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Comparative study of two techniques on some nonlinear problems based using conformable derivative

<https://doi.org/10.1515/nleng-2020-0030>

Received Feb 24, 2020; accepted Nov 19, 2020.

Abstract: In this paper, three eminent types of time-fractional nonlinear partial differential equations are considered, which are the fractional foam drainage equation, fractional Gardner equation, and fractional Fornberg–Whitham equation in the sense of conformable fractional derivative. The approximate solutions of these considered problems are constructed and discussed using the conformable fractional variational iteration method and conformable fractional reduced differential transform method. The conformable derivative is one of the admirable choices to handle nonlinear physical problems of different fields of interest. Comparisons of approximate solution obtained by two techniques, to each other and with the exact solutions are also presented and affirm that the considered methods are efficient and reliable techniques to study other nonlinear fractional equations and models in the sense of conformable derivative. To explain the effects of several parameters and variables on the movement, the approximate results are shown in tables and two- and three-dimensional surface graphs.

Keywords: fractional calculus, conformable fractional derivative, reduced differential transform method, variational iteration method

1 Introduction

Fractional partial differential equations are used to model most of the natural problems, such as finance, physical, chemical, fluid dynamics, biological and other fields of engineering. For diagnosing these physical models as well as in addition follow these physical models in realistic med-

ical studies, it is essential to discover their approximate and exact solutions. In calculus a lot of numerical, analytic and approximate methods are developed for nonlinear models [1–7].

There are miscellaneous researches correlated to this derivative. Atangana *et al.* [8] give a few definitions of the conformable derivative and time-space fractional heat differential equations being resolved with the aid of the conformable derivative delivered by Cenesiz and Kurt [9]. New exact solutions of conformable Burgers' type equations obtained by Çenesiz *et al.* [10]. Conformable differential method was introduced by Ünal and Gökdoğan [11]. In addition, the conformable fractional method has been carried out to many models with the aid of many authors [12–15]. By using conformable operator Yavuz explains the dynamical behaviors of models [16, 17]. The conformable fractional derivative is taken into consideration for this examine because it is simple for the calculation. A Chinese professor He [18, 19] introduced variational iteration method. Such as linear and nonlinear wave equation, systems of fourth-order differential equations are solved by the variational iteration method [20]. Conformable variational iteration method is built on the conformable derivative for fractional ordinary differential equations and fractional partial differential equations [21].

Keskin and Oturanc [22, 23] introduced reduced differential transform method. The technique gives a dependable and proficient process for extensive kind of chemical engineering, medical and fluid model problems, inclusive of fractional order or non-fractional partial differential equations [24]. By the definition of conformable derivative, Acan *et al.* [25] offered a new form of reduced differential transform technique that is known as conformable reduced differential transform, implemented to a few fractional differential equations.

Take a look, foam drainage equation is very momentous because many technological and commercial programs were developed, which consist cleansing, water purification, mineral extraction. Numerical solution of fractional differential equations is supplied in [26]. Traveling Wave solution of fractional differential equations with the

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aid of using rational (G'/G)-expansion technique is presented in [27].

We analyzed the time fractional Gardner equation. The Gardner equation is an advantageous example for the definition of interior solitary waves in shallow water, while Buckmaster's equation is applied in thin viscous fluid sheet flows. Gardner equation is extensively used in several physics and engineering problems consisting of fluid physics, chemical engineering, and quantum discipline principle. The equation presents outstanding function in ocean waves. In shallow seas Gardner equation defines internal solitary waves. This equation scrutinized within the literature because it's far used to model a diffusion of nonlinear phenomena. Gardner equation is solved by way of using F-expansion in [28] and has been generally examined by several other methods [29–31].

The Fornberg–Whitham equation has been found to require peakon results as a simulation for limiting wave heights as well as the frequency of wave breaks. In fractional calculus has gained considerable significance and popularity, primarily because of its well-shown applications in a wide range of apparently disparate areas of engineering and science. Fractional Fornberg–Whitham equation is solved by using fractional variational iteration method in [32]. Kumar [33] to resolve the fractional Fornberg–Whitham equation via Laplace transform. Besides, the Laplace Adomian decomposition approach is likewise used in [34]. Many scholars have therefore researched the fractional extensions of the Fornberg–Whitham model for the Caputo fractional-order derivative [35–37].

