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On the exact and numerical solutions to a new $(2 + 1)$ -dimensional Korteweg-de Vries equation with conformable derivative

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Abstract: The aim of this paper is to introduce a novel study of obtaining exact solutions to the $(2+1)$ -dimensional conformable KdV equation modeling the amplitude of the shallow-water waves in fluids or electrostatic wave potential in plasmas. The reduction of the governing equation to a simpler ordinary differential equation by wave transformation is the first step of the procedure. By using the improved $\tan(\phi/2)$ -expansion method (ITEM) and Jacobi elliptic function expansion method, exact solutions including the hyperbolic function solution, rational function solution, soliton solution, traveling wave solution, and periodic wave solution of the considered equation have been obtained. We achieve also a numerical solution corresponding to the initial value problem by conformable variational iteration method (C-VIM) and give comparative results in tables. Moreover, by using Maple, some graphical simulations are done to see the behavior of these solutions with choosing the suitable parameters.

Keywords: Solitons, Korteweg-de Vries equation, exact solutions, improved $\tan(\phi/2)$ -expansion method, Jacobi elliptic function expansion method

1 Introduction

Nonlinear evolution equations (NLEEs) have been used for many years to express the modern world phenomena we encounter in nonlinear sciences such as mathematical biology, plasma physics, elastic media, finance, fluid mechanics, control theory, chemistry, optics, engineering sciences, etc. The concept of fractional derivative operator

dates back to the work of L'hospital in the 17th century. We need fractional partial differential equations (FPDEs) to be able to and interpret physical models that occur in most applied sciences. Recently, we have been observing these equations, especially in physical models that contain space and time variables. In fact, fractional derivatives better explain the various physical phenomena encountered. It is quite crucial to examine fractional space-time differential equations which are nonlocal operator instead of integer ordered differential operator which is a local operator. For, as is known, this means that the next state of a system depends not only on its present state but also on all its former states. This is the main advantage of fractional differential equations over integer-order differential equations and this makes the model created more realistic [1]. There are several definitions of fractional derivative, such as Riemann-Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu [2–6].

In 2014, a new derivative called conformable derivative by Khalil [7] was identified and developed by Abdel-javad [8]. To date, there are many studies in the literature on this subject [9–16]. Over the past few years, many methods have been proposed to find exact solutions of equations with fractional derivatives. For example first integral method [17–19], the modified trial equation method [20, 21], auxiliary equation method [22, 23], the modified Kudryashov method [24–26], Jacobi elliptic function expansion method [27] and so on [28–31].

The classical $(1 + 1)$ -dimensional KdV equation

$$u_t + 6u_x u + u_{xxx} = 0 \quad (1)$$

is a well known equation that is utilized to characterize the waves on shallow water surfaces. In addition, this equation has a wide range of applications in various branches such as bubble liquid mixtures, waves in enharmonic crystals, ion acoustic wave and magneto-hydrodynamic waves in a warm plasma, cold lossless (collisionless) plasma as well as shallow water waves [32–37]. One of the main features of the KdV equation is that the speed of solitary wave is related to the magnitude of the solitary wave, and the other is that the solutions can represent solitary wave solutions noted as solitons which have quantum mechani-

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cal effects which occur in particle physics and quantum field theory [33, 37]. Many researchers have investigated numerous versions of this famous equation with different procedures and techniques [38–49]. A few of the various interesting features of this well-known equation are an infinite number of conservation laws (higher order), bi-Hamilton structures, symmetries and the Lax pair. The $(2 + 1)$ -dimensional KdV equation

$$u_t + 3(uv)_x + u_{xxx} = 0, \quad u_x - v_y = 0 \quad (2)$$

was derived by Boiti et al. using the idea of the weak Lax pair [50]. For $v = u$ and $y = x$, Eq. (2) reads the $(1 + 1)$ -dimensional KdV equation. Compared to $(1 + 1)$ -dimensional case, the $(2 + 1)$ -dimensional case explains a more involved nonlinear phenomenon. As highlighted in the lines above that physical systems can be better expressed with fractional order derivatives used for global modeling.

The main objective of this paper is to use ITEM to find the new exact solutions of the time fractional $(2 + 1)$ -dimensional KdV equation given as follows

$$D_t^\alpha u - 6u_x u + 6u_y u - u_{xxx} + u_{yyy} + 3u_{xyy} - 3u_{xyx} = 0, \\ t \geq 0, 0 < \alpha \leq 1. \quad (3)$$

where $D_t^\alpha(\cdot)$ is conformable derivative of order α . When $\alpha = 1$, Eq. (3) changes to the $(2 + 1)$ -dimensional KdV equation which has been proved based on the extended Lax pair and has been announced in [46].

The organization of this paper is as follows: In Section 2, some basic definitions of the conformable derivative are recalled. In Sections 3, 4 and 5, the key idea of ITEM, Jacobi elliptic function expansion method and conformable variational iteration method is described. In section 6, the acquisition of the considered physical model is briefly presented. In Sections 7, 8 and 9, applications of the methods are given. Finally, some conclusions are presented in the last section.

2 Conformable derivative

R. Khalil et al. proposed a derivative that coincides with the classical derivative when $\alpha = 1$ and that can rectify the shortcomings of the previous definitions. Here, definition and some properties of the conformable derivative are presented [7, 8].

Definition [7]: Let $f : [0, \infty) \rightarrow \mathbb{R}$. Then the α order conformable derivative of f is defined as

$$T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0, \alpha \in (0, 1)$.

Theorem 1: Let T_α be a conformable derivative operator with order α and $\alpha \in (0, 1]$, f, g be α -differentiable at point $t > 0$. Then [6, 7],

- (i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \forall a, b \in \mathbb{R}$.
- (ii) $T_\alpha(t^p) = pt^{p-\alpha}, \forall p \in \mathbb{R}$.
- (iii) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
- (iv) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
- (v) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
- (vi) If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

3 Algorithm of the improved $\tan(\phi/2)$ -expansion method for NLEEs

This method was summarized and improved for achieving the analytic solutions of NLEEs by Manafian et al in 2015 [51]. Assume a nonlinear partial differential equation is given in general form as follows

$$\mathfrak{N}(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0. \quad (4)$$

After simple algebraic operations, this equation is transformed into an ordinary differential equation (ODE) with $\xi = x - \mu t$ transformation

$$\mathfrak{Q}(u, u', -\mu u', u'', \mu^2 u'', \dots) = 0. \quad (5)$$

Then, assume that the searched wave solutions of Eq. (5) have the following representation

$$u(\xi) = S(\phi) = \sum_{k=0}^m A_k [p + \tan(\phi/2)]^k + \sum_{k=1}^m B_k [p + \tan(\phi/2)]^{-k} \quad (6)$$

where A_k ($0 \leq k \leq m$) and B_k ($1 \leq k \leq m$) are constants to be determined and p is arbitrary constant, such that $A_m \neq 0$, $B_m \neq 0$ and $\phi = \phi(\xi)$ is the solution of the following first order differential equation:

$$\phi'(\xi) = a \sin(\phi(\xi)) + b \cos(\phi(\xi)) + c. \quad (7)$$

If we try to find the solution of the (7), then we obtain special solutions that vary according to the state of the coefficients:

Family 1. When $\Delta = a^2 + b^2 - c^2 < 0$ and $b - c \neq 0$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{a}{b-c} - \frac{\sqrt{-\Delta}}{b-c} \tan\left(\frac{\sqrt{-\Delta}}{2} \xi\right) \right]$

Family 2. When $\Delta = a^2 + b^2 - c^2 > 0$ and $b - c \neq 0$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{a}{b-c} + \frac{\sqrt{\Delta}}{b-c} \tanh\left(\frac{\sqrt{\Delta}}{2} \xi\right) \right]$

Family 3. When $\Delta = a^2 + b^2 - c^2 > 0, b \neq 0$ and $c = 0$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{a}{b} + \frac{\sqrt{b^2 + a^2}}{b} \tanh\left(\frac{\sqrt{b^2 + a^2}}{2} \xi\right) \right]$

Family 4. When $\Delta = a^2 + b^2 - c^2 < 0$, $c \neq 0$ and $b = 0$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{-a}{c} + \frac{\sqrt{c^2 - a^2}}{c} \tanh\left(\frac{\sqrt{c^2 - a^2}}{2} \bar{\xi}\right) \right]$

Family 5. When $\Delta = a^2 + b^2 - c^2 > 0$, $b - c \neq 0$ and $a = 0$, then $\phi(\xi) = 2 \tan^{-1} \left[\sqrt{\frac{b+c}{b-c}} \tanh\left(\frac{\sqrt{b^2 - c^2}}{2} \bar{\xi}\right) \right]$

Family 6. When $a = 0$ and $c = 0$, then $\phi(\xi) = \tan^{-1} \left[\frac{e^{2b\bar{\xi}} - 1}{e^{2b\bar{\xi}} + 1}, \frac{e^{2b\bar{\xi}}}{e^{2b\bar{\xi}} + 1} \right]$

