Research Article

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Novel results on conformable Bessel functions

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Abstract: Novel results on conformable Bessel functions are proposed in this study. We complete this study by proposing and proving certain properties of the Bessel functions of first order involving their conformable derivatives or their zeros. We also establish the orthogonality of such functions in the interval $[0,1]$. This study is essential due to the importance of these functions while modeling various physical and natural phenomena.

Keywords: conformable derivative, conformable Bessel functions, conformable Bessel equation

1 Introduction

Fractional calculus is theoretically a powerful analysis technique for investigating arbitrary order integrals and derivatives. At the beginning, this field of research has been only presented purely, and until very recently, researchers have realized the powerful applicability of this field in modeling many phenomena from natural sciences and engineering much better than using the ordinary usual calculus due to several properties in fractional calculus that can provide a good explanation of physical behavior of certain system (see refs. [1–7]). The applications of conformable and fractional calculus have been recently discussed in some research studies [8–15]. The analysis of conformable derivatives and integrals has been discussed in detail in earlier studies [16,17].

Riemann–Liouville and Caputo fractional operators [18,19] have been initially introduced, and recently, various new or generalized definitions have been proposed

by researchers. In 2014, Khalil *et al.* proposed a local definition of fractional derivative, known as conformable derivative [20]. This topic has been discussed in some research studies whether conformable derivatives are considered fractional derivatives or not. While Tarasov has discussed in an earlier study [21] that some recent definitions of fractional derivatives are not fractional derivatives, Almeida *et al.* have concluded in an earlier study [22] that conformable derivative is still interesting to study it further. A new generalized definition of conformable derivative that coincides with the definitions of Caputo and Riemann–Liouville fractional derivatives has been proposed in an earlier study [23]. Some mathematicians such as Zhao and Luo and Khalil *et al.* discussed the conformable derivative's physical and geometrical meanings in an earlier studies [24,25], respectively. Therefore, the conformable derivative in this study will only be considered as a modified form of local usual derivative. The analysis of Bessel functions in the context of conformable derivative has been rarely discussed in any other research studies. Therefore, our results in this article are novel and worthy due to the importance of Bessel functions in various modeling scenarios in science and engineering.

The article is outlined as follows: some essential notions of the conformable calculus and conformable Bessel functions are presented in Section 2. In Section 3, we propose and prove new properties of Bessel functions of the first kind involving their conformable derivatives or their zeros. We conclude with Section 4, where the orthogonality of such functions in the interval $[0,1]$ is studied.

2 Preliminaries

Definition 2.1. [20] For a function $f : [0, \infty) \rightarrow \mathbb{R}$, the conformable derivative of order α is expressed as follows:

$$(T_\alpha f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \quad (1)$$

$\forall t > 0; 0 < \alpha \leq 1$. If f is α -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} (T_\alpha f)(t)$ exists, then it is defined as follows:

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$$(T_\alpha f)(0) = \lim_{t \rightarrow 0^+} (T_\alpha f)(t). \quad (2)$$

As a result, we have Theorem 2.1 as follows:

Theorem 2.1. [20]. If a function $f : [0, \infty) \rightarrow R$ is α -differentiable at $t_0 > 0$; $0 < \alpha \leq 1$, then f is continuous at t_0 .

Hence, T_α satisfies some properties as follows:

Theorem 2.2. [20]. Suppose that $0 < \alpha \leq 1$ and f, g are α -differentiable at a point $t > 0$. Then, we get the following:

- (i) $T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g)$, for all $a, b \in R$.
- (ii) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in R$.
- (iii) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
- (iv) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.
- (v) $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$; $g \neq 0$.
- (vi) Additionally, if f is a differentiable function, then $(T_\alpha f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

From the above definition, the conformable derivatives of certain functions are written as follows:

- (i) $T_\alpha(1) = 0$.
- (ii) $T_\alpha(\sin(at)) = at^{1-\alpha} \cos(at)$, $a \in R$.
- (iii) $T_\alpha(\cos(at)) = -at^{1-\alpha} \sin(at)$, $a \in R$.
- (iv) $T_\alpha(e^{at}) = ae^{at}$, $a \in R$.