2 Basic concept for fractional calculus

Some primary definitions of fractional calculus are recalled which can be practiced in the calculation. It is famous that there are precise definitions of fractional Integral and fractional derivatives, which includes, Grünwald-Letnikov, Riesz, Riemann-Liouville, Caputo, Hadamard and Erdélyi-Kober and lots of others [38–40].

Definition 2.1 The fractional integral operator of RL of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-v)^{\alpha-1} dv, \quad \alpha > 0, \quad x > 0 \quad (1)$$

$$J^0 f(x) = f(x). \quad (2)$$

Definition 2.2 The Caputo's fractional derivative is [41]

$$D^\alpha t^\xi = \begin{cases} \frac{\Gamma(\xi+1)}{\Gamma(\xi+1-\alpha)} t^{\xi-\alpha}, & \xi \in \mathbb{N}_0, \xi \geq [\alpha] \\ 0, & \xi \in \mathbb{N}_0, \xi < [\alpha], \end{cases} \quad (3)$$

$$D^\alpha c = 0, \quad c \in \mathbb{R}.$$

Definition 2.3 Let f be an n time differentiable at x then the conformable fractional derivative is well-defined by [41]

$$(T_\alpha f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{f^{([a]-1)}(x + \varepsilon x^{[a]-\alpha}) - f^{([a]-1)}(x)}{\varepsilon}, \quad x > 0, \quad \alpha \in (0, 1]. \quad (4)$$

If the above limit exists, then f is called α -differentiable. Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $x > 0$, then T_α satisfies the following properties:

1.

$$T_\alpha (af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad \text{for all } a, b \in \mathbb{R}$$

2.

$$T_\alpha(x^p) = px^{p-\alpha}, \quad \text{for all } p \in \mathbb{R}$$

3. $T_\alpha(\lambda) = 0$, for all constant functions $f(x) = \lambda$

4.

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$$

5.

$$T_\alpha(f/g) = \{gT_\alpha(f) - fT_\alpha(g)\} / g^2$$

6. If, in addition, f is differentiable $T_\alpha(f)(x) = x^{1-\alpha} \frac{df}{dx}(x)$

3 Analysis of conformable fractional variational iteration method (CFVIM)

Firstly the conformable fractional variational iteration method is discussed for the solution of the following nonlinear time fractional partial differential equation

$$T_t^\alpha w(x, t) + L(w(x, t)) + N(w(x, t)) = h(x, t), \quad n < \alpha \leq n+1, \quad (5)$$

where L and N is linear and non-linear operator respectively, $h(x)$ is source term and T_t^α is conformable fractional derivative of order α . To solve differential Equation (5) via conformable fractional variational iteration method, write the differential Eq. (5) in the form,

$$t^{[\alpha]-\alpha} \frac{\partial^{([a])} w(x, t)}{\partial t^{([a])}} + L(w(x, t)) + N(w(x, t)) = h(x, t). \quad (6)$$

As in conformable fractional variational iteration method, the correction functional for Eq. (6) can be

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \mu(\tau) \left(\tau^{[\alpha]-\alpha} \frac{\partial^{([\alpha])} w_n(x, \tau)}{\partial \tau^{([\alpha])}} + L(w_n(x, \tau)) + N(\tilde{w}_n(x, \tau)) - h(x, \tau) \right) d\tau. \quad (7)$$

Finally, the solution is

$$w(x, t) = \lim_{n \rightarrow \infty} w_n(x, t). \quad (8)$$

4 Analysis of conformable fractional reduced differential transform method (CFRDTM)

Let $w(x, t)$ is continuous and analytic with respect to time t and space x . conformable fractional reduce differential transform of $w(x, t)$ is defined as

$$W_k^\alpha(x) = \frac{1}{\alpha^k k!} [T_t^{\alpha(k)} w]_{t=t_0}. \quad (9)$$

Where some $0 < \alpha \leq 1$, α is a parameter labeling the order of CFD, $T_t^{\alpha(k)} w = \underbrace{T_t^\alpha T_t^\alpha T_t^\alpha \dots T_t^\alpha}_{k\text{-times}} w(x, t)$.