Family 7. When $b = 0$ and $c = 0$, then $\phi(\xi) = \tan^{-1} \left[\frac{e^{2a\bar{\xi}}}{e^{2a\bar{\xi}} + 1}, \frac{e^{2a\bar{\xi}} - 1}{e^{2a\bar{\xi}} + 1} \right]$

Family 8. When $a^2 + b^2 = c^2$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{a\bar{\xi} + 2}{(b-c)\bar{\xi}} \right]$

Family 9. When $a = b = c = ka$, then $\phi(\xi) = 2 \tan^{-1} [e^{ka\bar{\xi}} - 1]$

Family 10. When $a = c = ka$ and $b = -ka$, then $\phi(\xi) = -2 \tan^{-1} \left[\frac{e^{ka\bar{\xi}}}{e^{ka\bar{\xi}} - 1} \right]$

Family 11. When $c = a$, then $\phi(\xi) = -2 \tan^{-1} \left[\frac{(a+b)e^{b\bar{\xi}} - 1}{(a-b)e^{b\bar{\xi}} - 1} \right]$

Family 12. When $a = c$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{(c+b)e^{b\bar{\xi}} + 1}{(b-c)e^{b\bar{\xi}} - 1} \right]$

Family 13. When $c = -a$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{e^{b\bar{\xi}} + b - a}{e^{b\bar{\xi}} - b - a} \right]$

Family 14. When $b = -c$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{ae^{a\bar{\xi}}}{1 - de^{a\bar{\xi}}} \right]$

Family 15. When $b = 0$ and $a = c$, then $\phi(\xi) = -2 \tan^{-1} \left[\frac{c\bar{\xi} + 2}{c\bar{\xi}} \right]$

Family 16. When $a = 0$ and $b = c$, then $\phi(\xi) = 2 \tan^{-1} [c\bar{\xi}]$

Family 17. When $a = 0$ and $b = -c$, then $\phi(\xi) = -2 \tan^{-1} \left[\frac{1}{c\bar{\xi}} \right]$

Family 18. When $a = 0$ and $b = 0$, then $\phi(\xi) = c\bar{\xi} + C$

Family 19. When $b = c$, then $\phi(\xi) = 2 \tan^{-1} \left[\frac{e^{a\bar{\xi}} - c}{a} \right]$, where $\bar{\xi} = \xi + C$, $p, A_0, A_k, B_k (k = 1, 2, \dots, m)$, a, b and c are constants to be determined later.

As usual, for determining m , the highest order derivative should be balanced with the highest order nonlinear terms in Eq. (5). In the case of $m = q/p$ (where $m = q/p$ be a fraction in the lowest term), we need to do a conversion on the unknown function u as follows:

$$u(\xi) = (v(\xi))^{q/p}. \quad (8)$$

Then substitute Eq. (8) into Eq. (5). By using of new Eq. (5), the value of m can be determined. If m be a negative integer, similar process can be followed with the transformation

$$u(\xi) = (v(\xi))^m.$$

Following these operations, according to the m value obtained above, let substitute (6) into Eq. (5). Therefore we obtain a set of algebraic equations that contains $\tan(\phi/2)^k, \cot(\phi/2)^k, (k = 0, 1, 2, \dots)$. Then setting each coefficients of $\tan(\phi/2)^k, \cot(\phi/2)^k$ to zero, we can get a set of over-determined equations for $A_0, A_k, B_k (k =$

$1, 2, \dots, m)$, a, b, c and p . Since obtained algebraic equations system will be difficult to solve manually, symbolic computation as Maple can be used at this stage. Finally, $A_0, A_1, B_1, \dots, A_m, B_m, \mu, p$ are replaced in the Eq. (6).

4 Algorithm of the Jacobi elliptic function expansion method

In this section, we recall the Jacobi elliptic function expansion method [52]. Consider a nonlinear partial differential equation is

$$\mathfrak{N}(u, u_x, u_t, u_{xx}, u_{tt}, \dots) = 0. \quad (9)$$

Applying the transformation $\xi = x - \mu t$, (9) is transformed into an ODE

$$\mathfrak{Q}(u, u', -\mu u', u'', \mu^2 u'', \dots) = 0, \quad (10)$$

where $u = u(\xi)$, $u' = \frac{du}{d\xi}, \dots$. In order to construct more general periodic and solitary wave solution of Eq. (3) by employing the Jacobi elliptic function expansion method, it is assumed that $u(\xi)$ can be formulated as a finite series of Jacobi elliptic sine and cosine functions. The ansatz are given below

$$u(\xi) = \sum_{j=0}^n a_j \operatorname{sn}^j(\xi), \quad (11)$$

and

$$u(\xi) = \sum_{j=0}^n b_j \operatorname{cn}^j(\xi), \quad (12)$$

where n, a_j and $b_j (j = 0, 1, 2, 3, \dots)$ are constants. $\operatorname{sn}(\xi) = \operatorname{sn}(\xi|m)$ and $\operatorname{cn}(\xi) = \operatorname{cn}(\xi|m)$ where $m (0 < m < 1)$ is called a modulus of the elliptic function, are double periodic and satisfy the following properties:

$$\operatorname{sn}^2(\xi) + \operatorname{cn}^2(\xi) = 1, \quad \operatorname{dn}^2(\xi) + m^2 \operatorname{sn}^2(\xi) = 1,$$

$$\frac{d}{d\xi} \operatorname{sn}(\xi) = \operatorname{cn}(\xi) \operatorname{dn}(\xi), \quad \frac{d}{d\xi} \operatorname{cn}(\xi) = -\operatorname{sn}(\xi) \operatorname{dn}(\xi),$$

$$\frac{d}{d\xi} \operatorname{dn}(\xi) = -m^2 \operatorname{cn}(\xi) \operatorname{sn}(\xi).$$

The value of n is determined again by balancing the nonlinear term and the highest derivative. Therefore, the highest degree of $\frac{d^p u}{d\xi^p}$ is taken as

$$O\left(\frac{d^p u}{d\xi^p}\right) = n + p, \quad p = 1, 2, 3, \dots \quad (13)$$

and the nonlinear term as

$$O\left(u^q \frac{d^p u}{d\xi^p}\right) = (q + 1)n + p, \quad q = 0, 1, 2, 3, \dots \quad (14)$$

Then substituting the ansatz (11) and (12) into Eq. (10) and equating the coefficients of all powers of elliptic functions to zero, we get a system of algebraic equations for a_j and b_j ($j = 0, 1, 2, 3, \dots$).

5 Succinct overview of the conformable variational iteration method

In this section, we will present how the conformable variational iteration method (C-VIM) works for conformable nonlinear evolution type equations [53, 54]. Let us assume that the following conformable nonlinear evolution equations in operator form

$${}_t T_\alpha u(x, t) + L(u(x, t)) + N(u(x, t)) = g(x, t), \quad n < \alpha \leq n+1 \quad (15)$$

where L is a linear operator, N is a non-linear operator, g is an non-homogeneous term and ${}_t T_\alpha$ is conformable derivative of order α . To solve differential equation (15) via C-VIM write the differential equation (15) in the form by Theorem 1 (property (vi)),

$${}_t^{[\alpha]-\alpha} \frac{\partial^{([\alpha])} u(x, t)}{\partial t^{([\alpha])}} + L(u(x, t)) + N(u(x, t)) = g(x, t). \quad (16)$$

As in classical variational iteration method, the trial functional for (16) can be constructed as

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(\zeta) \left[\zeta^{[\alpha]-\alpha} \frac{\partial^{([\alpha])} u_n(x, \zeta)}{\partial \zeta^{([\alpha])}} + L(u_n(x, \zeta)) + N(u_n(x, \zeta)) - g(x, \zeta) \right] d\zeta$$

where λ is a general Lagrangian multiplier and it can be optimally determined by the aid of variational theory [55–57]. Here $\tilde{u}_n$ is a restricted variation [55–57] where $\delta \tilde{u}_n = 0$. As the first step of this approach, λ multiplier should be derived by the help of variational theory and integration by parts. Using the determined Lagrangian multiplier and any selected function u_0 , the u_{n+1} iteration, which is the successive approximations of $u(x)$ for $n \geq 0$, will be obtained readily. Hence, we get the solution as