Theorem 2.3. (Rolle's theorem). [20]. Suppose that $a > 0$; $\alpha \in (0, 1]$, and $f : [a, \infty) \rightarrow R$ is a function that satisfies the following:

- f is continuous on a, b .
- f is α -differentiable on (a, b) .
- $f(a) = f(b)$.

Then, $\exists c \in (a, b)$ such that $(T_\alpha f)(c) = 0$.

Theorem 2.4. [26]. Suppose that $a > 0$; $\alpha \in (0, 1]$, and $f : [a, \infty) \rightarrow R$ is function that satisfies the following:

- f is continuous in $[a, b]$.
- f is α -differentiable on (a, b) .

If $(T_\alpha f)(c) = 0 \forall t \in (a, b)$, then f is a constant on $[a, b]$.

Definition 2.2. The left-conformable derivative beginning from a of a function $f : [a, \infty) \rightarrow R$ of f of order $0 < \alpha \leq 1$, [27], is expressed as follows:

$$(T_\alpha^a f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon(t - a)^{1-\alpha}) - f(t)}{\varepsilon}. \quad (3)$$

When $a = 0$, it is denoted as $(T_\alpha f)(t)$. If f is α -differentiable in some (a, b) , then we get the following equation:

$$(T_\alpha^a f)(a) = \lim_{t \rightarrow a^+} (T_\alpha^a f)(t). \quad (4)$$

It is noticeable that if f is differentiable function, then $(T_\alpha^a f)(a) = (t - a)^{1-\alpha} \frac{df}{dt}(t)$. Theorem 2.2 holds for definition 2.2 when changing by $(t - a)$.

Theorem 2.5. (Chain rule). [27]. Assume $f, g : (a, \infty) \rightarrow R$ be left α -differentiable function functions, where $0 < \alpha \leq 1$. Suppose that $h(t) = f(g(t))$. The $h(t)$ is α -differentiable function $\forall t \neq a$ and $g(t) \neq 0$, hence we get the following equation:

$$(T_\alpha^a h)(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot (g(t))^{\alpha-1}. \quad (5)$$

If $t = a$, then we define as follows:

$$(T_\alpha^a h)(a) = \lim_{t \rightarrow a^+} (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t) \cdot (g(t))^{\alpha-1}. \quad (6)$$

Remark 2.1. In an earlier study [27], the left-conformable derivative at a for some smooth functions has been investigated. Suppose that $0 < \alpha \leq 1$ and $n \in \mathbb{Z}^+$, then the left sequential conformable derivative of order n is expressed as follows: ${}^{(n)}T_\alpha^a f(t) = \underbrace{T_\alpha^a T_\alpha^a \dots T_\alpha^a f(t)}_{n\text{-times}}$.

We show via induction that if f is continuously α -differentiable and $0 < \alpha \leq \frac{1}{n}$ then, the n th order sequential conformable derivative is continuous and vanishes at the end point a .

Theorem 2.6. [27]. Suppose that f is infinitely α -differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point t_0 . Then, f has a conformable power series expansion as follows:

$$f(t) = \sum_{k=0}^{\infty} \frac{((k)T_\alpha^{t_0})(t_0)}{\alpha^k k!} (t - t_0)^{k\alpha}, \quad t_0 < t < t_0 + R^{\frac{1}{\alpha}}. \quad (7)$$

Here, $((k)T_\alpha^{t_0})(t_0)$ indicates that we are applying the conformable derivative k times.

The α -integral of a function f beginning from $a \geq 0$ is expressed as follows:

Definition 2.3. [20]. $I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \cdot dx$, where it is a usual Riemann improper integral, and $\alpha \in (0, 1]$.

As a result, we obtain the following:

Theorem 2.7. $T_\alpha^a I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α .