The inverse conformable fractional reduce differential transform of $W_k^\alpha(x)$ is defined as

$$w(x, t) = \sum_{k=0}^{\infty} W_k^\alpha(x) (t - t_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{1}{\alpha^k k!} [T_t^{\alpha(k)} w]_{t=t_0} (t - t_0)^{\alpha k} \quad (10)$$

For integer order derivatives the initial conditions for conformable fractional reduce differential transform are defined as

$$W_k^\alpha(x) = \begin{cases} \frac{1}{(\alpha k)!} \left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}} w(x, t) \right] & \text{if } \alpha k \in \mathbb{Z}^+ \\ 0 & \text{if } \alpha k \notin \mathbb{Z}^+ \end{cases} \quad \text{for } k = 0, 1, 2, \dots, \left(\frac{n}{\alpha} - 1 \right) \quad (11)$$

where n is the order of conformable fractional partial differential equation.

By considering $W_k^\alpha(x) = f(x)$ as the initial condition

$$w(x, 0) = f(x). \quad (12)$$

and

$$\tilde{w}_n(x, t) = \sum_{k=0}^n W_k^\alpha(x) t^{\alpha k}. \quad (13)$$

The solution is written as

$$w(x, t) = \lim_{n \rightarrow \infty} \tilde{w}_n(x, t). \quad (14)$$

Numerical applications

4.1 The foam drainage equation

$$T_t^\alpha w = \frac{1}{2} w w_{xx} - 2w^2 w_x + (w_x)^2, \quad (15)$$

subject to condition

$$w(x, 0) = -\sqrt{c} \tanh(\sqrt{cx}). \quad (16)$$

According to conformable fractional variational iteration method, the correctional functional of Eq. (15) is read as

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \mu \left(\tau^{1-\alpha} (w_{n\tau}) - \frac{1}{2} w_n w_{nxx} + 2w_n^2 w_{nx} - (w_{nx})^2 \right) d\tau \quad (17)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left(\tau^{1-\alpha} (w_{n\tau}) - \frac{1}{2} w_n w_{nxx} + 2w_n^2 w_{nx} - (w_{nx})^2 \right) d\tau \quad (18)$$

Consequently, we get

$$w_0(x, t) = -\sqrt{c} \tanh(\sqrt{cx}), \quad (19)$$

$$w_1(x, t) = -\sqrt{c} \tanh(\sqrt{cx}) - \frac{c^2 t^{2-\alpha} \text{Sech}(\sqrt{cx})^2}{\alpha - 2}, \quad (20)$$

$$w_2(x, t) = -\frac{2}{3} c^5 t^3 \text{Sech}(\sqrt{cx})^4 - \sqrt{c} \text{Tanh}(\sqrt{cx}) + c^5 t^3 \text{Sech}(\sqrt{cx})^4 \left(1 + c^{\frac{3}{2}} t \text{Tanh}(\sqrt{cx}) \right) + \frac{1}{\alpha - 2} \left(c^2 t \text{Sech}(\sqrt{cx})^2 \left(t^{1-\alpha} + (\alpha - 2) \left(2 + c^{\frac{3}{2}} t \text{Tanh}(\sqrt{cx}) \right) \right) \right), \quad (21)$$

Now, by applying conformable fractional reduce differential transform on Eq. (15) and (16), we get

$$W_0^\alpha(x) = -\sqrt{c} \tanh(\sqrt{cx}), \quad (22)$$

$$W_1^\alpha(x) = \frac{1}{\alpha} \left(c^2 \text{Sech}(\sqrt{cx})^2 \right), \quad (23)$$

$$W_2^\alpha(x) = \frac{1}{\alpha^2} \left(c^{\frac{7}{2}} \text{Sech}(\sqrt{cx})^2 \text{Tanh}(\sqrt{cx}) \right), \quad (24)$$

$$W_3^\alpha(x) = \frac{c^5}{3\alpha^3} \left(-2 + \text{Cosh}(2\sqrt{cx}) \text{Sech}(\sqrt{cx})^4 \right), \quad (25)$$

The series solution is

$$\tilde{w}(x, t) = \sum_{k=0}^n W_k^{\alpha}(x) t^{k\alpha}, \quad (26)$$

$$\tilde{w}(x, t) = W_0^{\alpha} + W_1^{\alpha} t + W_2^{\alpha} t^2 + W_3^{\alpha} t^3 + \dots, \quad (27)$$

$$\begin{aligned} \tilde{w}(x, t) = & -\sqrt{c} \tanh(\sqrt{c}x) + \frac{1}{\alpha} \left(c^2 \operatorname{Sech}(\sqrt{c}x)^2 \right) t + \frac{1}{\alpha^2} \left(c^{\frac{7}{2}} \operatorname{Sech}(\sqrt{c}x)^2 \operatorname{Tanh}(\sqrt{c}x) \right) t^2 \\ & + \frac{c^5}{3\alpha^3} \left(-2 + \operatorname{Cosh}(2\sqrt{c}x) \operatorname{Sech}(\sqrt{c}x)^4 \right) t^3 + \dots, \end{aligned} \quad (28)$$

