations based on constant fluid properties are assumptions that are only valid in cases when the heat flow has been neglected or in small but finite thermal ranges. In the context of thin film flow, several works have been documented in the literature on thermal instability with or without temperature-dependent properties. Closely related to this work is the brilliant effort by Makinde [16, 17], in which thin film along heated plate with inclination and the thermal instabilities associated with the gravity-driven flow was accurately determined based on aspect ratio approximation of the governing equations. However, apart from the dynamic viscosity, other fluid properties are very sensitive to temperature changes as confirmed in [11–16]. As a result, the specific objective of this article is to investigate the effect of variation of thermal conductivity, fluid temperature and internal heat generation on the flow and thermal structure of the thin gravity-driven flow which to the best of our understanding has not been addressed in spite of its various application in the thermal and lubrication community. The present work is coupled and nonlinear, therefore, the exact solution of the problem is not possible. The numerical solution of the problem has been obtained by using the combination of shooting method to convert the boundary value problem to its equivalent initial value problem and solved using the fourth order Runge-Kutta integration scheme. The result revealed the non-existence of solution as the critical values of the nonlinear terms are exceeded.

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The next section of the article is devoted to the mathematical analysis, which consist of the model formulation, dimensionless process and the numerical solution of the couple problem including thermal stability. Numerical results are presented in the third section of the paper while the article is concluded in section four.

2 Mathematical analysis

Consider the gravity-driven flow of a temperature-dependent viscous and incompressible fluid down an inclined heated surface. The heated plate is assumed to be at the angle of inclination, ϑ , to the horizontal with uniform film thickness δ as shown in Figure 1 below. The fluid dynamic viscosity, thermal conductivity, and the internal heat generation depends on temperature.

Figure 1: Flow geometry

Using the aspect ratio condition, the balance equations for the wave-less flow can be written as [16]:

$$0 = \frac{d}{dy} \left(\mu \frac{du}{dy} \right) + \rho g \sin \vartheta \quad (1)$$

$$0 = \frac{d}{dy} \left(k \frac{dT}{dy} \right) + \frac{\mu}{k} \left(\frac{du}{dy} \right)^2 + Q \quad (2)$$

Subject to free and adiabatic flow conditions

$$\begin{aligned} \bar{u} &= 0, & T &= T_0 & \text{on } \bar{y} &= 0 \\ \frac{d\bar{u}}{d\bar{y}} &= 0, & \frac{dT}{d\bar{y}} &= 0 & \text{on } \bar{y} &= \delta \end{aligned} \quad (3)$$

Additional terms in (1)-(3) are temperature dependent internal heat generation and thermal conductivity. In equations (1)-(3), u, ρ, g represent the fluid dimensional velocity, density, and gravitational acceleration respectively, the fluid dynamic viscosity μ , thermal conductivity k depends on temperature T in an exponential manner while internal heat generation Q is a linear function as follows:

$$\mu = \mu_0 e^{-\alpha'(T-T_a)}, \quad k = k_0 e^{-\beta'(T-T_a)}, \quad Q = Q_0 (T - T_0) \quad (4)$$

where μ_0, k_0, T_a are the referenced dynamic viscosity, thermal conductivity, and temperature (associated to a) respectively. Also, α', β', Q_0 are variation constants which could take positive or negative values depending on the fluid under consideration. Using the following dimensionless variables and parameters

$$\begin{aligned} y &= \frac{\bar{y}}{\delta}, \quad U = \frac{\mu_0 u}{\delta^2 \rho g \sin \vartheta}, \quad \theta = \frac{T - T_a}{T_a}, \quad \alpha = \alpha' T_a, \\ \beta &= \beta' T_a, \quad \lambda = \frac{(\delta^2 \rho g \sin \vartheta)^2}{\mu_0 T_a k}, \quad \eta = \frac{\delta^2 Q_0}{k_0} \end{aligned} \quad (5)$$

we get

$$\frac{d^2 U}{dy^2} = \alpha \frac{d\theta}{dy} \frac{dU}{dy} - e^{\alpha\theta}; \quad U(0) = 0 = \frac{dU}{dy}(1) \quad (6)$$

$$\begin{aligned} \frac{d^2 \theta}{dy^2} &= \beta \left(\frac{d\theta}{dy} \right)^2 - \lambda e^{(\beta-\alpha)\theta} \left(\frac{dU}{dy} \right)^2 - \eta \theta e^{\beta\theta}; \\ \theta(0) &= 0 = \frac{d\theta}{dy}(1) \end{aligned} \quad (7)$$