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t).$$

6 Governing equation

Recently in 2019, a new (2+1)-dimensional KdV equation has been proved based on the extended Lax pair [46]. To

derive of the (2 + 1)-dimensional KdV equation, (2 + 1)-dimensional zero curvature equation [42, 58, 59] considered

$$X_t - X_x + T_x - T_y + XT - TX = 0, \quad (17)$$

where

$$X = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix}, T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (18)$$

with the compatibility conditions and $\zeta_t = 0$. Plugging (18) into Eq. (17), a system of algebraic equations is obtained. Based on the work done by Ablowitz [42], A, B, C, D is assumed to have a series expansion in $K = \sum_{j=0}^n k_j(x, y, t) \zeta^{n-j}$, where $K = A, B, C, D$ and $k = a, b, c, d$. Relations between coefficients is found by using algebraic equation system and series expansions of functions [46]. If $n = 2$, the new (2 + 1)-dimensional Schrödinger equation [59] is obtained

$$i(q_x - q_t) + q_{xx} + q_{yy} - 2q_{xy} + 2|q|^2 q = 0. \quad (19)$$

If $n = 3$, with the appropriate selection of coefficients [46], it immediately generates the following new (2 + 1)-dimensional KdV equation

$$q_t - q_x - 6qq_x + 6qq_y - q_{xxx} + q_{yyy} + 3q_{xy} - 3q_{xyy} = 0.$$

If $q = u$, by the help of the Galilean transformation $X = x - t, T = t, Y = y$, one can get

$$u_t - 6uu_x + 6uu_y - u_{xxx} + u_{yyy} + 3u_{xy} - 3u_{xyy} = 0. \quad (20)$$

If $y = x$, (20) is reduced to (1 + 1)-dimensional KdV equation (1). In the next sections, the (2 + 1)-dimensional KdV equation with conformable derivative (3) obtained via Eq.

(20) will be discussed.

7 Application of ITEM to conformable (2+1) dimensional KdV equation

In this section, we apply the ITEM to Eq. (3) to obtain the traveling wave solutions. In this context, let us consider $u(x, t) = u(\xi)$, $\xi = kx + ry - \frac{w}{\alpha} t^\alpha$ and therefore Eq. (3) becomes

$$-wu' - (6k - 6r)u'u - (k^3 - r^3 - 3rk^2 + 3r^2k)u''' = 0 \quad (21)$$

where " ' " shows the derivative according to ξ . By integrating (21) once with respect to ξ , we obtain

$$-wu - (3k - 3r)u^2 - (k^3 - r^3 - 3rk^2 + 3r^2k)u'' = 0. \quad (22)$$

With balancing procedure, where u'' derivative is balanced by u^2 , $m + 2 = 2m$, then $m = 2$ is obtained. Therefore, by considering $p = 0$ in (6), we get the following finite series expansion for unknown function of $u(\xi)$

$$u(\xi) = A_0 + A_1 \tan\left(\frac{\phi(\xi)}{2}\right) + A_2 \left(\tan\left(\frac{\phi(\xi)}{2}\right)\right)^2 + B_1 \left(\cot\left(\frac{\phi(\xi)}{2}\right)\right) + B_2 \left(\cot\left(\frac{\phi(\xi)}{2}\right)\right)^2. \quad (23)$$

We substitute the expression of u in (23) into (22) and collect all terms with the same order of $\tan(\phi(\xi)/2)$, $\cot(\phi(\xi)/2)$ together. Then by equating the coefficient of each polynomial to zero, we obtain a set of algebraic equations

$$\begin{aligned} & 3 B_2 b^2 k^3 - 9 B_2 b^2 k^2 r + 9 B_2 b^2 k r^2 - 3 B_2 b^2 r^3 + 6 B_2 b c k^3 - 18 B_2 b c k^2 r + 18 B_2 b c k r^2 \\ & - 6 B_2 b c r^3 + 3 B_2 c^2 k^3 - 9 B_2 c^2 k^2 r + 9 B_2 c^2 k r^2 - 3 B_2 c^2 r^3 + 6 B_2^2 k - 6 B_2^2 r = 0, \\ & 2 B_1 b^2 k^3 - 6 B_1 b^2 k^2 r + 6 B_1 b^2 k r^2 - 2 B_1 b^2 r^3 + 4 B_1 b c k^3 - 12 B_1 b c k^2 r - 24 B_1 B_2 r \\ & + 12 B_1 b c k r^2 - 4 B_1 b c r^3 + 2 B_1 c^2 k^3 - 6 B_1 c^2 k^2 r + 6 B_1 c^2 k r^2 - 2 B_1 c^2 r^3 + 24 B_1 B_2 k \\ & + 20 B_2 a b k^3 - 60 B_2 a b k^2 r + 60 B_2 a b k r^2 - 20 B_2 a b r^3 + 20 B_2 a c k^3 - 20 B_2 a c r^3 \\ & - 60 B_2 a c k^2 r + 60 B_2 a c k r^2 = 0, \end{aligned} \quad (24)$$

⋮

Solving the above algebraic equations (24) by help of Maple, we have numerous sets of coefficients for the solutions of (22). We only choose some of them as follows:

SET 1

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = \frac{1}{6} r^2 b^2 - \frac{1}{6} k^2 c^2 + \frac{1}{6} k^2 b^2 - \frac{1}{6} r^2 c^2 - \frac{1}{3} k r b^2 + \frac{1}{3} k r c^2, A_1 = 0, \\ A_2 = -\frac{1}{2} (-b r + k b + r c - k c)^2, B_1 = 0, B_2 = 0, a = 0, b = b, c = c, k = k, r = r, \\ w = k^3 b^2 - 3 r k^2 b^2 + 3 r^2 k b^2 - r^3 b^2 - k^3 c^2 + 3 r k^2 c^2 - 3 r^2 k c^2 + r^3 c^2 \end{array} \right\} \quad (25)$$

By using *Family 1*, (23) becomes

$$u_1(x, y, t) = \frac{(k-r)^2 (b-c)(b+c)}{6} - \frac{(-b r + k b + r c - k c)^2 (-b^2 + c^2)}{2(b-c)^2} \left(\tan\left(\frac{\sqrt{-b^2+c^2}}{2} \left(kx + ry - \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2.$$

By using *Family 2*, (23) reads

$$\begin{aligned} u_2(x, y, t) &= \frac{(k-r)^2 (b-c)(b+c)}{6} \\ &- \frac{(k-r)^2 (b-c)(b+c)}{2} \left(\tanh\left(\frac{\sqrt{b^2-c^2}}{2} \left(kx + ry - \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2. \end{aligned}$$

By using *Family 5*, one constructs for (23)

$$\begin{aligned} u_3(x, y, t) &= \frac{(k-r)^2 (b^2 - c^2)}{6} \\ &- \frac{(-b r + k b + r c - k c)^2 (b+c)}{2(b-c)} \left(\tanh\left(\frac{\sqrt{b^2-c^2}}{2} \left(kx + ry - \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2. \end{aligned} \quad (26)$$

By using *Family 8*, (23) can be written as

$$u_4(x, y, t) = -\frac{-24 k r + 12 r^2 + 12 k^2}{6(kx + ry + C)^2}. \quad (27)$$

By using *Family 11*, we can write

$$u_5(x, y, t) = \frac{(k-r)^2 b^2}{6} - \frac{\left(\frac{(-b r + k b)^2}{2} \left(b e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - 1 \right) \right)^2}{\left(-b e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - 1 \right)^2}. \quad (28)$$

By using *Family 12*, (23) becomes

$$u_6(x, y, t) = \frac{(k-r)^2 b^2}{6} - \frac{(-br + kb)^2 \left(be^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + 1 \right)^2}{2 \left(be^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} - 1 \right)^2}. \quad (29)$$

By using *Family 13*, (23) reads

$$u_7(x, y, t) = \frac{(k-r)^2 b^2}{6} - \frac{(-br + kb)^2 \left(e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + b \right)^2}{2 \left(e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} - b \right)^2}. \quad (30)$$

By using *Family 17*, one constructs for (23)

$$u_8(x, y, t) = -\frac{(2r - 2k)^2}{2(kx + ry + C)^2}. \quad (31)$$

By using *Family 18*, we get

$$u_9(x, y, t) = -\frac{(k-r)^2 c^2}{6} - \frac{(r-k)^2 c^2}{2} \left(\tan \left(\frac{c}{2} \left(kx + ry + \frac{(k-r)^3 c^2 t^\alpha}{a} \right) + \frac{C}{2} \right) \right)^2. \quad (32)$$

SET 2

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = \frac{(-r+k)^2(b^2-c^2)}{2}, A_1 = A_1, A_2 = -\frac{(-r+k)^2(-c+b)^2}{2}, B_1 = 0, B_2 = 0, \\ a = \frac{A_1}{(-r+k)^2(-c+b)}, b = b, c = c, k = k, r = r, w = \frac{-(-r+k)^4(b+c)(-c+b)^3 + A_1^2}{(-r+k)(-c+b)^2} \end{array} \right\}$$