Lemma 2.1. [27]. Let $f : (a, b) \rightarrow R$ be differentiable and $\alpha \in (0, 1]$. Then, $\forall a > 0$, and we get the following:

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a). \quad (9)$$

Theorem 2.8. [27]. Let $f : [a, b] \rightarrow R$ be two functions $\ni fg$ is differentiable. Then we have the following equation:

$$\int_a^b f(x) T_\alpha^a(g)(x) d\alpha(x) = fg|_a^b - \int_a^b g(x) T_\alpha^a(f)(x) d\alpha(x). \quad (10)$$

Now, consider the sequential conformable Bessel equation [28]:

$$t^{2\alpha} T_\alpha T_\alpha y + \alpha t^\alpha T_\alpha y + \alpha^2 (t^{2\alpha} - p^2) y = 0. \quad (11)$$

where $0 < \alpha \leq 1$ and p is any real number. If $\alpha = 1$, then Eq. (1) is a usual Bessel equation [29]. $t = 0$ is a α -regular singular point for the equation. In this case, for $t > 0$, the solution can be investigated via a conformable Fröbenius series as:

$$y = t^{r\alpha} \sum_{n=0}^{\infty} c_n t^{n\alpha} = \sum_{n=0}^{\infty} c_n t^{(n+r)\alpha}$$

We let:

$$T_\alpha y = \sum_{n=0}^{\infty} \alpha(n+r) c_n t^{(n+r-1)\alpha}$$

$$T_\alpha T_\alpha y = \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1) c_n t^{(n+r-2)\alpha}$$

Let substitute these expressions in Eq. (11) to get the following:

$$l(r) c_0 t^{r\alpha} + l(r+1) c_1 t^{(r+1)\alpha} + \sum_{n=0}^{\infty} [l(r+n) c_n + \alpha^2 c_{n-2}] t^{(r+n)\alpha} = 0$$

where $l(r) = r(r-1)\alpha^2 + r\alpha^2 - \alpha^2 p^2$.

Let us $c_0 \neq 0$, then we obtain the following:

$$l(r) = 0$$

Since $\alpha^2 \neq 0$, the following can be written as follows:

$$l(r) = 0 \Rightarrow r(r-1)\alpha^2 + r\alpha^2 - \alpha^2 p^2 = 0 \Rightarrow r^2 - p^2 = 0.$$

Hence, we find:

$$r_1 = p, \quad r_2 = -p.$$

Now, for $p > 0$, the solutions of the conformable Bessel equation of order p are analyzed as follows:

In this case, for $r_1 = p$, we have:

$$l(r_1 + 1) c_1 = [\alpha^2 p(p+1) + \alpha^2(p+1) - \alpha^2 p^2] c_1 = 0.$$

$$(2p+1) c_1 = 0.$$

Due to $p > 0$, it follows that $c_1 = 0$. The recurrence relation is as follows:

$$c_n = -\frac{c_{n-2}}{n(n+p)}.$$

From $c_1 = 0$ and last recurrence relation, we obtain the following:

$$c_3 = c_5 = \dots = 0.$$

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n! (p+1)(p+2)\dots(p+n)}, \quad n \geq 1.$$

Let us $c_0 = \frac{c}{\Gamma(p+1)}$. Thus, the first solution of the conformable Bessel equation of order p has the following form:

$$y_1(t) = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+1)} \left(\frac{t^\alpha}{2}\right)^{2n+p}. \quad (12)$$

Besides, the Bessel function of order p is valid.

$$(J_\alpha)_p(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(p+n+1)} \left(\frac{t^\alpha}{2}\right)^{2n+p}. \quad (13)$$

For $r_2 = -p$, if $r_1 - r_2 = 2p$ is not a positive integer, then the second solution of the conformable Bessel equation of order p is written as follows:

$$y_2(t) = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(-p+n+1)} \left(\frac{t^\alpha}{2}\right)^{2n-p}. \quad (14)$$

The second type Bessel functions of order p has the following form:

$$(J_\alpha)_{-p}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(-p+n+1)} \left(\frac{t^\alpha}{2}\right)^{2n-p}. \quad (15)$$

3 Some basic properties of conformable Bessel functions

First, we will study the convergence of series (13) and (15).

Theorem 3.1. *The series that defines the conformable Bessel functions of the first kind of order p and $-p$ are absolutely convergent for all $t > 0$.*

Proof. The radius of convergence of conformable series is as follows:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\pm p + n + 1)} x^{n\alpha}$$

which is easily found by ratio test [30],

$$\begin{aligned} \frac{1}{R} &= \lim_{n \rightarrow \infty} \frac{n! \Gamma(\pm p + n + 1)}{(n+1)! \Gamma(\pm p + n + 2)} \\ &= \lim_{n \rightarrow \infty} \frac{1}{(n+1)(\pm p + n + 1)} = 0 \end{aligned}$$

which implies $R = \infty$. $\square$

Remark 3.1. As series (13) and (15) are both convergent $\forall t > 0$, we may do a term-by-term differentiation for them [29].

