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A Generalized ML-Hyers-Ulam Stability of Quadratic Fractional Integral Equation

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Abstract: An interesting quadratic fractional integral equation is investigated in this work via a generalized Mittag-Leffler (ML) function. The generalized ML-Hyers-Ulam stability is established in this investigation. We study both of the Hyers-Ulam stability (HUS) and ML-Hyers-Ulam-Rassias stability (ML-HURS) in detail for our proposed differential equation (DEq). Our proposed technique unifies various differential equations' classes. Therefore, this technique can be further applied in future research works with applications to science and engineering.

Keywords: Stability; quadratic fractional integral equation; Mittag-Leffler function

MSC: 45M10, 65P40, 65L20, 70H14

1 Introduction

Several types of integral equations (IEs) are considered very important in various functional analysis topics because of their essential role in engineering, physics, economics, and natural sciences. Many real-life applications can be well described using a suitable quadratic fractional integral equation. For instance, quadratic integral

equations (QIEs) play an important role in kinetic molecular, radiative, neutron transport, traffic, and queuing theories [1, 2].

The fundamental idea behind the theory of functional equations' stability was first initiated in 1940 when Ulam studied the group homomorphism problem's stability in [3]. Then, in the following year, that problem was investigated by Hyers for the Banach spaces (BaSps) case. As a result, this particular stability is named as HUS. A generalized form of the Hyers' theorem was studied by Rassias [4] in 1978 where the stability was investigated via unbounded Cauchy differences. Thus, this contribution is named as HURS. By interchanging the functional equation with an inequality that acts as an initial equation's perturbation, the functional equation's stability problem has been given a special priority in research studies [5]. The HUS of DEq with an integer order was first studied by Alsina and Ger [6]. An extension of analysis to fractional differential equations was investigated in many research works [7, 8]. Various ML-Hyers-Ulam stability's types have been proposed in [9] for the fractional integral equation (FIE), and its exact solution can be approximated from every FIE's mapping.

Fractional calculus (FC) constitutes an essential tool for analysis when the arbitrary-order integrals and derivatives are investigated via the generalizations of integer-order differentiation and n -fold integration [10–14]. Some interesting research works concerning the relation between symmetry and generalized fractional calculus properties have been conducted [15]. Although studying FC was focused on the pure mathematical investigation, various natural sciences and technology fields have studied them such as mathematical physics, fluid mechanics, image processing, biology, and entropy which reveal the powerful applicability of FC in modeling scientific phenomena [16–24]. Theoretically and practically, FC has received a global research interest [25, 26]. There are equivalent types of differential equations to quadratic fractional integral equation which acquire soliton solutions to have more effective model equation than other related ones.

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A notable approximation type is Padé-type approximation with lump solutions (see [27]). The N -soliton solutions to integrable equations have been recently investigated via the Hirota bilinear method for both $(1 + 1)$ -dimensional integrable equations [28] and $(2 + 1)$ -dimensional integrable equations [29] (see also [30–32] for examples). For more background information and examples of soliton solutions, we refer to [33–35]. While QIEs have several applications in natural sciences and engineering, investigating QIEs with the help of fractional calculus can provide us with a very powerful tool in modeling phenomena in science and engineering, particularly queuing theory and biology [36] (see also [37–39]), due to the importance of nonlocality and memory effect of the proposed fractional models when we discuss the systems' dynamics and fractional behavior. Some numerical approaches for the ML distribution have been studied such as Monte Carlo approach for a random number generator for the one-parameter ML distribution [40], Grünwald–Letnikov approach in the sense of equivalent fractional integrator [41], rational approximation for the two-parameter ML function with discrete elliptic operator [42]. In addition, multi-parameter extensions of the ML distribution have been recently studied in [43].

This work investigates the following quadratic FIE's stability:

$$y(t) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r} ((t - \xi)^q) \mathcal{W}(\xi, y(\xi)) d\xi \right], \quad (1.1)$$

where $\mathcal{V}, \mathcal{W} : J \times \mathbf{R} \rightarrow \mathbf{R}$ are continuous functions (CFs), $q \in [1, 2)$, Γ represents Gamma function (GF), and $\mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}$ is the generalized ML function. The quadratic operator equations' existence can be shown under the conditions of mixed Lipschitz and compactness along with a certain growth condition on the nonlinearities included in the quadratic operator.