By using *Family 1*, (23) can be written as

$$u_{10}(x, y, t) = \frac{(-r+k)^2(-c+b)(b+c)}{2} + A_1 \left(\frac{A_1}{(-r+k)^2(-c+b)^2} - \sqrt{c^2 - b^2 - \frac{A_1^2}{(-c+b)^2(-r+k)^4(-c+b)}} \tan(F) \right) - \frac{(-r+k)^2(-c+b)^2}{2} \left(\frac{A_1}{(-r+k)^2(-c+b)^2} - \sqrt{c^2 - b^2 - \frac{A_1^2}{(-c+b)^2(-r+k)^4(-c+b)}} \tan(F) \right)^2, \quad (33)$$

where

$$F = \frac{\sqrt{c^2 - b^2 - \frac{A_1^2}{(-c+b)^2(-r+k)^4}} \left(kx + ry + \frac{((-r+k)^4(b+c)(-c+b)^3 + A_1^2)t^\alpha}{(-r+k)(-c+b)^2 a} + C \right)}{2}.$$

By using *Family 2*, (23) becomes

$$u_{11}(x, y, t) = \frac{(-r+k)^2(-c+b)(b+c)}{2} + A_1 \left(\frac{A_1}{(-r+k)^2(-c+b)^2} + \sqrt{-c^2 + b^2 + \frac{A_1^2}{(-c+b)^2(-r+k)^4(-c+b)}} \tan(F) \right) - \frac{(-r+k)^2(-c+b)^2}{2} \left(\frac{A_1}{(-r+k)^2(-c+b)^2} + \sqrt{-c^2 + b^2 + \frac{A_1^2}{(-c+b)^2(-r+k)^4(-c+b)}} \tan(F) \right)^2, \quad (34)$$

where

$$F = \frac{\sqrt{-c^2 + b^2 + \frac{A_1^2}{(-c+b)^2(-r+k)^4}}}{2} \left(kx + ry + \frac{((-r+k)^4(b+c)(-c+b)^3 + A_1^2)t^\alpha}{(-r+k)(-c+b)^2 a} + C \right).$$

By using *Family 3*, we obtain

$$u_{12}(x, y, t) = \frac{(-r+k)^2 b^2}{2} + A_1 \left(\frac{A_1}{(-r+k)^2 b^2} + \frac{1}{b} \sqrt{b^2 + \frac{A_1^2}{b^2(-r+k)^4}} \tanh(F) \right) \quad (35)$$

$$-1/2 (-r+k)^2 b^2 \left(\frac{A_1}{(-r+k)^2 b^2} + \frac{1}{b} \sqrt{b^2 + \frac{A_1^2}{b^2 (-r+k)^4}} \tanh(F) \right)^2$$

where

$$F = \frac{\sqrt{b^2 + \frac{A_1^2}{b^2 (-r+k)^4}}}{2} \left(kx + ry + \frac{((-r+k)^4 b^4 + A_1^2) t^\alpha}{(-r+k) b^2 \alpha} + C \right).$$

By using *Family 4*, one constructs for (23)

$$\begin{aligned} u_{13}(x, y, t) = & -\frac{(-r+k)^2 c^2}{2} + A_1 \left(\frac{A_1}{(-r+k)^2 c^2} + \frac{1}{c} \sqrt{c^2 - \frac{A_1^2}{c^2 (-r+k)^4}} \tan(F) \right) \\ & - \frac{(-r+k)^2 c^2}{2} \left(\frac{A_1}{(-r+k)^2 c^2} + \frac{1}{c} \sqrt{c^2 - \frac{A_1^2}{c^2 (-r+k)^4}} \tan(F) \right)^2, \end{aligned} \quad (36)$$

where

$$F = \frac{\sqrt{c^2 - \frac{A_1^2}{c^2 (-r+k)^4}}}{2} \left(kx + ry + \frac{(-c^4 (-r+k)^4 + A_1^2) t^\alpha}{(-r+k) c^2 \alpha} + C \right).$$

By using *Family 5*, (23) can be written as

$$u_{14}(x, y, t) = \frac{(-r+k)^2 (b^2 - c^2)}{2} \left(1 - \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry + \frac{(-r+k)^3 (b+c)(-c+b) t^\alpha}{\alpha} + C \right) \right) \right)^2 \right). \quad (37)$$

By using *Family 11*, we can write

$$u_{15}(x, y, t) = -\frac{2(-r+k)^2 (a-b) e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)} b^2}{\left((a-b) e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)} - 1 \right)^2}. \quad (38)$$

By using *Family 12*, (23) reads

$$u_{16}(x, y, t) = -\frac{2(-r+k)^2 (-c+b) e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)} b^2}{\left((b-c) e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)} - 1 \right)^2}. \quad (39)$$

By using *Family 13*, one constructs for (23)

$$u_{17}(x, y, t) = -\frac{2(-r+k)^2 (a+b) b^2 e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)}}{\left(e^{b \left(kx + ry + \frac{b^2 (-r+k)^3 t^\alpha}{\alpha} + C \right)} - b - a \right)^2}. \quad (40)$$

By using *Family 14*, we get

$$u_{18}(x, y, t) = -\frac{A_1^2 e^F}{2c(-r+k)^2 (-1 + ce^F)^2}, \quad (41)$$

where

$$F = -\frac{A_1 (4(-r+k) c^2 \alpha (kx + ry + C) + A_1^2 t^\alpha)}{c^3 (-r+k)^3 \alpha}.$$

By using *Family 15*, we can write

$$u_{19}(x, y, t) = -2 \frac{(-r+k)^2}{(kx + ry + C)^2}. \quad (42)$$

By using *Family 17*, (23) becomes

$$u_{20}(x, y, t) = -2 \frac{(-r + k)^2}{(kx + ry + C)^2}. \quad (43)$$

By using *Family 18*, (23) reads

$$u_{21} = -\frac{(-r + k)^2 c^2}{2} - \frac{(-r + k)^2 c^2}{2} \left(\tan \left(\frac{c}{2} \left(kx + ry - \frac{(-r + k)^3 c^2 t^\alpha}{a} \right) + \frac{C}{2} \right) \right)^2. \quad (44)$$

SET 3

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = -\frac{(k-r)^2(c^2-b^2+2a^2)}{6}, A_1 = 0, A_2 = 0, B_1 = -a(k-r)^2(b+c), \\ B_2 = -\frac{(k-r)^2(b+c)^2}{2}, a = a, b = b, c = c, k = k, r = r, w = (k-r)^3(b^2-c^2+a^2) \end{array} \right\}$$

By using *Family 1*, (23) can be written as

$$u_{22}(x, t) = \frac{(k-r)^2 \left((b^2 - c^2 + a^2) \left((c^2 - b^2 + 2a^2) (\tan(F))^2 + 4a\sqrt{-b^2 + c^2 - a^2} \tan(F) - (3b^2 + 2a^2 - 3c^2) \right) \right)}{6 \left(-a + \sqrt{-b^2 + c^2 - a^2} \tan(F) \right)^2}, \quad (45)$$

where

$$F = \frac{\sqrt{-b^2 + c^2 - a^2}}{2} \left(kx + ry - \frac{(k-r)^3(b^2 - c^2 + a^2)t^\alpha}{a} + C \right).$$

By using *Family 2*, (23) becomes

$$u_{23}(x, y, t) = -\frac{(k-r)^2(b^2 - c^2 + a^2)((c^2 - b^2 + 2a^2)(\tanh(F))^2 + 4a \tanh(F) + 3b^2 + 2a^2 - 3c^2)}{6 \left(a + \sqrt{b^2 - c^2 + a^2} \tanh(F) \right)^2}, \quad (46)$$

where

$$F = \frac{\sqrt{b^2 - c^2 + a^2}}{2} \left(kx + ry - \frac{(k-r)^3(b^2 - c^2 + a^2)t^\alpha}{a} + C \right).$$

By using *Family 8*, (23) reads

$$u_{24}(x, y, t) = -\frac{2(k-r)^2 a^2}{(akx + ary + aC + 2)^2}. \quad (47)$$

By using *Family 11*, one constructs for (23)

$$u_{25}(x, y, t) = -\frac{(k-r)^2}{3} \frac{\left((4b^2a + 4b^3) e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + (b^4 + a^2 b^2 + 2b^3 a) e^{2b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + b^2 \right)}{\left((a + b) e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} - 1 \right)^2}. \quad (48)$$

By using *Family 12*, (23) can be written as

$$u_{26}(x, y, t) = -\frac{(k-r)^2}{3} \frac{\left((-4b^2c - 4b^3) e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + (c^2 b^2 + 2b^3 c + b^4) e^{2b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + b^2 \right)}{\left(1 + (b + c) e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} \right)^2}. \quad (49)$$

By using *Family 13*, (23) can be written as

$$u_{27}(x, y, t) = \frac{(k-r)^2}{3} \frac{\left((-4ab^2 + 4b^3) e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} - a^2 b^2 + 2b^3 a - b^4 - b^2 e^{2b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} \right)}{\left(e^{b \left(kx + ry - \frac{(k-r)^3 b^2 t^\alpha}{a} + C \right)} + b - a \right)^2}. \quad (50)$$