As $(J_\alpha)_p$ and $(J_\alpha)_{-p}$ are the second order conformable linear differential equation's solutions, our goal is to see that they are linearly independent, and thus they form a basis for the vector space of the solutions of Eq. (11).

Theorem 3.2. *If $2p$ is not a positive integer, then the conformable Bessel functions of the first kind $(J_\alpha)_p$ and $(J_\alpha)_{-p}$, are linearly independent. In that case, for any solution y of (11), $\exists A, B \in \mathbb{R}$*

$$y(t) = A(J_\alpha)_p(t) + B(J_\alpha)_{-p}(t). \quad (16)$$

Proof. It is enough to see that the α -Wronskian of $(J_\alpha)_p(t)$ and $(J_\alpha)_{-p}(t)$

$$W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) = \begin{vmatrix} (J_\alpha)_p(t) & (J_\alpha)_{-p}(t) \\ T_\alpha(J_\alpha)_p(t) & T_\alpha(J_\alpha)_{-p}(t) \end{vmatrix},$$

does not vanish at any point. Since $(J_\alpha)_p$ and $(J_\alpha)_{-p}(t)$ satisfy Eq. (11)

$$t^{2\alpha}T_\alpha T_\alpha(J_\alpha)_p(t) + \alpha t^\alpha T_\alpha(J_\alpha)_p(t) + \alpha^2(t^{2\alpha} - p^2)(J_\alpha)_p(t) = 0,$$

$$t^{2\alpha}T_\alpha T_\alpha(J_\alpha)_{-p}(t) + \alpha t^\alpha T_\alpha(J_\alpha)_{-p}(t) + \alpha^2(t^{2\alpha} - p^2)(J_\alpha)_{-p}(t) = 0.$$

By the multiplication of the two equations by $(J_\alpha)_{-p}(t)$ and $(J_\alpha)_p(t)$, respectively. Subtracting one from the other, and dividing by t^α , we get the following:

$$\begin{aligned} & t^\alpha((J_\alpha)_p(t)T_\alpha T_\alpha(J_\alpha)_{-p}(t) - (J_\alpha)_{-p}(t)T_\alpha T_\alpha(J_\alpha)_p(t)) \\ & - \alpha((J_\alpha)_p(t)T_\alpha(J_\alpha)_{-p}(t) - (J_\alpha)_{-p}(t)T_\alpha(J_\alpha)_p(t)) = 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} & T_\alpha(t^\alpha((J_\alpha)_p(t)T_\alpha(J_\alpha)_{-p}(t) - (J_\alpha)_{-p}(t)T_\alpha(J_\alpha)_p(t))) \\ & = T_\alpha(t^\alpha W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t))) = 0. \end{aligned}$$

This implies $W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) = \frac{C}{t^\alpha}$, (see Theorem 2.4), C is a constant that needs to be found. By considering the first term in the series (13), we have the following:

$$\begin{aligned} (J_\alpha)_p(t) &= \frac{\left(\frac{t^\alpha}{2}\right)^p}{\Gamma(p+1)}(1 + O(t^{2\alpha})), \\ T_\alpha(J_\alpha)_p(t) &= \frac{\left(\frac{t^\alpha}{2}\right)^{p-1}}{2\Gamma(p)}(1 + O(t^{2\alpha})). \end{aligned}$$

The same applies to $(J_\alpha)_{-p}(t)$. Then, with the help of Euler's reflection formula, $\left(\Gamma(z)\Gamma(1-p) = \frac{\pi}{\sin \pi z}, \forall z \in \mathbb{C} - \mathbb{Z}\right)$

$$\begin{aligned} & W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) \\ &= \frac{1}{t^\alpha} \left(\frac{1}{\Gamma(p+1)\Gamma(-p)} - \frac{1}{\Gamma(-p+1)\Gamma(p)} \right) \\ &+ O(t^\alpha) = -\frac{2\alpha \sin p\pi}{\pi t^\alpha} + O(t^\alpha). \end{aligned}$$