This work is outlined as follows: some essential notions about generalized metric and ML function are discussed in Sec. 2. Our main results are obtained in both Sec. 3 and Sec. 4. Our work is concluded in Sec. 5.

2 Essential Concepts

For a nonempty set $\mathcal{Y}$, the generalized metric on $\mathcal{Y}$ is introduced in this section. Given a function: $\hat{\rho} : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, +\infty]$, named as a generalized metric on $\mathcal{Y}$ iff the assumptions below are satisfied:

- (A1) $\hat{\rho}(y_1, y_2) = 0$ iff $y_1 = y_2$.
- (A2) $\hat{\rho}(y_1, y_2) = \hat{\rho}(y_2, y_1) \forall y_1, y_2 \in \mathcal{Y}$.
- (A3) $\hat{\rho}(y_1, y_2) \leq \hat{\rho}(y_1, y_3) + \hat{\rho}(y_3, y_2) \forall y_i \in \mathcal{Y}$ with $i = 1, 2, 3$.

It is obvious that the above definition is different from the usual complete metric space definition where not every two points in $\mathcal{Y}$ have necessarily a finite distance. Therefore, this space can be named as a generalized complete metric space (GCMSp).

Banach's fixed point theorem (BFPTm) in a GCMSp is expressed as:

Theorem 2.1. Suppose that $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let us assume that $\mathcal{O} : \mathcal{Y} \rightarrow \mathcal{Y}$ is a strictly contractive operator with the Lipschitz constant $\ell < 1$. If $\exists$ a nonnegative integer $k \in \mathbb{N}$

$$\hat{\rho}(\mathcal{O}^{k+1}(y), \mathcal{O}^k(y)) < \infty,$$

for some $y \in \mathcal{Y}$, then the following are true:

- (I) The sequence $\mathcal{O}^n(y)$ converges to a fixed point y^* of $\mathcal{O}$.
- (II) y^* is the unique fixed point of $\mathcal{O}$ in

$$\mathcal{Y}^* = \left\{ y \in \mathcal{Y} \mid \hat{\rho}(\mathcal{O}^k(y), y) < \infty \right\}.$$

- (III) If $y \in \mathcal{Y}^*$, then we have:

$$\hat{\rho}(y, y^*) \leq \frac{1}{1 - \ell} \hat{\rho}(\mathcal{O}(y), y).$$

Definition 2.2. [44] (ML function) The one-parameter ML function, denoted by $\mathcal{E}_\alpha(z)$, is written as:

$$\mathcal{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+\alpha k)} z^k, \quad (2.1)$$

where $z, \alpha \in \mathbf{C}$, $\operatorname{Re}(\alpha) > 0$. If we substitute $\alpha = 1$ in the above equation, then we have:

$$\mathcal{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

Definition 2.3. In 1905, Wiman proposed the generalized form of $\mathcal{E}_\alpha(z)$ [45]. Then, both Agarwal [46], and Humbert and Agarwal [47] introduced a function as follows:

$$\mathcal{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta + \alpha k)} z^k, \quad (2.2)$$

where $z, \alpha, \beta \in \mathbf{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$.

Prabhakar generalized in 1971 this function in the following form:

$$\mathcal{E}_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta + \alpha k)} z^k.$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$, $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, $\operatorname{Re}(\gamma) > 0$, such that $\gamma \neq 0$,

$$(\gamma)_k = \prod_{i=0}^{k-1} (\gamma + i), \quad (2.3)$$

which is called the Pochhammer symbol [48] $\ni (\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. Another generalization of this function was introduced in 2007 by Shulka and Prajapati in [48] as:

$$\mathcal{E}_{\alpha,\beta}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{k! \Gamma(\beta + \alpha k)} z^k. \quad (2.4)$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$,

$$\min \{ \operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(\gamma) \} > 0, \quad (2.5)$$

and $q \in (0, 1) \cup \mathbf{N}$. In 2009, again Shalka and Prajapati [49] introduced a generalized ML function. In 2012, a novel generalized form of ML function was proposed by both Salim in [50], and by Amit and Saraswat in [51] as:

