Research Article

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The role of relaxation and retardation phenomenon of Oldroyd-B fluid flow through Stehfest's and Tzou's algorithms

<https://doi.org/10.1515/nleng-2022-0006>

received September 28, 2021; accepted December 11, 2021

Abstract: Delayed response (delay of the elasticity) and time needed for initial stress can lead to relaxation and retardation phenomenon; this is because of the consistent behavior of viscoelastic fluid on thermodynamic principles. In this context, the aim of this article is to investigate the unsteady, incompressible, and Oldroyd-B viscoelastic fluid under wall slip conditions to know the hidden aspects of relaxation and retardation. The motion of the liquid is assumed over a flat vertical plate which moves through an oscillating velocity. A fractional model is developed by using the modern definition of the non-singular kernel proposed by Caputo and Fabrizio. We have obtained a semi-analytical solution of the non-dimensional model by using the Laplace transformation that satisfies our imposed suitable boundary conditions. We have tackled the Laplace inverse by employing Stehfest's and Tzou's algorithms. The velocity is enhanced by decreasing the estimations of relaxation time λ as well as slip parameter, and the temperature is also increasing for a considerable measure of the fractional factor. The effects of different fractional and physical parameters are plotted using Mathcad software based on the relaxation and retardation phenomenon of Oldroyd-B viscoelastic fluid.

Keywords: viscoelastic fluid, slip condition, fractional model, Laplace transform, non-singular kernel, semi-analytical solution

1 Introduction

In recent years, non-Newtonian fluids have become more important due to their applications in the industrial and engineering fields. Non-Newtonian fluids includes paint, suspension, colloidal solutions, specific oil, exotic lubricants, clay coatings, cosmetic products, and polymer solutions. There is not a single constitutive demonstration that can foresee all the notable highlights of non-Newtonian liquids due to different physical structures of these liquids. We analyzed and worked on the rate type fluid model, called Oldroyd-B fluid. The best subclass of rate type liquid is Maxwell liquid; in any case, this liquid demonstrates as it was depicted in terms of its relaxation time, whereas there is no evidence of its retardation time. Fetecau *et al.* [1] demonstrated Oldroyd-B fluid flow over a plate. This idea gained attention of many researchers. Vieru *et al.* [2] inspected the influences of Oldroyd-B fluid due to a constantly accelerating plate. Chang *et al.* [3] investigated the Walters-B viscoelastic flow at wall suction. They examined the numerical results of convective heat transport of fluid flow at the wall and gained the most important results. Hayat *et al.* [4] discussed and analyzed the flow of Oldroyd-B fluid in a porous channel. Azeem Khan *et al.* [5] highlighted the Oldroyd-B nanomaterial fluid flow effects due to stretching sheets. Awan *et al.* [6–9] examined the flow of non-Newtonian liquids by varying shear stress in different circumstances. In recent days, similar studies have been carried out in various circumstances but few researchers have developed interest in analyzing the non-Newtonian fluid's effects on the stretching surface due to various assumptions, see latest attempts [10–15] and references therein.

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Boundary layer flow is the most significant application in routine life. The liquids used in technologies and industries do not follow Newton's law of viscosity, for example, greases, shampoo, food, yogurt, ketchup, and polymer melts. These fluids revealed the complicated relationship between the rate of strain and shear stress. The boundary layer flow and heat transfer examination of these liquids on a persistently moving surface have a wide range of applications in building and mechanical forms, for example, fabrication of plastic sheets, polymeric sheets, artificial fibers, plastic froth preparing, the expulsion of polymer sheet from a pass on, warm materials voyaging between a bolster roll, and so on. Soundalgekar [16] pioneered and analyzed the fluid flow at an infinite oscillating plate in the presence of impacts of free convection. Mass transfer of fluid flow at the oscillating vertical plate in the presence of the free convection impacts has been investigated by Soundalgekar and Akolkar [17]. Hocking [18] investigated the waving flow in the oscillating vertical plate. Chang and Lin [19] analyzed the reverse flow at the oscillating channel. Recently, a few investigators have analyzed the flow over oscillatory sheet, see references [20–26].

From the last three decades, fractional differential conditions have picked up significance and ubiquity, primarily because of their exhibited applications in various material science and designing fields. Numerous significant phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, material science, likelihood and measurements, electrochemistry of erosion, concoction physical science, and sign preparing are all depicted by fractional differential equations [27–30]. Consequently, special consideration has been given to discover solutions of fractional differential equations. When all is said in done, it is hard to get an exact answer. The fractional studies on viscoelastic fluid [27–30] and numerical and analytical studies on viscoelastic fluid [31–40] can be overviewed.