The exact solution of Eq. (15) is

$$w(x, t) = \begin{cases} -\sqrt{c} \tanh(\sqrt{c}(x - ct)) & ; \quad x \leq ct \\ 0 & ; \quad x > ct \end{cases} \quad (29)$$

Here c is the speed of wave.

Table 1: Numerical analysis of solution Eq. (21), (28) and exact solution (29) for example 1 at $c=1$ and $\alpha=1$

x	t	CFRDTM	CFVIM	Exact Sol	Absolute Error	
					CFRDTM	CFVIM
-0.2	0.01	0.206967	0.206967	0.206966	1.2631×10^{-5}	1.2011×10^{-6}
	0.001	0.198336	0.198336	0.198336	2.0421×10^{-7}	1.1052×10^{-8}
	0.0001	0.197471	0.197471	0.197471	2.0531×10^{-7}	1.5014×10^{-9}
-0.5	0.01	0.469945	0.469945	0.469945	2.2421×10^{-8}	1.2540×10^{-10}
	0.001	0.462903	0.462903	0.462903	2.5321×10^{-8}	1.4251×10^{-10}
	0.0001	0.462196	0.462196	0.462196	2.6173×10^{-8}	2.2530×10^{-10}
-0.7	0.01	0.610677	0.610677	0.610677	1.2003×10^{-8}	1.3046×10^{-10}
	0.001	0.605002	0.605002	0.605002	1.5631×10^{-8}	1.7613×10^{-10}
	0.0001	0.604431	0.604431	0.604431	1.8641×10^{-8}	1.9141×10^{-10}
-0.9	0.01	0.721132	0.721132	0.721132	1.5042×10^{-8}	2.2931×10^{-10}
	0.001	0.716784	0.716784	0.716784	1.9402×10^{-8}	2.4641×10^{-10}
	0.0001	0.716347	0.716347	0.716347	2.0047×10^{-8}	3.0021×10^{-10}
-1	0.01	0.765762	0.765762	0.765762	3.0042×10^{-8}	4.0001×10^{-10}
	0.001	0.762014	0.762014	0.762014	3.9442×10^{-8}	4.6851×10^{-10}
	0.0001	0.761636	0.761636	0.761636	4.0723×10^{-8}	5.0031×10^{-10}

Figure1

Figure 1: Approximate solution for the foam drainage model Eq. (15) for different values of α with fixed values $0 \leq t \leq 1, -10 \leq x \leq 0$.

4.2 Time-fractional Gardner equation

$$T_t^\alpha u + 6(u - \varepsilon^2 u^2)u_x + u_{xxx} = 0, \quad (30)$$

with the initial condition

$$w(x, 0) = \frac{1}{2} + \frac{1}{2} \tanh \left[\frac{x}{2} \right]. \quad (31)$$

For $\varepsilon = 1$, (30) will reduce into

$$T_t^\alpha w + 6(w - w^2)w_x + w_{xxx} = 0, \quad (32)$$

According to conformable fractional variational iteration method, the correctional functional of Eq. (32) is

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \mu \left(\tau^{1-\alpha} (w_{n\tau}) + 6w_n w_{nx} - 6w_n^2 w_{nx} - w_{nxxx} \right) d\tau \quad (33)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left(\tau^{1-\alpha} (w_{n\tau}) + 6w_n w_{nx} - 6w_n^2 w_{nx} - w_{nxxx} \right) d\tau \quad (34)$$