$$\mathcal{E}_{\alpha,\beta,\delta}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}}{(\delta)_{qk} \Gamma(\beta + \alpha k)} z^k. \quad (2.6)$$

where $z, \alpha, \beta, \gamma \in \mathbf{C}$, Eq. (2.5) holds, $q \in (0, 1) \cup \mathbf{N}$ and

$$(\gamma)_{qk} = \frac{\Gamma(\gamma + qk)}{\Gamma(\gamma)}, \quad (\delta)_{qk} = \frac{\Gamma(\delta + qk)}{\Gamma(\delta)}, \quad (2.7)$$

denote the generalized Pochhammer symbol [48]. After them, in 2016 Desai *et al.* [52] introduced another definition of generalized ML function.

Definition 2.4. [53]. The generalized ML function, denoted by $\mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(y)$, can be expressed as:

$$\begin{aligned} \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(z) &= \mathcal{Q}_{\alpha,\beta,\delta}^{\gamma,q,r}(a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_r, z) \\ &= \sum_{s=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, s) (\gamma)_{qs}}{\prod_{n=1}^r \beta(b_n, s) (\delta)_{qs} \Gamma(\beta + \alpha s)} z^s, \end{aligned} \quad (2.8)$$

where $y, \alpha, \beta, \gamma, \delta, a_i, b_i \in \mathbf{C}$, Equation (2.5) holds, $q \in (0, 1) \cup \mathbf{N}$, $(\gamma)_{qk}$ and $(\delta)_{qk}$ are defined in (2.7).

3 Hyers–Ulam–Rassias Stability

The HURS and HUS of equation (1.1) are investigated in this section on a compact interval $[0, a]$.

Definition 3.1. If for each given function y satisfies

$$\left| y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t - \xi)^q) \mathcal{W}(\xi, y(\xi)) d\xi \right] \right| \leq \varepsilon \varphi(t),$$

$\exists$ an equation's (1.1) solution u_0 and a constant, $c > 0$, which is independent of both y and $u_0 \ni$

$$|y(t) - u_0(t)| \leq c\varepsilon \varphi(t),$$

for $t \in [a, b]$, then equation (1.1) is named as Hyers–Ulam–Rassias stable. On the other hand, when φ is formed as a constant function, equation (1.1) is named as Hyers–Ulam stable.

Theorem 3.2. For a closed and bounded interval $J = [0, a]$ of the real line $\mathbf{R}$ for some $a > 0$, suppose that $\mathcal{V}$ and $\mathcal{W} : J \times \mathbf{R} \rightarrow \mathbf{R}$ are CFs, $q \in [1, 2)$ and a gamma function, denoted by Γ , the following are satisfied:

$$|\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \leq M_v |y(t) - u(t)| \quad (3.1)$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)|, \quad (3.2)$$

for each $t \in J$, $y, u \in \mathbf{R}$, and suppose that

$$\left| y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}((t - \xi)^q) g(\xi, y(\xi)) d\xi \right] \right| \leq \varepsilon, \quad (3.3)$$

and we also suppose that

$$0 < (M_v M_w K \varepsilon + M_v \|\mathcal{W}\| + M_w \|\mathcal{V}\|) \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \cdot \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \right] = K' < 1.$$

Then, the quadratic FIE is Hyers–Ulam stable.

Proof. Let us consider the continuous functions space: $\mathcal{Y} = C([0, a], \mathbf{R})$ with a generalized metric (GMr), expressed as:

$$\hat{\rho}(g, h) = \inf \left\{ K \in [0, \infty] : |g(x) - h(x)| \leq K\varepsilon, \forall t \in J \right\}.$$