By using *Family 15*, (23) reads

$$u_{28}(x, y, t) = -2 \frac{(k-r)^2 c^2}{(c(kx+ry) + cC + 2)^2}. \quad (51)$$

By using *Family 16*, (23) becomes

$$u_{29}(x, y, t) = -2 \frac{(k-r)^2}{(kx+ry+C)^2}. \quad (52)$$

By using *Family 18*, one constructs for (23)

$$u_{30}(x, y, t) = -\frac{(k-r)^2}{6} \left(c^2 + 3 \left(\cot \left(\frac{c}{2} \left(kx+ry + \frac{(k-r)^3 c^2 t^\alpha}{\alpha} \right) + \frac{C}{2} \right) \right)^2 c^2 \right). \quad (53)$$

By using *Family 19*, (23) can be written as

$$u_{31}(x, y, t) = -\frac{(k-r)^2}{6} \frac{\left(2a^2 e^{2a \left(kx+ry - \frac{(k-r)^3 a^2 t^\alpha}{\alpha} + C \right)} + 8a^2 e^{a \left(kx+ry - \frac{(k-r)^3 a^2 t^\alpha}{\alpha} + C \right)} c + 2a^2 c^2 \right)}{\left(e^{a \left(kx+ry - \frac{(k-r)^3 a^2 t^\alpha}{\alpha} + C \right)} - c \right)^2}. \quad (54)$$

SET 4:

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = \frac{(k-r)^2(b^2-c^2)}{2}, A_1 = 0, A_2 = 0, B_1 = -a(k-r)^2(b+c), \\ B_2 = -\frac{(k-r)^2(b+c)^2}{2}, a = a, b = b, c = c, k = k, r = r, w = -(k-r)^3(b^2-c^2+a^2) \end{array} \right\}$$

By using *Family 1*, (23) becomes

$$\begin{aligned} u_{32}(x, y, t) &= \frac{(k-r)^2(b^2-c^2)}{2} - \frac{a(k-r)^2(b^2-c^2)}{a - \sqrt{-b^2+c^2-a^2} \tan \left(\frac{\sqrt{-b^2+c^2-a^2}}{2} \left(kx+ry + \frac{(k-r)^3(b^2-c^2+a^2)t^\alpha}{\alpha} + C \right) \right)} \\ &\quad - \frac{(k-r)^2(b^2-c^2)^2}{2 \left(a - \sqrt{-b^2+c^2-a^2} \tan \left(\frac{\sqrt{-b^2+c^2-a^2}}{2} \left(kx+ry + \frac{(k-r)^3(b^2-c^2+a^2)t^\alpha}{\alpha} + C \right) \right) \right)^2}. \end{aligned} \quad (55)$$

By using *Family 2*, (23) reads

$$\begin{aligned} u_{33}(x, y, t) &= \frac{(k-r)^2(b^2-c^2)}{2} - \frac{a(k-r)^2(b^2-c^2)}{a + \sqrt{b^2-c^2+a^2} \tanh \left(\frac{\sqrt{b^2-c^2+a^2}}{2} \left(kx+ry + \frac{(k-r)^3(b^2-c^2+a^2)t^\alpha}{\alpha} + C \right) \right)} \\ &\quad - \frac{(k-r)^2(b^2-c^2)^2}{2 \left(a + \sqrt{b^2-c^2+a^2} \tanh \left(\frac{\sqrt{b^2-c^2+a^2}}{2} \left(kx+ry + \frac{(k-r)^3(b^2-c^2+a^2)t^\alpha}{\alpha} + C \right) \right) \right)^2}. \end{aligned}$$

By using *Family 5*, (23) can be written as

$$u_{34}(x, y, t) = \frac{(k-r)^2(b^2-c^2)}{2} - \frac{(k-r)^2(b^2-c^2)}{2} \left(\tanh \left(\frac{\sqrt{b^2-c^2}}{2} \left(kx+ry + \frac{(k-r)^3(b^2-c^2)t^\alpha}{\alpha} + C \right) \right) \right)^{-2}.$$

By using *Family 8*, one constructs for (23)

$$u_{35}(x, y, t) = -2 \frac{(k-r)^2 a^2}{(akx+ary+aC+2)^2}.$$

By using *Family 11*, (23) reads

$$u_{36}(x, y, t) = \frac{(k-r)^2(b^2-a^2)}{2} + a(k-r)^2(a+b) \left(\frac{(a-b)e^{b \left(kx+ry + \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - 1}{(a+b)e^{b \left(kx+ry + \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - 1} \right)$$

$$-\frac{(k-r)^2(a+b)^2}{2} \left(\frac{(a-b)e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}-1}{(a+b)e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}-1} \right)^2.$$

By using *Family 12*, we get

$$\begin{aligned} u_{37}(x, y, t) &= \frac{(k-r)^2(b^2-c^2)}{2} - c(k-r)^2(b+c) \left(\frac{(b-c)e^{b\left(kx+ry+\frac{b^2(k-r)^3t^\alpha}{a}+C\right)}-1}{(b+c)e^{b\left(kx+ry+\frac{b^2(k-r)^3t^\alpha}{a}+C\right)}+1} \right) \\ &\quad - \frac{(k-r)^2(b+c)^2}{2} \left(\frac{(b-c)e^{b\left(kx+ry+\frac{b^2(k-r)^3t^\alpha}{a}+C\right)}-1}{(b+c)e^{b\left(kx+ry+\frac{b^2(k-r)^3t^\alpha}{a}+C\right)}+1} \right)^2. \end{aligned}$$

By using *Family 13*, (23) becomes

$$\begin{aligned} u_{38}(x, y, t) &= \frac{(k-r)^2(b^2-c^2)}{2} + c(k-r)^2(b+c) \left(\frac{e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}-b+c}{e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}+b+c} \right) \\ &\quad - \frac{(k-r)^2(b+c)^2}{2} \left(\frac{e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}-b+c}{e^{b\left(kx+ry+\frac{(k-r)^3b^2t^\alpha}{a}+C\right)}+b+c} \right)^2. \end{aligned}$$

By using *Family 15*, (23) can be written as

$$u_{39}(x, y, t) = -\frac{(k-r)^2c^2}{2} + \frac{c^3(k-r)^2(kx+ry+C)}{c(kx+ry+C)+2} - \frac{(k-r)^2c^4(kx+ry+C)^2}{2(c(kx+ry+C)+2)^2}.$$

By using *Family 16*, (23) becomes

$$u_{40}(x, y, t) = -2 \frac{(k-r)^2}{(kx+ry+C)^2}.$$

By using *Family 18*, (23) reads

$$u_{41}(x, y, t) = -\frac{(k-r)^2c^2}{2} - \frac{(k-r)^2c^2}{2} \left(\cot \left(\frac{c}{2} \left(kx+ry - \frac{(k-r)^3c^2t^\alpha}{a} \right) + \frac{C}{2} \right) \right)^2. \quad (56)$$

By using *Family 19*, one constructs for (23)

$$u_{42}(x, y, t) = -2a^2(k-r)^2c \left(e^{a\left(kx+ry+\frac{(k-r)^3a^2t^\alpha}{a}+C\right)} - c \right)^{-1} - 2(k-r)^2c^2a^2 \left(e^{a\left(kx+ry+\frac{(k-r)^3a^2t^\alpha}{a}+C\right)} - c \right)^{-2}.$$

SET 5:

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = -\frac{(k-r)^2(b-c)(b+c)}{3}, A_1 = 0, A_2 = -\frac{(k-r)^2(b-c)^2}{2}, B_1 = 0, \\ B_2 = -\frac{(k-r)^2(b+c)^2}{2}, a = 0, b = b, c = c, k = k, r = r, w = 4(k-r)^3(b-c)(b+c) \end{array} \right\} \quad (57)$$

By using *Family 1*, (23) becomes

$$\begin{aligned} u_{43}(x, y, t) &= \frac{(k-r)^2(b-c)^2(b+c)^2}{3(-b^2+c^2)} \\ &\quad - \frac{(k-r)^2(b-c)^2(b+c)^2}{2(-b^2+c^2)} \left(\tan \left(\frac{\sqrt{-b^2+c^2}}{2} \left(kx+ry - 4 \frac{(k-r)^3(b-c)(b+c)t^\alpha}{a} + C \right) \right) \right)^2 \\ &\quad - \frac{(k-r)^2(b-c)^2(b+c)^2}{2(-b^2+c^2)} \left(\tan \left(\frac{\sqrt{-b^2+c^2}}{2} \left(kx+ry - 4 \frac{(k-r)^3(b-c)(b+c)t^\alpha}{a} + C \right) \right) \right)^{-2}. \end{aligned} \quad (58)$$