Consequently, we get

$$w_0(x, t) = \frac{1}{2} + \frac{1}{2} \tanh \left[\frac{x}{2} \right], \quad (35)$$

$$w_1(x, t) = \frac{1}{8} \left(4 - 9t \operatorname{Sech} \left(\frac{x}{2} \right)^2 + t \operatorname{Sech} \left(\frac{x}{2} \right)^4 - 96t \operatorname{Csch}(x)^3 \operatorname{Sinh} \left(\frac{x}{2} \right)^4 - 80t \operatorname{Csch}(x)^4 \operatorname{Sinh} \left(\frac{x}{2} \right)^6 + 4 \operatorname{Tanh} \left(\frac{x}{2} \right) \right), \quad (36)$$

$$w_2(x, t) = -\frac{1}{8} t \operatorname{Sech} \left(\frac{x}{2} \right)^4 \left(1 + 7 \operatorname{Cos}(x) + 6 \operatorname{Sin}(x) \right) + \frac{1}{2} \left(1 + \operatorname{Tanh} \left(\frac{x}{2} \right) \right) + \frac{1}{98304} t \operatorname{Sech} \left(\frac{x}{2} \right)^{12} \left(1 + \operatorname{Tanh} \left(\frac{x}{2} \right) \right) \\ \left(\begin{aligned} & -64512 + 36960t - 8012t^3 - \frac{23688}{-2+\alpha} t^{1-\alpha} + 4(-27738 + 20265t - 4064t^2 \\ & + 1356t^3 - \frac{9936}{-2+\alpha} t^{1-\alpha}) \operatorname{Cosh}(x) + \left(\begin{aligned} & -70020 + 74694t - 9712t^2 \\ & + 15924t^3 - \frac{23220}{-2+\alpha} t^{1-\alpha} \end{aligned} \right) \operatorname{Cosh}(2x) \\ & -31620 \operatorname{Cosh}(3x) + 39594t \operatorname{Cosh}(3x) - 128t^2 \operatorname{Cosh}(3x) - 6576t^3 \operatorname{Cosh}(3x) \\ & - \frac{9936}{-2+\alpha} t^{1-\alpha} \operatorname{Cosh}(3x) - 9696 \operatorname{Cosh}(4x) + 9648t \operatorname{Cosh}(4x) + 3t^3 \operatorname{Cosh}(4x) \\ & - \frac{2232}{-2+\alpha} t^{1-\alpha} \operatorname{Cosh}(4x) - 1812 \operatorname{Cosh}(5x) + 690t \operatorname{Cosh}(5x) - \frac{288}{-2+\alpha} t^{1-\alpha} \operatorname{Cosh}(5x) \\ & -156 \operatorname{Cosh}(6x) + 42t \operatorname{Cosh}(6x) - \frac{12}{-2+\alpha} t^{1-\alpha} \operatorname{Cosh}(6x) - 1944 \operatorname{Sinh}(x) \\ & -11336t^2 \operatorname{Sinh}(x) + 4170t^3 \operatorname{Sinh}(x) - \frac{4464}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(x) - 2052 \operatorname{Sinh}(2x) \\ & 17118t \operatorname{Sinh}(2x) - 8360t^2 \operatorname{Sinh}(2x) + 15510t^3 \operatorname{Sinh}(2x) - \frac{4932}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(2x) \\ & -948 \operatorname{Sinh}(3x) + 7962t \operatorname{Sinh}(3x) + 24t^2 \operatorname{Sinh}(3x) - 6606t^3 \operatorname{Sinh}(3x) \\ & - \frac{2568}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(3x) - 144 \operatorname{Sinh}(4x) + 1488t \operatorname{Sinh}(4x) + 1364t^2 \operatorname{Sinh}(4x) \\ & -3t^3 \operatorname{Sinh}(4x) - \frac{624}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(4x) + 36 \operatorname{Sinh}(5x) - 66t \operatorname{Sinh}(5x) \\ & - \frac{24}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(5x) + 12 \operatorname{Sinh}(6x) - 42t \operatorname{Sinh}(6x) + \frac{12}{-2+\alpha} t^{1-\alpha} \operatorname{Sinh}(6x) \end{aligned} \right) \quad (37)$$

And so on.