By referring to the discussed notions in Sec. 2, it is very clear that $(\mathcal{Y}, \hat{\rho})$ is a GCMSp (see Theorem 2.1). Let us now construct an operator: $\mathcal{O} : \mathcal{Y} \rightarrow \mathcal{Y}$ as:

$$\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right].$$

From the definition of $\mathcal{O}$ and equations (3.1) and (3.2), we obtain:

$$\begin{aligned} |\mathcal{O}(y(t)) - \mathcal{O}(u(t))| &= \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] \right. \\ &\quad \left. - \mathcal{V}(t, u(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \mathcal{Q}_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, u(\xi)) d\xi \right] \right| \end{aligned}$$

$$\begin{aligned}
& \leq |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| d\xi \right] \\
& \quad + |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q |\mathcal{W}(\xi, u(\xi))| d\xi \right] \\
& \quad + |\mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| d\xi \right] \\
& \leq M_v |y(t) - u(t)| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q M_w |y(\xi) - u(\xi)| d\xi \right] \\
& \quad + M_v |y(t) - u(t)| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q \|\mathcal{W}\| d\xi \right] \\
& \quad + \|\mathcal{V}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q M_w |y(\xi) - u(\xi)| d\xi \right] \\
& \leq (M_v M_w K^2 \varepsilon^2 + M_v K \varepsilon \|\mathcal{W}\| + M_v K \varepsilon \|\mathcal{V}\|) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q d\xi \right] \\
& \leq (M_v M_w K^2 \varepsilon^2 + M_v K \varepsilon \|\mathcal{W}\| + M_w K \varepsilon \|\mathcal{V}\|) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} \right. \\
& \quad \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} (t-\xi)^{mq} d\xi \Big] \\
& = (M_v M_w K^2 \varepsilon^2 + M_v K \varepsilon \|\mathcal{W}\| + M_w K \varepsilon \|\mathcal{V}\|) \\
& \quad \times \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \int_0^t (t-\xi)^{q-1} (t-\xi)^{mq} d\xi \right] \\
& \leq K \varepsilon (M_v M_w + M_v \|\mathcal{W}\| + M_w \|\mathcal{V}\|) \\
& \quad \times \left[\frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \cdot \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \right].
\end{aligned}$$

Because $0 < K' < 1$, we conclude that $\mathcal{O}$ is contraction mapping. Let us take $y'_0 \in \mathcal{Y}$, from the continuous property of $y'_0 \in \mathcal{Y}$ and $\mathcal{O}(y'_0) \in \mathcal{Y}$, $\exists$ a constant $0 < C_1 < \infty$ with

$$|(\mathcal{O}y'_0)(t) - y'_0(t)| \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] - y'_0(t) \right| \leq C_1 \varepsilon, \quad (3.4)$$

$\forall t \in [0, a]$. so $\hat{\rho}(\mathcal{O}(y'_0), y'_0) < \infty$. Therefore, Theorem 2.1(I) indicates that $\exists$ a CF: $y'_0 : [0, a] \rightarrow \mathbf{R}$ such that $\mathcal{O}^n y'_0 \rightarrow y'_0$ in $(\mathcal{Y}, \hat{\rho})$ as $n \rightarrow \infty$, $y'_0 = \mathcal{O}(y'_0)$ where y'_0 satisfies equation (1.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y'_0 and y are CFs defined on a compact interval $[0, a]$. Thus, $\exists$ a constant $C_y > 0$ with

$$|y'_0(t) - y(t)| \leq C_x \varepsilon,$$

$\forall t \in [0, a]$. This indicates that $\hat{\rho}(y'_0, y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently

$$\{y \in \mathcal{Y} : \hat{\rho}(y'_0, y) < \infty\} = \mathcal{Y}.$$

Hence, from Theorem 2.1(II) y'_0 is a unique continuous function (UqCF) with property (1.1). Also, it implies from (3.1)

$$\hat{\rho}(\mathcal{O}(y(t)), y(t)) \leq \varepsilon,$$

$\forall t \in [0, a]$. At last,

$$\hat{\rho}(y, y'_0) \leq \frac{1}{1-K'} \hat{\rho}(\mathcal{O}y, y) \leq \frac{1}{1-K'} \varepsilon.$$

This means that the quadratic FIE is Hyers-Ulam stable. $\square$

The HURS of equation (1.1) is investigated below.