By using *Family 2*, (23) can be written as

$$\begin{aligned}
 u_{44}(x, y, t) = & -\frac{(k-r)^2 (-4b^2c^2 + 2b^4 + 2c^4)}{6(b^2 - c^2)} \\
 & -\frac{(k-r)^2 (-6b^2c^2 + 3b^4 + 3c^4)}{6(b^2 - c^2)} \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry - 4 \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2 \\
 & -\frac{(k-r)^2 (-6b^2c^2 + 3b^4 + 3c^4)}{6(b^2 - c^2)} \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry - 4 \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^{-2}.
 \end{aligned} \quad (59)$$

By using *Family 8*, (23) reads

$$u_{45}(x, y, t) = -2 \frac{(k-r)^2}{(kx + ry + C)^2}.$$

By using *Family 11*, one constructs for (23)

$$u_{46}(x, y, t) = -\frac{2(k-r)^2}{3} \frac{2b^2 + 8b^4 e^{2b \left(kx + ry - 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} + 2b^6 e^{4b \left(kx + ry - 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)}}{\left(-1 - e^{b \left(kx + ry - 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} b \right)^2 \left(-1 + e^{b \left(kx + ry - 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} b \right)^2}.$$

By using *Family 12*, (23) becomes

$$u_{47}(x, y, t) = -\frac{2(k-r)^2}{3} \frac{\left(c^2 + 2b^2 + (4c^3 - 4b^2c) e^F + (6c^4 - 14b^2c^2 + 8b^4) e^{2F} \right) + (4c^5 + 4cb^4 - 8c^3b^2) e^{3F} + (c^6 - 3b^4c^2 + 2b^6) e^{4F}}{(1 + (b-c)e^F)^2 (1 + (b+c)e^F)^2} \quad (60)$$

where

$$F = b \left(kx + ry - 4 \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right).$$

By using *Family 13*, (23) can be written as

$$u_{48}(x, y, t) = \frac{2(k-r)^2}{3} \frac{\left(-12ab^4 e^F + (-8b^4 - 10a^2b^2) e^{2F} - 12ab^2 e^{3F} + (-2b^2 - a^2) e^{4F} - 2b^6 - b^4a^2 \right)}{(e^F - b)^2 (e^F + b)^2}$$

where

$$F = b \left(kx + ry - 4 \frac{(k-r)^3 (b+a)(b-a)t^\alpha}{\alpha} + C \right).$$

By using *Family 16*, (23) becomes

$$u_{49}(x, y, t) = -2 \frac{(k-r)^2}{(kx + ry + C)^2}.$$

SET 6:

We have yielded the arbitrary constants as

$$\left\{ \begin{array}{l} A_0 = (k-r)^2 (b-c)(b+c), A_1 = 0, A_2 = -\frac{(k-r)^2 (b-c)^2}{2}, B_1 = 0, \\ B_2 = -\frac{(k-r)^2 (b+c)^2}{2}, a = 0, b = b, c = c, k = k, r = r, w = -4(k-r)^3 (b-c)(b+c) \end{array} \right\} \quad (61)$$

By using *Family 1*, (23) reads

$$\begin{aligned}
 u_{50}(x, y, t) = & (k-r)^2 (b-c)(b+c) - \frac{(k-r)^2 (-b^2 + c^2)}{2} \left(\tan \left(\frac{\sqrt{-b^2 + c^2}}{2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2 \\
 & - \frac{(k-r)^2 (b+c)^2 (b-c)^2}{2(-b^2 + c^2) \left(\tan \left(\frac{\sqrt{-b^2 + c^2}}{2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c)(b+c)t^\alpha}{\alpha} + C \right) \right) \right)^2}.
 \end{aligned} \quad (62)$$

By using *Family 2*, we get

$$u_{51}(x, y, t) = (k-r)^2 (b-c) (b+c) - \frac{(k-r)^2 (b^2 - c^2)}{2} \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c) (b+c) t^\alpha}{\alpha} + C \right) \right) \right)^2 \\ - \frac{(k-r)^2 (b+c)^2 (b-c)^2}{2 (b^2 - c^2) \left(\tanh \left(1/2 \sqrt{b^2 - c^2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c) (b+c) t^\alpha}{\alpha} + C \right) \right) \right)^2}.$$

By using *Family 4*, one constructs for (23)

$$u_{52}(x, y, t) = -(k-r)^2 c^2 - \frac{(k-r)^2 c^2}{2} \left(\tan \left(\frac{\sqrt{c^2}}{2} \left(kx + ry - 4 \frac{(k-r)^3 c^2 t^\alpha}{\alpha} + C \right) \right) \right)^2 \\ - \frac{(k-r)^2 c^2}{2 \left(\tan \left(\frac{\sqrt{c^2}}{2} \left(kx + ry - 4 \frac{(k-r)^3 c^2 t^\alpha}{\alpha} + C \right) \right) \right)^2}.$$

By using *Family 5*, (23) reads

$$u_{53}(x, y, t) = (k-r)^2 (b-c) (b+c) - \frac{(k-r)^2 (b-c) (b+c)}{2} \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c) (b+c) t^\alpha}{\alpha} + C \right) \right) \right)^2 \\ - \frac{(k-r)^2 (b-c) (b+c)}{2 \left(\tanh \left(\frac{\sqrt{b^2 - c^2}}{2} \left(kx + ry + 4 \frac{(k-r)^3 (b-c) (b+c) t^\alpha}{\alpha} + C \right) \right) \right)^2}.$$

By using *Family 8*, (23) can be written as

$$u_{54}(x, y, t) = -2 \frac{(k-r)^2}{(kx + ry + C)^2}.$$

By using *Family 11*, (23) becomes

$$u_{55}(x, y, t) = (k-r)^2 b^2 - \frac{(k-r)^2 b^2}{2} \left(\frac{-1 + b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)}}{-1 - b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)}} \right)^2 \\ - \frac{(k-r)^2 b^2}{2} \left(\frac{-1 - b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)}}{-1 + b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)}} \right)^2.$$

By using *Family 12*, one constructs for (23)

$$u_{56}(x, y, t) = -8 b^4 (k-r)^2 e^{2b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} \left(b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - 1 \right)^{-2} \left(b e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} + 1 \right)^{-2}$$

By using *Family 13*, we get

$$u_{57}(x, y, t) = (k-r)^2 b^2 - \frac{(k-r)^2 b^2}{2} \left(e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} + b \right)^2 \left(e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - b \right)^{-2} \\ - \frac{(k-r)^2 b^2}{2} \left(e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} - b \right)^2 \left(e^{b \left(kx + ry + 4 \frac{(k-r)^3 b^2 t^\alpha}{\alpha} + C \right)} + b \right)^{-2}.$$

By using *Family 17*, (23) becomes

$$u_{58}(x, y, t) = -2 \frac{(k-r)^2}{(kx + ry + C)^2}.$$

By using *Family 18*, (23) reads

$$u_{59}(x, y, t) = -(k-r)^2 c^2 - \frac{(k-r)^2 c^2}{2} \left(\tan \left(\frac{c}{2} \left(kx + ry - 4 \frac{(k-r)^3 c^2 t^\alpha}{\alpha} \right) + \frac{C}{2} \right) \right)^2 \\ - \frac{(k-r)^2 c^2}{2} \left(\cot \left(\frac{c}{2} \left(kx + ry - 4 \frac{(k-r)^3 c^2 t^\alpha}{\alpha} \right) + \frac{C}{2} \right) \right)^2.$$

Figure 1: Graph of $u_5(x, y, t)$ of Eq. (28) corresponding to $b = 0.4, r = 2, k = -3, C = 0, y = 0$ for $\alpha = 0.43, \alpha = 0.61$ and $\alpha = 0.96$, respectively.

Figure 2: Graph of $u_{18}(x, y, t)$ of Eq. (41) corresponding to $A_1 = 0, c = -2, r = 2, k = 3, C = 0, y = 0$ for $\alpha = 0.1, \alpha = 0.13$ and $\alpha = 0.5$, respectively.

Figure 3: Graph of $u_{23}(x, y, t)$ of Eq. (46) corresponding to $a = 1, b = 2, c = 1, r = 1, k = 2, C = 0, y = 0$ for $\alpha = 0.89, \alpha = 0.91$ and $\alpha = 1$, respectively.

Figure 4: Graph of $u_{32}(x, y, t)$ of Eq. (55) corresponding to $a = 1, b = 1, c = 2, r = 1, k = 2, C = 0, y = 0$ for $\alpha = 0.1, \alpha = 0.5$ and $\alpha = 1$, respectively.

Figure 5: Graph of $u_{41}(x, y, t)$ of Eq. (56) corresponding to $a = 0, b = 0, c = 1, r = -2, k = 3, C = 0, y = 0$ for $\alpha = 0.1, \alpha = 0.55$ and $\alpha = 1$, respectively.