However, $W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) = \frac{C}{t^\alpha}$ have been stated previously, so the last $O(t^\alpha)$ must be zero and

$$W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) = -\frac{2\alpha \sin p\pi}{\pi t^\alpha},$$

which only vanishes when p is an integer. From the hypothesis, $2p$ is not an integer, so neither is p nor

$$W^\alpha((J_\alpha)_p(t), (J_\alpha)_{-p}(t)) \neq 0, \forall t > 0.$$

Therefore, $(J_\alpha)_p(t)$ and $(J_\alpha)_{-p}(t)$ are linearly independent solutions of (Eq. (11)) which is a second-order conformable linear differential equation. Due to the fact that solutions constructing a two-dimensional vector space, and $(J_\alpha)_p(t)$ and $(J_\alpha)_{-p}(t)$ being linearly independent, any solution can be expressed as a linear combination of them.

We will derive some basic facts about the zeros of the conformable Bessel function: $(J_\alpha)_p(t)$ and its α -derivative $T_\alpha(J_\alpha)_p(t)$. $\square$

Remark 3.2. As in the case of classical Bessel functions [30,31], the positive zeros of the conformable Bessel function: $(J_\alpha)_p(t)$ can be arranged as a sequence:

$$0 < a_{p,1} < a_{p,2} < \dots < a_{p,n} < \dots, \lim_{n \rightarrow \infty} a_{p,n} = \infty. \quad (17)$$

Theorem 3.3. *All zeros of $(J_\alpha)_p(t)$, except $t = 0$ possibly, are simple.*

Proof. If $t_0 \neq 0$ is a multiple zero of $(J_\alpha)_p(t)$, then we have at least that $(J_\alpha)_p(t_0) = 0$ and $T_\alpha(J_\alpha)_p(t_0) = 0$. As $t_0 \neq 0$, it follows from the conformable differential Eq. (11) that also $T_\alpha T_\alpha(J_\alpha)_p(t_0) = 0$. Iteration then leads to $(^{(n)}T_\alpha(J_\alpha)_p(t_0) = 0$ for all $n = 0, 1, 2, \dots$, which implies that is identically zero. This is a trivial contradiction. $\square$

Theorem 3.4. *Let $p > 0$. All zeros of $T_\alpha(J_\alpha)_p(t)$, except $t = 0$ or $t = p$ possibly, are simple.*

Proof. If t_0 is a multiple zero of $T_\alpha(J_\alpha)_p(t)$, then we have at least that $T_\alpha(J_\alpha)_p(t_0) = 0$ and $T_\alpha T_\alpha(J_\alpha)_p(t_0) = 0$. For $t_0 \neq 0$ and $t_0 \neq p$, then it follows from the conformable differential Eq. (11) that also $(J_\alpha)_p(t_0) = 0$. In addition, this leads to $(J_\alpha)_p(t)$ being identically zero which is clearly not true. $\square$

Theorem 3.5. Let p is a nonnegative integer. It is verified that between any two consecutive zeros of $(J_\alpha)_p(t)$, $\exists$ precisely one zero of $(J_\alpha)_{p-1}(t)$ and precisely one zero of $(J_\alpha)_{p+1}(t)$.

Proof. Let $0 < a < b$ be two consecutive zeros of $(J_\alpha)_p(t)$. Therefore, $t^{p\alpha}(J_\alpha)_p(t)$ vanishes at a and b . By theorem 2.3, we have:

$$T_\alpha(t^{p\alpha}(J_\alpha)_p(c)) = 0 \text{ for some } c \in (a, b).$$

As given by Yazici and Gözütok [32],

$$T_\alpha(t^{p\alpha}(J_\alpha)_p(t)) = \alpha t^{p\alpha}(J_\alpha)_{p-1}(t).$$

Hence, we obtain $(J_\alpha)_{p-1}(c) = 0$.

Repeating the above argument with the identity given by Uddin *et al.* [33], $T_\alpha(t^{-p\alpha}(J_\alpha)_p(t)) = -\alpha t^{-p\alpha}(J_\alpha)_{p+1}(t)$, we get that $(J_\alpha)_{p+1}(t)$ has a root in (a, b) .

Thus, we have proved that both $(J_\alpha)_{p-1}(t)$ and $(J_\alpha)_{p+1}(t)$ have at least one root in (a, b) .