Theorem 3.3. For a closed and bounded interval $J = [0, a]$ of the real line $\mathbb{R}$ for some $a > 0$, suppose that $\mathcal{V}, \mathcal{W} : J \times \mathbb{R} \rightarrow \mathbb{R}$ are CFs, $q \in [1, 2)$ and a gamma function, denoted by Γ , the following are satisfied:

$$|\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \leq M_v |y(t) - u(t)| \quad (3.5)$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)| \quad (3.6)$$

and

$$\left[\int_0^t (\varphi(\xi))^{1/p} d\xi \right]^p \leq C \varphi(t),$$

for any $t \in J, y, u \in \mathbb{R}$ and suppose that

$$\left| y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t-\xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] \right| \leq \varepsilon \varphi(t), \quad (3.7)$$

and we also suppose that

$$\begin{aligned} 0 &< (M_v M_w K C \varepsilon \varphi(t) + M_w C \|\mathcal{V}\|) \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right]^{1/w} \left(\frac{w t^{\frac{mq+q+w-1}{w}}}{mq+q+w-1} \right) \\ &+ M_v \|\mathcal{W}\| \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \cdot \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \\ &= K' < 1. \end{aligned}$$

Then, the quadratic FIE is Hyers-Ulam stable.

Proof. Let us consider the continuous functions space: $\mathcal{Y} = C([0, a], \mathbb{R})$ and $g \in \mathcal{Y}$, with a GMr, expressed as:

$$\hat{\rho}(g, h) = \inf \left\{ K \in [0, \infty] : |g(t) - h(t)| \leq K \varepsilon \varphi(t), \forall t \in J \right\}.$$

Obviously, $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let us now form an operator: $\mathcal{O} : \mathcal{Y} \rightarrow \mathcal{Y}$ as:

$$\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t-\xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right]. \quad (3.8)$$

From the definition $\mathcal{O}$ and equations (3.5) and (3.6), we get:

$$\begin{aligned} |\mathcal{O}(y(t)) - \mathcal{O}(u(t))| &= \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t-\xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] \right. \\ &\quad \left. - \mathcal{V}(t, u(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t-\xi)^q \mathcal{W}(\xi, u(\xi)) d\xi \right] \right| \end{aligned}$$

$$\begin{aligned}
& \leq |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| \, d\xi \right] \\
& \quad + |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q |\mathcal{W}(\xi, u(\xi))| \, d\xi \right] \\
& \quad + |\mathcal{V}(t, u(t))| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| \, d\xi \right] \\
& \leq M_v |y(t) - u(t)| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q M_w |y(\xi) - u(\xi)| \, d\xi \right] \\
& \quad + M_v |y(t) - u(t)| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \|\mathcal{W}\| \, d\xi \right] \\
& \quad + \|\mathcal{V}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q M_v |y(\xi) - u(\xi)| \, d\xi \right] \\
& \leq M_v M_w K^2 \varepsilon^2 \varphi(t) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \varphi(\xi) \, d\xi \right] \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \varphi(t) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \, d\xi \right] \\
& \quad + M_v K \varepsilon \|\mathcal{V}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \varphi(\xi) \, d\xi \right] \\
& \leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \|\mathcal{V}\| \right) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \varphi(\xi) \, d\xi \right] \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \varphi(t) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \, d\xi \right] \\
& \leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \|\mathcal{V}\| \right) \frac{1}{\Gamma(q)} \left[\int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} ((t - \xi)^q)^{1/w} \, d\xi \right]^w \\
& \quad \times \left[\int_0^t (\varphi(\xi))^{1/p} \, d\xi \right]^p \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \, d\xi \right] \\
& \leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \|\mathcal{V}\| \right) C \varphi(t) \frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{(q-1)/w} \\
& \quad \times \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right]^{1/w} ((t - \xi)^q)^{mq/w} \, d\xi \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \varphi(t) \frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \, d\xi
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right]^{1/w} (t - \xi)^{mq} d\xi \\
& \leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \|\mathcal{V}\| \right) C \varphi(t) \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right]^{1/w} \\
& \quad \times \int_0^t (t - \xi)^{(mq+q-1)/w} d\xi \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \varphi(t) \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \int_0^t (t - \xi)^{mq+q-1} d\xi \\
& \leq \left(M_v M_w K^2 \varepsilon^2 \varphi(t) + M_v K \varepsilon \|\mathcal{V}\| \right) C \varphi(t) \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \left[\frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right]^{1/w} \\
& \quad \times \frac{w}{mq + q - 1} t^{(mq+q+w-1)/w} \\
& \quad + M_v K \varepsilon \|\mathcal{W}\| \varphi(t) \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \frac{t^{(m+1)q}}{(m+1)q} \\
& \leq K \varepsilon \varphi(t) K'.
\end{aligned}$$