Figure 6: Graph of $u_{47}(x, y, t)$ of Eq. (60) corresponding to $b = 1, c = 0, r = 1, k = -1, C = 0, y = 0$ for $\alpha = 0.6, \alpha = 0.89$ and $\alpha = 0.999$, respectively.

8 Application of Jacobi elliptic function expansion method to conformable (2+1) dimensional KdV equation

To solve Eq. (3) by the Jacobian elliptic function expansion method, considering $u(x, t) = u(\xi)$, $\xi = kx + ry - \frac{w}{\alpha}t^\alpha$ and integrating once with respect to ξ , Eq. (3) becomes

$$-wu - (3k - 3r)u^2 - (k^3 - r^3 - 3rk^2 + 3r^2k)u'' = 0. \quad (63)$$

The Solutions in Terms of $sn(\xi)$:

By balancing the highest-order linear term and the highest-order nonlinear term we obtain $n = 2$, thus the solution of Eq. (63) can be expressed as

$$u(\xi) = a_0 + a_1 sn(\xi) + a_2 sn^2(\xi). \quad (64)$$

Substituting Eq. (64) into Eq. (63) and collecting various powers of $sn(\xi)$, we get

$$\begin{aligned} 18kr^2a_2m^2 + 6k^3a_2m^2 - 6r^3a_2m^2 - 3ra_2^2 - 18k^2ra_2m^2 + 3ka_2^2 &= 0, \\ 2k^3a_1m^2 + 6kr^2a_1m^2 - 2r^3a_1m^2 - 6k^2ra_1m^2 + 6ka_1a_2 - 6ra_1a_2 &= 0, \\ wa_2 + 4r^3a_2m^2 + 4r^3a_2 - 4k^3a_2m^2 - 12kr^2a_2m^2 - 12kr^2a_2 - 4k^3a_2 \\ + 6ka_0a_2 + 3ka_1^2 + 12k^2ra_2 - 6ra_0a_2 + 12k^2ra_2m^2 - 3ra_1^2 &= 0, \\ -3kr^2a_1m^2 + wa_1 + 3k^2ra_1m^2 + 6ka_0a_1 - k^3a_1m^2 + r^3a_1 \\ + 3k^2ra_1 - k^3a_1 + r^3a_1m^2 - 6ra_0a_1 - 3kr^2a_1 &= 0 \\ 2k^3a_2 + wa_0 + 6kr^2a_2 - 2r^3a_2 - 3ra_0^2 - 6k^2ra_2 + 3ka_0^2 &= 0. \end{aligned}$$

By solving the above system of equations using any package of symbolic computations, we can determine the values of the coefficients as:

$$a_0 = 4 \left(\frac{m^2}{6} + \frac{1}{6} \pm \frac{1}{6} \sqrt{m^4 - m^2 + 1} \right) (k - r)^2,$$

$$\begin{aligned} a_1 &= 0, a_2 = -2(k^2 - 2kr + r^2)m^2, \\ w &= \mp 4\sqrt{m^4 - m^2 + 1}(k - r)^3. \end{aligned} \quad (65)$$

Substituting Eq. (65) into Eq. (64), we obtain the exact solution of Eq. (3) in the form

$$u(\xi) = 4 \left(\frac{m^2}{6} + \frac{1}{6} \pm \frac{1}{6} \sqrt{m^4 - m^2 + 1} \right) (k - r)^2 - 2(k^2 - 2kr + r^2)m^2 \operatorname{sn}^2(\xi|m) \text{ with } \xi = kx + ry - \frac{w}{\alpha} t^\alpha, \quad (66)$$

which is an exact periodic solution of the KdV equation with conformable derivative. For $m \rightarrow 1$, $\operatorname{sn}(\xi|m) \rightarrow \tanh(\xi)$ and the above exact periodic solution is degenerated into a new form of solution can be written as

$$u(x, y, t) = 4 \left(\frac{1}{3} \pm \frac{1}{6} \right) (k - r)^2 - 2(k^2 - 2kr + r^2) \tanh^2(kx + ry \pm \frac{4(k - r)^3}{\alpha} t^\alpha). \quad (67)$$

Figure 7: Graph of Eq. (67) corresponding to $k = 1.3$, $r = 2.1$, $y = 0$ for $\alpha = 0.35$.

The Solutions in Terms of $\operatorname{cn}(\xi)$:

By balancing the highest-order linear term and the highest-order nonlinear term we obtain $n = 2$, thus the solution of Eq. (63) can be expressed by Jacobi elliptic cosine function

$$u(\xi) = b_0 + b_1 \operatorname{cn}(\xi) + b_2 \operatorname{cn}^2(\xi). \quad (68)$$

Substituting Eq. (68) into Eq. (63) and collecting various powers of $\operatorname{cn}(\xi)$, we get an algebraic equations system by solving this system of equations using any package of symbolic computations, we can determine the values of the coefficients as:

$$\begin{aligned} b_0 &= 4 \left(-\frac{m^2}{3} + \frac{1}{6} \pm \frac{1}{6} \sqrt{m^4 - m^2 + 1} \right) (k - r)^2, \\ b_1 &= 0, b_2 = 2(k^2 - 2kr + r^2)m^2, \\ w &= \mp 4\sqrt{m^4 - m^2 + 1}(k - r)^3. \end{aligned} \quad (69)$$

Substituting Eq. (69) into Eq. (68), we obtain the exact solution of Eq. (3) in the form

$$u(\xi) = 4 \left(-\frac{m^2}{3} + \frac{1}{6} \pm \frac{1}{6} \sqrt{m^4 - m^2 + 1} \right) (k - r)^2 + 2(k^2 - 2kr + r^2)m^2 \operatorname{cn}^2(\xi|m) \text{ with } \xi = kx + ry - \frac{w}{\alpha} t^\alpha \quad (70)$$

which is an exact periodic solution of the KdV equation with conformable derivative. For $m \rightarrow 1$, $\operatorname{cn}(\xi|m) \rightarrow \operatorname{sech}(\xi)$ and the above exact periodic solution is degenerated into a new form of solution can be written as

$$u(x, y, t) = 4 \left(-\frac{1}{6} \pm \frac{1}{6} \right) (k - r)^2 + 2(k^2 - 2kr + r^2) \operatorname{sech}^2(kx + ry \pm \frac{4(k - r)^3}{\alpha} t^\alpha). \quad (71)$$

Figure 8: Graph of Eq. (71) corresponding to $k = 0.9$, $r = 1.6$, $y = 0$ for $\alpha = 0.9$.

9 Numerical application of C-VIM to conformable (2 + 1) -dimensional KdV equation

In this section, we consider the exact solution Eq. (32) (namely $u_9(x, y, t)$) of (2+1) -dimensional conformable derivative KdV equation (3) for special values of parameters ($c = 2$, $k = 2$, $r = 1$, $C = 0$) subject to initial condition

$$u(x, y, 0) = -\frac{2}{3} - 2 (\tan(2x + y))^2. \quad (72)$$

For $\alpha = 1$, the exact solution of (3) is

$$u(x, y, t) = -\frac{2}{3} - 2 (\tan(2x + y + 4t))^2.$$

Solution by C-VIM:

For solving Eq. (3) numerically corresponding to the initial condition (72) by C-VIM, we obtain the recurrence relation

$$\begin{aligned} u_{n+1}(x, y) = & u_n(x, y, t) + \int_0^t \lambda(\zeta) \left[\zeta^{1-\alpha} \frac{\partial u_n(x, y, \zeta)}{\partial \zeta} - 6 \frac{\partial u_n(x, y, \zeta)}{\partial x} u_n(x, y, \zeta) + 6 \frac{\partial u_n(x, y, \zeta)}{\partial y} u_n(x, y, \zeta) - \frac{\partial^3 u_n(x, y, \zeta)}{\partial x^3} \right. \\ & \left. + \frac{\partial^3 u_n(x, y, \zeta)}{\partial y^3} + 3 \frac{\partial^3 u_n(x, y, \zeta)}{\partial x^2 \partial y} - 3 \frac{\partial^3 u_n(x, y, \zeta)}{\partial x \partial y^2} \right] d\zeta \end{aligned} \quad (73)$$

and following conditions:

$$\lambda'(\zeta)|_{\zeta=t} = 0, \quad \lambda(\zeta)|_{\zeta=t} = -1.$$

Therefore, the Lagrange multiplier can be identified as $\lambda = -1$. As a result we obtain the following iteration formula:

$$\begin{aligned} u_{n+1}(x, y) = & u_n(x, y, t) - \int_0^t \left[\zeta^{1-\alpha} \frac{\partial u_n(x, y, \zeta)}{\partial \zeta} - 6 \frac{\partial u_n(x, y, \zeta)}{\partial x} u_n(x, y, \zeta) + 6 \frac{\partial u_n(x, y, \zeta)}{\partial y} u_n(x, y, \zeta) - \frac{\partial^3 u_n(x, y, \zeta)}{\partial x^3} \right. \\ & \left. + \frac{\partial^3 u_n(x, y, \zeta)}{\partial y^3} + 3 \frac{\partial^3 u_n(x, y, \zeta)}{\partial x^2 \partial y} - 3 \frac{\partial^3 u_n(x, y, \zeta)}{\partial x \partial y^2} \right] d\zeta. \end{aligned} \quad (74)$$