If $(J_\alpha)_{p-1}(t)$ has two roots in (a, b) , then from the above, we conclude that $(J_\alpha)_p(t)$ would have a root in (a, b) . However, this contradicts the assumption that c and d are consecutive roots. Thus, $(J_\alpha)_{p-1}(t)$ has exactly one root in (a, b) . Similarly, $(J_\alpha)_{p+1}(t)$ has exactly one root in (a, b) . $\square$

4 Orthogonality of conformable Bessel's function

In the classical sense, two functions f, g are orthogonal on an interval $[a, b]$; if $\int_a^b f(t)g(t)dt = 0$. For the case of Bessel functions, the interval that mathematicians consider is $[0, 1]$ for physical applications [30,31]. Now, we want to study the orthogonality of conformable Bessel functions on such interval.

Theorem 4.1. Let p is a nonnegative integer. If a and b be roots of the equation $(J_\alpha)_p(t) = 0$, then we have the following:

$$\int_0^1 t^\alpha (J_\alpha)_p(at)(J_\alpha)_p(bt) \frac{dt}{t^{1-\alpha}} = \begin{cases} 0, & \text{if } a \neq b, \\ \frac{\alpha}{2} (J_\alpha)_{p+1}^2(a), & \text{if } a = b. \end{cases} \quad (18)$$

Proof. First, let us write the conformable Bessel Eq. (11) as follows:

$$x^\alpha T_\alpha(x^\alpha T_\alpha(J_\alpha)_p(x)) - \alpha^2(x^{2\alpha} - p^2)(J_\alpha)_p(x) = 0. \quad (19)$$

The variable p needs to be a noninteger. It turns out to be useful to define a new variable t by $x = at$, where a is a constant, which we will take to be a zero of $(J_\alpha)_p$, that is, $(J_\alpha)_p(a) = 0$. Let us define:

$$u(t) = (J_\alpha)_p(at). \quad (20)$$

which implies $u(1) = 0$, and by substituting into Eq. (18), the following is obtained:

$$t^\alpha T_\alpha(t^\alpha(T_\alpha u)(t)) - \alpha^2(a^{2\alpha}t^{2\alpha} - p^2)u(t) = 0. \quad (21)$$

We can also write down the equation obtained by picking another zero, b say. Let us define:

$$v(t) = (J_\alpha)_p(bt) \text{ so } v(1) = 0. \quad (22)$$

We have:

$$t^\alpha T_\alpha(t^\alpha(T_\alpha v)(t)) - \alpha^2(b^{2\alpha}t^{2\alpha} - p^2)v(t) = 0. \quad (23)$$

To derive the orthogonality relation, we multiply Eq. (21) by v , and Eq. (23) by u . Subtracting and dividing by t^α gives the following:

$$v(t)T_\alpha(t^\alpha(T_\alpha u)(t)) - u(t)T_\alpha(t^\alpha(T_\alpha v)(t)) + (a^{2\alpha} - b^{2\alpha})t^\alpha u(t)v(t) = 0. \quad (24)$$

The first two terms in Eq. (24) can be combined as follows:

$$T_\alpha(t^\alpha v(t)(T_\alpha u)(t)) - t^\alpha u(t)(T_\alpha v)(t)). \quad (25)$$

As the extra terms are presented in Eq. (25), but not in Eq. (24), when the derivatives are expanded out, which are equal and opposite, and hence, they are cancelled. Hence, we get the following:

$$T_\alpha(t^\alpha v(t)(T_\alpha u)(t)) - t^\alpha u(t)(T_\alpha v)(t)) + (a^{2\alpha} - b^{2\alpha})t^\alpha u(t)v(t) = 0.$$

In addition, we α -integrate this over the range of t from 0 to 1, which gives the following:

$$[t^\alpha v(t)(T_\alpha u)(t) - t^\alpha u(t)(T_\alpha v)(t)]_0^1 + (a^{2\alpha} - b^{2\alpha}) \int_0^1 t^\alpha u(t)v(t) \frac{dt}{t^{1-\alpha}} = 0. \quad (26)$$

The integrated term vanishes at the lower limit because $t = 0$, and it also vanishes at the upper limit because $u(1) = v(1) = 0$. Hence, if $a \neq b$, Eq. (26) gives the following:

$$\int_1^0 t^\alpha u(t)v(t) \frac{dt}{t^{1-\alpha}} = 0. \quad (27)$$

which by using Eqs. (20) and (21), the following can be written as follows:

$$\int_0^1 t^\alpha (J_\alpha)_p(at) (J_\alpha)_p(bt) \frac{dt}{t^{1-\alpha}} = 0. \quad (28)$$

This is the desired orthogonality equation. It is good to remember that we require that a and b are distinct zeroes of $(J_\alpha)_p$; hence, both Bessel functions in Eq. (28) vanish at the upper limit.