Note that $0 < K' < 1$. We conclude that $\mathcal{O}$ is contraction mapping (CoMp). Let us take $y'_0 \in \mathcal{Y}$, from the continuous property of $y'_0 \in \mathcal{Y}$ and $\mathcal{O}(y'_0) \in \mathcal{Y} \exists$ a constant $0 < C_1 < \infty$ with

$$\begin{aligned}
|\mathcal{O}(y'_0)(t) - y'_0(t)| &= \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] - y'_0(t) \right| \\
&\leq C_1 \varepsilon \varphi(t),
\end{aligned}$$

$\forall t \in [0, a]$. So, $\hat{\rho}(\mathcal{O}(y_0), y'_0) < \infty$. Thus, Theorem 2.1(a) indicates that $\exists$ a CF: $u'_0 : [0, a] \rightarrow \mathbb{R}$ such that $\mathcal{O}^n(y_0) \rightarrow u'_0$ in $(\mathcal{Y}, \hat{\rho})$ as $n \rightarrow \infty$, $u'_0 = \mathcal{O}(u'_0)$; therefore, u'_0 satisfies equation (1.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y'_0 and y are CFs defined on a compact interval $[0, a]$. Thus, $\exists$ a constant $C_x > 0$ with

$$|y'_0(t) - y(t)| \leq C_y \varepsilon \varphi(t), \quad \forall t \in [0, a].$$

This indicates that $\hat{\rho}(y'_0, y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently $\{y \in \mathcal{Y} : \hat{\rho}(y'_0, y) < \infty\} = \mathcal{Y}$. Hence, from Theorem (2.1)(II) u'_0 is a UqCF with property (1.1). As a result, from (3.7), it implies that

$$\hat{\rho}(\mathcal{O}(u(t)), u(t)) \leq \varepsilon \varphi(t),$$

$\forall t \in [0, a]$. At last,

$$\hat{\rho}(u, u'_0) \leq \frac{1}{1 - K'} \hat{\rho}(\mathcal{O}(u), u) \leq \frac{1}{1 - K'} \varepsilon \varphi(t).$$

Thus, the quadratic FIE is Hyers–Ulam–Rassias stable. $\square$

4 ML–Hyers–Ulam Stability

The ML–HUS of equation (1.1) is studied in this section.

Definition 4.1. If for each function y satisfies

$$|y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r} (t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right]| \leq \varepsilon E_q(t^q),$$

∃ an equation's (1.1) solution y_0 , and a constant, $c > 0$, which is an independent of both y and y_0 such that

$$|y(t) - y_0(t)| \leq c\varepsilon E_q(t^q),$$

for each $t \in [0, a]$, then equation (1.1) is named as ML-Hyers-Ulam stable.

Theorem 4.2. For a closed and bounded interval $J = [0, a]$ of the real line $\mathbb{R}$ for some $a > 0$, suppose that $\mathcal{V}, \mathcal{W} : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $q \in [1, 2)$ and a gamma function, denoted by Γ , the following are satisfied:

$$|\mathcal{V}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_v |y(t) - u(t)| \quad (4.1)$$

and

$$|\mathcal{W}(t, y(t)) - \mathcal{W}(t, u(t))| \leq M_w |y(t) - u(t)|, \quad (4.2)$$

for any $t \in J$, $y, u \in \mathbb{R}$, and suppose that

$$\left| y(t) - \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] \right| \leq \varepsilon E_q(t^q). \quad (4.3)$$

Also, suppose that

$$\begin{aligned} 0 &< (M_v M_w K \varepsilon E_q(t^q) + M_w K \varepsilon \|\mathcal{V}\|) \\ &\times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \Gamma((m+1)q) \\ &+ M_v \|\mathcal{W}\| \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \left(\frac{t^{q(m+1)}}{q(m+1)} \right) \\ &= K' < 1, \end{aligned}$$

then quadratic FIE is Hyers-Ulam stable.