By the initial condition (72) we write

$$u_0(x, y, t) = u(x, y, 0) = -\frac{2}{3} - 2 (\tan(2x + y))^2. \quad (75)$$

Now, substituting (75) into (74) and then iterate for $n = 0$, $n = 1$, and $n = 2$ respectively, we yield

$$u_1(x, y, t) = -\frac{2}{3} - 2 (\tan(2x + y))^2 - \frac{16 \sin(2x + y) t}{(\cos(2x + y))^3},$$

$$u_2(x, y, t) = -\frac{2}{3} - 2 (\tan(2x + y))^2 - \frac{32 \sin(2x + y) t}{(\cos(2x + y))^3}$$

$$\begin{aligned}
& + \left(-\frac{16t^{2-\alpha}}{(-2+\alpha)(\cos(2x+y))^3} - \frac{1024t^3}{(\cos(2x+y))^5} + \frac{1536t^3}{(\cos(2x+y))^7} \right) \sin(2x+y) \\
& + \frac{64t^2}{(\cos(2x+y))^2} - \frac{96t^2}{(\cos(2x+y))^4}. \\
u_3(x, y, t) = & \left(\frac{64t - 16t^{2-\alpha} - 32t\alpha}{(-2+\alpha)} A^3 - 1024t^3 A^5 + 1536t^3 A^7 \right) B \\
& + \frac{4}{3} + \left(64t^2 - 2 \right) A^2 - 96t^2 A^4 + 14155776t^7 BA^{15} - 28311552t^7 BA^{13} - 1622016t^6 A^{12} \\
& + \left(\frac{196608t^5(-21+640t^2)}{7} + \frac{1474560t^{6-\alpha}}{(-2+\alpha)(-6+\alpha)} \right) BA^{11} \\
& + 50688t^4(64t^2+3)A^{10} - \left(\frac{24576t^5(-1183+5120t^2)}{35} + \frac{1966080t^{6-\alpha}}{(-2+\alpha)(-6+\alpha)} \right) BA^9 \\
& - \left(512t^4(411+3968t^2) + 64512 \frac{t^{5-\alpha}}{(-5+\alpha)(-2+\alpha)} \right) A^8 \\
& - \left(\frac{4608t^3(-5+304t^2)}{5} + \frac{4608t^{5-2\alpha}}{(-5+2\alpha)(-2+\alpha)^2} - \frac{589824t^{6-\alpha}}{(-2+\alpha)(-6+\alpha)} - \frac{4608(-6+\alpha)t^{4-\alpha}}{(-4+\alpha)(-2+\alpha)} \right) BA^7 \\
& + \left(2048t^4(29+192t^2) + 86016 \frac{t^{5-\alpha}}{(-5+\alpha)(-2+\alpha)} \right) A^6 \\
& + \left(\frac{512t^3(-35+96t^2)}{5} - \frac{3072(-6+\alpha)t^{4-\alpha}}{(-4+\alpha)(-2+\alpha)} + \frac{3072t^{5-2\alpha}}{(-5+2\alpha)(-2+\alpha)^2} \right) BA^5 \\
& + \left(64t^2(-3+64t^2) - 192 \frac{t^{3-\alpha}}{-2+\alpha} - 24576 \frac{t^{5-\alpha}}{(-5+\alpha)(-2+\alpha)} \right) A^4 \\
& + \left(\frac{16t(-3+32t^2)}{3} + \frac{16t^{-2\alpha+3}}{2\alpha-3} - \frac{32t^{2-\alpha}}{-2+\alpha} \right) BA^3 \\
& + 128 \frac{t^2\alpha - 2t^2 + t^{3-\alpha}}{-2+\alpha} A^2
\end{aligned}$$

where $A^{-1} = \cos(2x+y)$, $B = \sin(2x+y)$.

Remark 1. The accuracy of all the results obtained in this study was provided by using maple.

10 Conclusion

In this work, we investigated the $(2 + 1)$ -dimensional conformable derivative KdV equation for which we constructed exact wave solutions with the help of ITEM and Jacobi elliptic function expansion method. ITEM is advantageous in terms of simplicity and diversity of solutions, the other method is advantageous in that it obtains the results directly, quickly and needs simple algorithms. Traveling wave solutions including soliton, periodic, kink and kink singular wave solutions of the model studied using the presented methods were found. Based on the ITEM devised, we have constructed dark soliton solutions such as $u_2, u_3, u_{12}, \dots$, exponential solutions such as $u_5, u_7, u_{15}, u_{31}, \dots$, trigonometric soliton solutions such as $u_9, u_{11}, u_{22}, \dots$ and rational solutions such as $u_4, u_8, u_{28}, \dots$. Dark ((67)) and bright ((71)) soliton solutions were also obtained as a result of the application of the Jacobi elliptic function expansion method. We use different values of coefficients for obtaining novel graphical representations of solutions which are helpful for researchers to understand the physical phenomena of underlying model.

In Figs. 1-6, we plot three dimensional graphics of $u_5, u_{18}, u_{23}, u_{32}, u_{41}$ and u_{47} , which denote the dynamics of solutions with appropriate parametric selections. These graphics were drawn according to different values of α , which is the order of fractional derivative, and it is tried to increase the intelligibility of dynamic behavior while changing α .

Table 1: Comparison of obtained result by 3-th iteration and exact solution (32) of (3).

$\alpha = 1.0$					
x	y	t	$C - VIM$	$Exact$	$Error$
0.001	0.001	0.001	-0.6667650	-0.6667647	-0.3×10^{-6}
		0.002	-0.6669090	-0.6669087	-0.3×10^{-6}
		0.003	-0.6671167	-0.6671168	0.1×10^{-6}
0.01	0.01	0.01	-0.6764517	-0.6764988	0.471×10^{-4}
		0.02	-0.6903011	-0.6910632	0.7621×10^{-3}
		0.03	-0.7084940	-0.7123504	0.38564×10^{-2}
		0.04	-0.7284763	-0.7406405	0.121642×10^{-1}
0.02	0.02	0.05	-0.7466970	-0.7763130	0.296160×10^{-1}
		0.01	-0.6867516	-0.6868008	0.492×10^{-4}
		0.02	-0.7055811	-0.7063847	0.8036×10^{-3}
		0.03	-0.7288268	-0.7328925	0.40657×10^{-2}
		0.04	-0.7538605	-0.7666780	0.128175×10^{-1}
0.01	0.02	0.05	-0.7770070	-0.8082016	0.311946×10^{-1}
		0.03	-0.7148429	-0.7187534	0.39105×10^{-2}
		0.04	-0.7365185	-0.7488494	0.123309×10^{-1}
0.02	0.01	0.03	-0.7216186	-0.7255988	0.39802×10^{-2}
		0.04	-0.7449777	-0.7575261	0.125484×10^{-1}

Table 2: Comparison of obtained result (32) of (3) by 3-th iteration for different α values.

			$\alpha = 0.8$	$\alpha = 0.9$
x	y	t	$C - VIM$	$C - VIM$
0.01	0.01	0.01	-0.6860734	-0.6821012
		0.02	-0.7155144	-0.7051462
		0.03	-0.7546394	-0.7356537
		0.04	-0.8001101	-0.7705637
		0.05	-0.8476016	-0.8058438
0.02	0.02	0.01	-0.7017910	-0.6953636
		0.02	-0.7404156	-0.7255605
		0.03	-0.7880061	-0.7628366
		0.04	-0.8411520	-0.8040732
		0.05	-0.8953551	-0.8450957

Solution u_{32} represent the soliton wave solution which are very important and remarkable as there are waves that can protect their identity in interaction with other waves. Solutions u_{23} and u_{41} represent the exact periodic wave solution. Solution u_5 , u_{18} and u_{47} represent the singular kink-type traveling wave solutions. In Figs. 7 and 8, under the choice of the suitable values of parameters, the 3D the contour graphs are plotted. Some obtained solutions behave as periodic traveling waves, bright soliton, and dark soliton solutions. When the modulus $m \rightarrow 1$ some of these obtained solutions degenerate as solitary wave solutions. Moreover, we applied the conformable variational iteration method by using of the exact solution (32) of $(2 + 1)$ -dimensional conformable KdV equation. For the appropriate values of x, y, t and different values of α ($\alpha = 0.8, \alpha = 0.9, \alpha = 1.0$), we compared the exact solution and iterated solution in Table 1 and 2. We hope this study will guide future research and be useful for engineers and scientists in this field.

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