Now, by applying conformable fractional reduce differential transform on Eq. (32) and (31), we get

$$W_0^\alpha(x) = \frac{1}{2} + \frac{1}{2} \operatorname{Tanh} \left(\frac{x}{2} \right), \quad (38)$$

$$W_1^\alpha(x) = \frac{1}{8\alpha} \left(\operatorname{Sech} \left(\frac{x}{2} \right)^4 \right) (-1 + 5 \operatorname{Cosh}(x) + 6 \operatorname{Sinh}(x)), \quad (39)$$

$$W_2^\alpha(x) = -\frac{1}{128\alpha^2} \left(\operatorname{Sech} \left(\frac{x}{2} \right)^7 \right) \left(\begin{aligned} & 168 \operatorname{Cosh} \left(\frac{x}{2} \right) - 420 \operatorname{Cosh} \left(\frac{3x}{2} \right) + 60 \operatorname{Cosh} \left(\frac{5x}{2} \right) \\ & -190 \operatorname{Sinh} \left(\frac{x}{2} \right) - 417 \operatorname{Sinh} \left(\frac{3x}{2} \right) + 61 \operatorname{Sinh} \left(\frac{5x}{2} \right) \end{aligned} \right), \quad (40)$$

$$W_3^\alpha(x) = -\frac{1}{3072\alpha^3} \left(\operatorname{Sech}\left(\frac{x}{2}\right)^{10} \right) \begin{pmatrix} -48737 + 3686\cosh(x) + 59336\cosh(2x) \\ -21278\cosh(3x) + 665\cosh(4x) + \\ 7909\sinh(x) + 49500\sinh(2x) \\ -21276\sinh(3x) + 666\sinh(4x) \end{pmatrix}, \quad (41)$$

The series solution is

$$\begin{aligned} \tilde{w}(x, t) = & \frac{1}{2} + \frac{1}{2} \tanh\left(\frac{x}{2}\right) + \frac{1}{8\alpha} \left(\operatorname{Sech}\left(\frac{x}{2}\right)^4 \right) (-1 + 5\cosh(x) + 6\sinh(x)) t + \\ & -\frac{1}{128\alpha^2} \left(\operatorname{Sech}\left(\frac{x}{2}\right)^7 \right) \begin{pmatrix} 168\cosh\left(\frac{x}{2}\right) - 420\cosh\left(\frac{3x}{2}\right) + 60\cosh\left(\frac{5x}{2}\right) - 190\sinh\left(\frac{x}{2}\right) - \\ 417\sinh\left(\frac{3x}{2}\right) + 61\sinh\left(\frac{5x}{2}\right) \end{pmatrix} t^2 + \\ & -\frac{1}{3072\alpha^3} \left(\operatorname{Sech}\left(\frac{x}{2}\right)^{10} \right) \begin{pmatrix} -48737 + 3686\cosh(x) + 59336\cosh(2x) - 21278\cosh(3x) + \\ 665\cosh(4x) + 7909\sinh(x) + 49500\sinh(2x) - 21276\sinh(3x) \\ + 666\sinh(4x) \end{pmatrix} t^3 + \dots, \end{aligned} \quad (42)$$

The exact solution of Eq. (30) is

$$u(x, t) = \frac{1}{2} + \frac{1}{2} \tanh\left[\frac{x-t}{2}\right]. \quad (43)$$

Table 2: Numerical analysis of solution Eq. (37), (42) and exact solution (43) for example 2 at $\alpha=1$

x	t	CFRDTM	CFVIM	Exact Sol	Absolute Error	
					CFRDTM	CFVIM
0.2	0.0001	0.549899	0.549509	0.549809	9.0131×10^{-5}	3.0520×10^{-4}
	0.00001	0.549841	0.549801	0.549832	9.0421×10^{-6}	3.1052×10^{-5}
	0.000001	0.549835	0.549831	0.549834	1.0531×10^{-6}	3.0114×10^{-6}
0.5	0.0001	0.622545	0.622086	0.622436	1.0921×10^{-4}	3.5254×10^{-4}
	0.00001	0.622468	0.622422	0.622457	1.1032×10^{-5}	3.5421×10^{-5}
	0.000001	0.622460	0.622456	0.622459	1.0013×10^{-6}	3.6253×10^{-6}
0.7	0.0001	0.668284	0.667784	0.668166	1.1803×10^{-4}	4.0231×10^{-4}
	0.00001	0.668197	0.668147	0.668186	1.1245×10^{-5}	3.9214×10^{-5}
	0.000001	0.668189	0.668184	0.668188	1.0021×10^{-6}	4.0241×10^{-6}
0.9	0.0001	0.711054	0.710517	0.710929	1.2504×10^{-4}	4.1234×10^{-4}
	0.00001	0.710960	0.710906	0.710947	1.3110×10^{-5}	4.1064×10^{-5}
	0.000001	0.710951	0.710945	0.710949	2.0047×10^{-6}	4.2021×10^{-6}
1	0.0001	0.731165	0.730613	0.731039	1.2604×10^{-4}	4.2601×10^{-4}
	0.00001	0.731069	0.731014	0.731057	5.5242×10^{-4}	2.5129×10^{-5}
	0.000001	0.731060	0.731054	0.731058	2.0123×10^{-6}	4.0062×10^{-6}