Proof. Let us consider the continuous functions space: $\mathcal{Y} = C([0, a], \mathbb{R})$ and $g \in \mathcal{Y}$, with a GMr, expressed as:

$$\hat{\rho}(g, h) = \inf \left\{ K \in [0, \infty] : |g(x) - h(x)| \leq K \varepsilon E_q(t^q), \forall t \in J \right\}.$$

Obviously, $(\mathcal{Y}, \hat{\rho})$ is a GCMSp. Let us express an operator: $\mathcal{O} : \mathcal{Y} \rightarrow \mathcal{Y}$ by

$$\mathcal{O}(y(t)) = \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right]. \quad (4.4)$$

From the definition $\mathcal{O}$ and equations (4.1) and (4.2), we get:

$$\begin{aligned} |\mathcal{O}(y(t)) - \mathcal{O}(u(t))| &= \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, y(\xi)) d\xi \right] \right. \\ &\quad \left. - \mathcal{V}(t, u(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q \mathcal{W}(\xi, u(\xi)) d\xi \right] \right| \\ &\leq |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \\ &\quad \times \left[\frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t - \xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| d\xi \right] \\ &\quad + |\mathcal{V}(t, y(t)) - \mathcal{V}(t, u(t))| \end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q |\mathcal{W}(\xi, u(\xi))| \, d\xi \right] \\
& + |\mathcal{V}(t, u(t))| \\
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q |\mathcal{W}(\xi, y(\xi)) - \mathcal{W}(\xi, u(\xi))| \, d\xi \right] \\
& \leq M_v |y(t) - u(t)| \\
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q M_w |y(\xi) - u(\xi)| \, d\xi \right] \\
& + M_v |y(t) - u(t)| \\
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q \|\mathcal{W}\| \, d\xi \right] \\
& + \|\mathcal{V}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q M_v |y(\xi) - u(\xi)| \, d\xi \right] \\
& \leq M_v M_w K^2 \varepsilon^2 E_q(t^q) \\
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q E_q(\xi^q) \, d\xi \right] \\
& + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q \, d\xi \right] \\
& + M_v K \varepsilon \|\mathcal{V}\| \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q E_q(\xi^q) \, d\xi \right] \\
& \leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K \varepsilon \|\mathcal{V}\| \right) \\
& \times \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q E_q(\xi^q) \, d\xi \right] \\
& + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha,\beta,\delta}^{\gamma,q,r} (t-\xi)^q \, d\xi \right] \\
& \leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K \varepsilon \|\mathcal{V}\| \right) \\
& \times \frac{1}{\Gamma(q)} \left[\int_0^t (t-\xi)^{q-1} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \right. \\
& \times (t-\xi)^{mq} \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn+1)} \, d\xi \\
& + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} \right. \\
& \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \, d\xi
\end{aligned}$$

$$\begin{aligned}
&\leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K \varepsilon \|\mathcal{V}\| \right) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \\
&\quad \times \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn + 1)} \int_0^t (t - \xi)^{q-1} (t - \xi)^{mq} \xi^{nq} d\xi \\
&\quad + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \frac{1}{\Gamma(q)} \int_0^t (t - \xi)^{q-1} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} (t - \xi)^{mq} d\xi \\
&\leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_v K \varepsilon \|\mathcal{V}\| \right) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \\
&\quad \times \sum_{n=0}^{\infty} \frac{s^{nq}}{\Gamma(qn + 1)} \int_0^t (t - \xi)^{mq+q-1} s^{nq} d\xi \\
&\quad + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \frac{1}{\Gamma(q)} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \\
&\quad \times \int_0^t (t - \xi)^{mq+q-1} d\xi \\
&\leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_w K \varepsilon \|\mathcal{V}\| \right) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \\
&\quad \times \Gamma((m + 1)q) \sum_{s=0}^{\infty} \frac{1}{\Gamma(qs + 1)} t^{sq} \Big) \\
&\quad + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \left(\frac{t^{(m+1)q}}{(m + 1)q} \right) \\
&\leq \left(M_v M_w K^2 \varepsilon^2 E_q(t^q) + M_w K \varepsilon \|\mathcal{V}\| \right) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \Gamma((m + 1)q) E_q(t^q) \\
&\quad + M_v K \varepsilon \|\mathcal{W}\| E_q(t^q) \\
&\quad \times \frac{1}{\Gamma(q)} \sum_{m=0}^{\infty} \frac{\prod_{n=1}^r \beta(b_n, m) (\gamma)_{qm}}{\prod_{n=1}^r \beta(a_n, m) (\delta)_{qm} \Gamma(\beta + \alpha m)} \left(\frac{t^{(m+1)q}}{(m + 1)q} \right) \\
&\leq K \varepsilon E_q(t^q) K'.
\end{aligned}$$