Now, we consider the case $a = b$. First, Eq. (26) can be rewritten as follows:

$$\begin{aligned} & (a^{2\alpha} - b^{2\alpha}) \int_0^1 t^\alpha (J_\alpha)_p(at) (J_\alpha)_p(bt) \frac{dt}{t^{1-\alpha}} \\ &= b^\alpha (J_\alpha)_p(a) T_\alpha (J_\alpha)_p(b) \\ &- a^\alpha (J_\alpha)_p(b) T_\alpha (J_\alpha)_p(a). \end{aligned} \quad (29)$$

Moreover, if $a = b$, $\int_0^1 t^\alpha (J_\alpha)_p^2(at) \frac{dt}{t^{1-\alpha}}$ is $\frac{0}{0}$ form. To overcome this challenge, let a be a root of the equation $(J_\alpha)_p(t) = 0$, so that $(J_\alpha)_p(a) = 0$, let also $b = a + \varepsilon$. By substituting $(J_\alpha)_p(a) = 0$, $b = a + \varepsilon$, and taking limit $\varepsilon \rightarrow 0$ in Eq. (29), we obtain the following:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^1 t^\alpha (J_\alpha)_p(at) (J_\alpha)_p((a + \varepsilon)t) \frac{dt}{t^{1-\alpha}} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{-a^\alpha (J_\alpha)_p(a + \varepsilon) T_\alpha (J_\alpha)_p(a)}{a^{2\alpha} - (a + \varepsilon)^{2\alpha}}. \end{aligned}$$

It is still $\frac{0}{0}$ form, so by applying L'Hopital's rule on the right-hand side:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_0^1 t^\alpha (J_\alpha)_p(at) (J_\alpha)_p((a + \varepsilon)t) \frac{dt}{t^{1-\alpha}} \\ &= \frac{1}{2\alpha} (T_\alpha (J_\alpha)_p(a))^2. \end{aligned}$$

Finally, if $a = b$, the corresponding integral is given by the following equation:

$$\int_0^1 t^\alpha (J_\alpha)_p^2(at) \frac{dt}{t^{1-\alpha}} = \frac{1}{2\alpha} (T_\alpha (J_\alpha)_p(a))^2. \quad \square \quad (30)$$

Remark 4.1. From Property 3.1 (iii) and (iv) in an earlier study [29], the above equation can be written as follows:

$$\begin{aligned} & \int_0^1 t^\alpha (J_\alpha)_p^2(at) \frac{dt}{t^{1-\alpha}} = \frac{1}{2\alpha} (T_\alpha (J_\alpha)_p(a))^2 \\ &= \frac{\alpha}{2} (J_\alpha)_{p+1}^2(a) \frac{\alpha}{2} = \frac{\alpha}{2} (J_\alpha)_{p-1}^2(a). \end{aligned} \quad (31)$$

Remark 4.2. Hence, we denote the positive zeros of $(J_\alpha)_p(t)$ by λ_n , arranging them as in Eq. (19). Then, on the interval $[0, 1]$, the conformable Bessel functions can be written as follows: $(J_\alpha)_p(a_1 t)$, $(J_\alpha)_p(a_2 t)$, ..., $(J_\alpha)_p(a_n t)$, ..., which constitutes as an orthogonal system with weight function t^α .

5 Conclusion

Novel results on conformable Bessel functions have been successfully obtained in this study. Some essential properties of the Bessel functions of first order involving their conformable derivatives or their zeros have been accomplished. These functions' orthogonality in $[0, 1]$ has been introduced and proven systematically. This study's outcomes are considered as an indication that the obtained results in the sense of conformable derivative coincide with the obtained classical integer order results. Our results can be further extended into applications of the obtained results and numerical analysis can be also studied in future research studies.

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