4.3 Time-fractional Fornberg–Whitham equation

$$D_t^\alpha w - w_{xxt} + w_x = ww_{xxx} - ww_x + 3w_x w_{xx}, \quad t > 0, \quad \alpha > 0, \quad (44)$$

Figure 2

Figure 2: Approximate solution for the time-fractional Gardner Eq. (32) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$.

subject to initial condition

$$w(x, 0) = e^{\frac{1}{2}x}. \quad (45)$$

According to conformable fractional variational iteration method, the correctional functional of Eq. (44) is

$$w_{n+1}(x, t) = w_n(x, t) + \int_0^t \mu \left(\tau^{1-\alpha} (w_{n\tau}) - w_{nxt} + w_{nx} - w_n w_{nxxx} + w_n w_{nx} + 3w_{nx} w_{nxx} \right) d\tau \quad (46)$$

$$w_{n+1}(x, t) = w_n(x, t) - \int_0^t \left(\tau^{1-\alpha} (w_{n\tau}) - w_{nxt} + w_{nx} - w_n w_{nxxx} + w_n w_{nx} + 3w_{nx} w_{nxx} \right) d\tau \quad (47)$$

Consequently, we get

$$w_0(x, t) = e^{\frac{1}{2}x}, \quad (48)$$

$$w_1(x, t) = e^{\frac{1}{2}x} - \left(\frac{e^{\frac{1}{2}x}}{2} + \frac{3e^x}{4} \right) t, \quad (49)$$

$$w_2(x, t) = \frac{1}{64(-2+\alpha)} \left(e^{\frac{1}{2}x} t^{-\alpha} \left(\begin{aligned} &-16 \left(2 + 3e^{\frac{1}{2}x} \right) t^2 - \\ &t^{\alpha} \left(-64 + t \left(\begin{aligned} &-8(-9+t) + 4e^{\frac{1}{2}x}(-6+t)^2 \\ &+ 21e^x(-3+t)t + 36e^{\frac{3}{2}x}t^2 \end{aligned} \right) \right) (-2+\alpha) \end{aligned} \right) \right), \quad (50)$$

And so on.

Now, by applying conformable fractional reduce differential transform on Eq. (44) and (45), we get

$$W_0^{\alpha}(x) = e^{\frac{1}{2}x}, \quad (51)$$

$$W_1^{\alpha}(x) = -\frac{1}{2\alpha} \left(e^{\frac{1}{2}x} \right), \quad (52)$$

$$W_2^{\alpha}(x) = \frac{1}{8\alpha^2} \left(e^{\frac{1}{2}x} \right), \quad (53)$$

$$W_3^{\alpha}(x) = -\frac{1}{48\alpha^3} \left(e^{\frac{1}{2}x} \right), \quad (54)$$

The series solution is

$$\tilde{w}(x, t) = e^{\frac{1}{2}x} + \frac{1}{2\alpha} \left(e^{\frac{1}{2}x} \right) t + \frac{1}{8\alpha^2} \left(e^{\frac{1}{2}x} \right) t^2 + \frac{1}{48\alpha^3} \left(e^{\frac{1}{2}x} \right) t^3 + \dots, \quad (55)$$

The closed form solution of Eq. (44) is

$$w(x, t) = e^{\frac{1}{2}|x-\frac{4}{3}t|}. \quad (56)$$

5 Results and discussions

From the above results, Figure 1 (a) represents the physical behavior of solution equation (21) by CFVIM or different values of α with fixed values $0 \leq t \leq 1, -10 \leq x \leq 0$. (b) Shows physical behavior of the solution using CFRDTM of equation (28) for different values of α with fixed values $0 \leq t \leq 1, -10 \leq x \leq 0$. (c) Depicts the comparison of CFVIM and CFRDTM with exact solution for $\alpha=1$ with fixed values $0 \leq t \leq 1, -10 \leq x \leq 0$. (d) Depicts the comparison of CFVIM and CFRDTM with exact solution with $\alpha=0.5$ for different values x and t . (e) Illustrations the comparison of CFVIM and CFRDTM with exact solution with for different values x and t for $\alpha=1$. Figure 2 (a) shows the physical behavior of CFVIM's

Figure 3

Figure 3: Approximate solution for the time-fractional Fornberg-Whitham Eq. (44) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$.