We note that $0 < K' < 1$. We conclude that $\mathcal{O}$ is contraction mapping. Let us take $y'_0 \in \mathcal{Y}$, from the continuous property of $y'_0 \in \mathcal{Y}$ and $\mathcal{O}(y'_0) \in \mathcal{Y}$, $\exists$ a constant $0 < C_1 < \infty$ with

$$\begin{aligned} |\mathcal{O}(y'_0)(t) - y'_0(t)| &= \left| \mathcal{V}(t, y(t)) \left[\frac{1}{\Gamma(q)} \int_0^t (t-\xi)^{q-1} Q_{\alpha, \beta, \delta}^{\gamma, q, r}(t-\xi)^q \mathcal{W}(\xi, y(\xi)) \, d\xi \right] - y'_0(t) \right| \\ &\leq C_1 \varepsilon E_q(t^q), \end{aligned}$$

$\forall t \in [0, a]$. So, $\hat{\rho}(\mathcal{O}(y'_0), y'_0) < \infty$. Thus, Theorem 2.1(I) indicates that $\exists$ a CF: $u'_0 : [0, a] \rightarrow \mathbb{R} \ni \mathcal{O}^n y'_0 \rightarrow u'_0$ in $(\mathcal{Y}, \hat{\rho})$ as $n \rightarrow \infty$, $u'_0 = \mathcal{O}(u'_0)$; therefore, u'_0 satisfies equation (1.1) for any $t \in J$. If $y \in \mathcal{Y}$, then y'_0 and y are CFs defined on a compact interval $[0, a]$. Thus, $\exists$ a constant $C_y > 0$ with

$$|y'_0(t) - y(t)| \leq C_y \varepsilon \mathcal{E}_q(t^q),$$

$\forall t \in [0, a]$. This indicates that $\hat{\rho}(y'_0, y) < \infty$ for every $y \in \mathcal{Y}$ or equivalently

$$\{y \in \mathcal{Y} : \hat{\rho}(y'_0, y) < \infty\} = \mathcal{Y}.$$

Hence, from Theorem (2.1)(II) u'_0 is a UqCF with property (1.1). From (4.3), it implies that

$$\hat{\rho}(\mathcal{O}(u(t)), u(t)) \leq \varepsilon \mathcal{E}_q(t^q),$$

$\forall t \in [0, a]$. At last,

$$\hat{\rho}(u, u'_0) \leq \frac{1}{1-K'} \hat{\rho}(\mathcal{O}(u), u) \leq \frac{1}{1-K'} \varepsilon \mathcal{E}_q(t^q).$$

This means that the quadratic FIE is ML-Hyers-Ulam stable. □

5 Conclusion

Functional equations, particularly quadratic fractional integral equations, have been applied in inner product spaces' characterization. An essential parallelogram equality:

$$\|y+z\|^2 + \|y-z\|^2 = 2 \left(\|y\|^2 + \|z\|^2 \right),$$

is satisfied by a square norm on an inner product space. Both of the HUS and ML-HURS have been investigated for the proposed DEq. All obtained results are new which show that ML is very important in proving differential equation's stability. Various differential equations' classes can be unified via our new generalized technique which can inspire interested engineers and scientists to work on future research studies, particularly in the field of environmental sciences, with other related applications based on this technique, and this technique will be extended further in our future research works by working on modeling various scientific phenomena.

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