Here the dimensionless parameters U, α, θ are dimensionless velocity, viscosity variation, and dimensionless temperature respectively while β, λ, η are thermal conductivity variation, viscous heating and internal heat generation parameter respectively. Moving forward, the boundary value problem (BVP) in (6)-(7) are coupled and non-linear. The exact solution of the problem is practically not possible; therefore, it is more convenient to obtain the numerical solution. We implement the famous shooting method on the coupled system by converting the boundary value problem to its equivalent initial value problem as follows.

$$x_1 = U, \quad x_2 = U', \quad x_3 = \theta, \quad x_4 = \theta' \quad (8)$$

With (8), the coupled BVP becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} x_2 \\ \alpha x_4 x_2 - e^{\alpha x_3} \\ x_4 \\ \beta x_4^2 - \lambda e^{(\beta-\alpha)x_3} - \eta x_3 e^{\beta x_3} \end{pmatrix} \quad (9)$$

Where (') represents derivative with respect to x_1 , equation (9) is subject to the initial conditions

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} 0 \\ D_1 \\ 0 \\ D_2 \end{pmatrix} \quad (10)$$

The system of equations (9) together with the initial conditions are elegantly obtained by using Runge-Kutta with the Fehlberg numerical integration approach. It is important to note that $D_{1,2}$ are the guess values that ensures that the boundary conditions at $y = 1$ are met.

Next, the skin interaction and the wall heat transfer rate along the heated wall can be expressed as follows:

$$\tau_w = \mu_0 e^{-\alpha'(T-T_a)} \left. \frac{du}{dy'} \right|_{y'=0} \quad (11)$$

With (5), (11) becomes

$$Sf = \frac{\tau_w}{\delta \rho g \sin \alpha} = e^{-\alpha \theta} \left. \frac{dU}{dy} \right|_{y=0} \quad (12)$$

3 Results and discussion

In this section, graphical representations of the numerical solutions are presented for various flow parameter values. In Figure 2, the temperature profile is represented against the thermal conductivity varying parameter. From the plot, rise in the variable thermal conductivity parameter leads to increase in the rate of heat transfer at the free end due to increasing heat flux, however, the heat at inclined plate affects the thermal conductivity of the fluid layer closer to the wall and that explains the minimal convective heat transfer in that region. Results in Figure 3 shows that convective heat transfer rate elevates the fluid velocity, and this follows the same trend with the temperature profile. In Figure 4, the effect of viscosity varying parameter on the temperature profile is presented. Result shows that decreasing fluid viscosity enhances the fluid temperature. This is correct because of increased molecular interaction of the fluid particles associated with decrease in the cohesive forces as the fluid viscosity decreases. Similarly, in Figure 5, thermal effect on the fluid viscosity is presented. From the result, increased viscosity variation parameter weakens cohesive force binding the fluid particles, as a result, the increased kinetic energy of the fluid particles enhances the deformation rate as shown in the plot.

Figures 6 and 7 represent the effect of viscous heating on the fluid temperature and velocity respectively. The

Figure 2: β effect on temperature

Figure 3: β effect on velocity

result shows heat generated from frictional interaction of fluid particles elevates the fluid temperature and the velocity respectively. Figures 8 and 9 depict the influence of temperature dependent internal heat generation parameter on the temperature and velocity profile respectively. From the plots, increasing the internal heat generation parameter helps elevate both fluid temperature and the thin film velocity along the heated plate. In Figure 10, the varia-

Figure 4: α effect on temperature Figure 6: λ effect on temperature Figure 5: α effect on velocity Figure 7: λ effect on velocity

tion of wall heat transfer (θ_{max}) with β and α is illustrated. An increase in β values shows a decrease in the criticality values (α_c), and this means that the rate of heat transfer at the wall is reduced as the viscosity of the fluid is increased, the same scenario is observed in Figure 11.

4 Conclusion

A numerical approach to investigate the thin film flow with temperature-dependent properties along heated inclined plates subjected to free and adiabatic surface has been presented. From the numerical computation, the temperature-dependent heat source, viscosity and thermal conductivity parameters are seen to enhance both

Figure 8: η effect on temperature Figure 9: η effect on velocity

fluid velocity and temperature significantly. Area of future research is not limited to the computation of critical values for thermal runaway.

Figure 10: Wall heat transfer rate variation with β and α Figure 11: Wall heat transfer rate variation with λ and α

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