Table 3: Numerical analysis of solution Eq. (50), (55) and exact solution (56) for example 3 at $\alpha=1$

x	t	CRDTM	CFVIM	Exact Sol	Absolute Error	
					CRDTM	CFVIM
0.01	0.01	0.999523	0.998376	0.998335	1.1885×10^{-3}	4.1541×10^{-5}
	0.001	1.004510	1.004287	1.004341	1.6942×10^{-4}	5.4102×10^{-5}
	0.0001	1.004960	1.004980	1.004951	9.0531×10^{-6}	2.9123×10^{-5}
0.05	0.01	1.020201	1.010333	1.018521	1.6832×10^{-3}	8.1882×10^{-3}
	0.001	1.024801	1.024311	1.024630	1.7151×10^{-4}	3.194×10^{-4}
	0.0001	1.025260	1.025390	1.025250	1.6173×10^{-5}	1.4253×10^{-4}
0.07	0.01	1.030450	1.030260	1.028741	1.7092×10^{-3}	1.5190×10^{-3}
	0.001	1.035123	1.033371	1.034930	1.9331×10^{-4}	1.5591×10^{-3}
	0.0001	1.035570	1.035539	1.035550	2.0041×10^{-5}	1.1041×10^{-5}
0.09	0.01	1.040810	1.038328	1.039081	1.7291×10^{-3}	7.5312×10^{-4}
	0.001	1.045510	1.045373	1.045331	1.7940×10^{-4}	4.2641×10^{-5}
	0.0001	1.045980	1.045901	1.045961	1.9001×10^{-5}	6.0021×10^{-5}
0.1	0.01	1.046030	1.044503	1.044291	1.7390×10^{-3}	2.1201×10^{-4}
	0.001	1.050751	1.050651	1.050571	1.8241×10^{-4}	8.2451×10^{-5}
	0.0001	1.051220	1.051210	1.051200	2.0012×10^{-5}	1.0031×10^{-5}

solution of equation (37) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (b) Depicts that the solution of CFRDTM of equation (42) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (c) Depicts the comparison of CFVIM and CFRDTM with exact solution for $\alpha=1$ with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (d) Displays the comparison of CFVIM and CFRDTM and exact solution with $\alpha=0.5$ for different values x and t . (e) Demonstrates the comparison of CFVIM and CFRDTM with exact solution for different values x and t for $\alpha=1$.

Fig 3 (a) shows surface behavior the solution by CFVIM of equation (50) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (b) Shows physical behavior of solution by CFRDTM of equation (55) for different values of α with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (c) Displays the comparison of CFVIM and CFRDTM results with exact solution for $\alpha=1$ with fixed values $0 \leq t \leq 1$, $0 \leq x \leq 1$. (d) Illustrate the comparison of CFVIM and CFRDTM and exact solution with $\alpha=0.5$ for different values x and t . (e) Confirmations the comparison of CFVIM and CFRDTM with exact solution for different values x and t for $\alpha=1$.

Table 1, 2 and 3 numerically explore the comparison of obtained solutions with the exact solution. Absolute errors also determine to show the worth of these techniques.

6 Conclusion

In this study, we successfully applied the conformable fractional differential transform method and conformable variational iteration method on the foam drainage equation, time- fractional Gardner equation and time-fractional Fornberg–Whitham equation. We showed the existence and uniqueness of the solutions. The solutions obtained by these methods are compared with the exact solution which is presented in graphs. From the results, given graphs, it is clear that for these models CFVIM gives more accurate results as compare to the CFRDTM using only a few iterates. Numerical results indicate that these techniques are very good results when looked carefully at the graphs and are very powerful, simple and compatible for solving the nonlinear FPDEs and allow us to understand the dynamics of nonlinear mathematical model of propagation. In this work Mathematica software is used for computing the results, tables and graphs. From these conclusions, we can say that the CFVIM with fractional derivative is suitable for examining many problems in the fields of science and engineering.

Conflict of interest: The authors declare no conflict of interest regarding the publication of this paper.

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