Motivated by the above discussions, the current study aims to deal with the concept for a fractional derivative on Oldroyd-B fluid over a flat vertical plate that moves to employ an oscillating velocity in its plane. We have calculated the temperature and velocity fields and shear stress with the constant wall and slip condition at the boundary using the Laplace transformation technique. We investigated the Laplace inverse through “Stehfest’s and Tzou’s algorithms.” The inspirations of different fractional as well

as physical parameters are plotted by using Mathcad software.

2 Methods

2.1 Problem formulation

Free convection flow of time-dependent Oldroyd-B fluid over a flat vertical and oscillating plate moving with an oscillating velocity is sketched in Figure 1(a).

Starting at $t = 0$ the fluid and plate have an ambient fluid temperature T_∞ . For the time $t_1 = 0^+$, the plate begins to oscillate with the velocity, $w = R_0 H_1(t_1) \cos(\omega t_1) i$, where, R_0 , $H_1(t_1)$, and i , are unit-step function and oscillating frequency, respectively, and i is the direction of vertical flow. The temperature is variable for the plate which can either be raised or be lowered to T_w . The governing equations of an Oldroyd-B fluid are described through the resulting differential equations [6]:

$$\rho \left(1 + \lambda_2 \frac{\partial}{\partial t_1} \right) \frac{\partial w}{\partial t_1} = \mu \left(1 + \lambda_1 \frac{\partial}{\partial t_1} \right) \frac{\partial^2 w}{\partial y_1^2} + \left(1 + \lambda_2 \frac{\partial}{\partial t_1} \right) \rho \beta g (T_1 - T_\infty), \quad (1)$$

$$\left(1 + \lambda_2 \frac{\partial}{\partial t_1} \right) \tau_1 = \mu \frac{\partial w}{\partial y_1}, \quad (2)$$

$$\rho C_p \frac{\partial T_1}{\partial t_1} = k \left(1 + \frac{16\sigma T_\infty}{3k\tilde{k}} \right) \frac{\partial^2 T_1}{\partial y_1^2}, \quad (3)$$

where w is the velocity in the x -direction and T_1 is the temperature field. The other parameters μ , ρ , β , g , k , and C_p denote viscosity, density, heat transmission factor, acceleration due to gravity, heat conduction, and heat capability at a particular pressure, respectively. Now, we define the Heaviside unit step function $H_1(t_1) = \frac{1}{2}(1 + \text{sign}(t_1))$; boundary conditions are as follows:

$$w(y_1, 0) = 0, T_1(y_1, 0) = T_\infty, \quad (4)$$

$$w(0, t_1) - b_1 \frac{\partial w(y_1, t_1)}{\partial t_1} = R_0 H_1(t_1) \cos(\omega t_1), \quad (5)$$

$$T_1(0, t_1) = T_w, \left. \frac{\partial w(y_1, t_1)}{\partial t_1} \right|_{t_1=0}, \quad (6)$$

$$w(\infty, y_1) = 0, T_1(\infty, t_1) = T_\infty. \quad (7)$$

Figure 1: (a) Flow analysis of oscillating vertical plate. (b) The velocity for the variants of dimensionless factor Gr at different times.

The non-dimensional parameters are as follows:

$$\left. \begin{aligned} w^* &= \frac{w}{W_0}, y_1^* = \frac{y_1 W_0}{\nu}, t_1^* = \frac{W_0^2 t_1}{\nu}, b_1^* = \frac{b_1 W_0}{\nu}, \\ \omega^* &= \frac{\omega}{W_0^2}, \tau_1^* = \frac{\tau_1}{\mu W_0^2}, \theta_1 = \frac{T_1 - T_\infty}{T_z - T_\infty}, \lambda^* = \frac{\lambda_2 W_0^2}{\nu}, \\ \text{Pr} &= \frac{\mu C_p}{k}, \text{Pr}_{\text{eff}} = \frac{\text{Pr}}{1 + \text{Nr}}, \text{Nr} = \frac{16 \sigma T_\infty^3}{3 k \tilde{k}}, \\ \text{Gr} &= \frac{\beta g (T_z - T_\infty)}{W_0^3}, \lambda_r^* = \frac{\lambda_r W_0^2}{\nu}. \end{aligned} \right\} \quad (8)$$

In Eqs. (1)–(7), we obtain it by dropping star notation:

$$\left(1 + \lambda_2 \frac{\partial}{\partial t_1}\right) \frac{\partial w}{\partial t_1} = \left(1 + \lambda_r \frac{\partial}{\partial t_1}\right) \frac{\partial^2 w}{\partial y_1^2} + \left(1 + \lambda_2 \frac{\partial}{\partial t_1}\right) \text{Gr} \theta_1, \quad (9)$$

$$\left(1 + \lambda_2 \frac{\partial}{\partial t_1}\right) \tau_1 = \frac{\partial w}{\partial y_1}, \quad (10)$$

$$\text{Pr} \frac{\partial \theta_1}{\partial t_1} = (1 + \text{Nr}) \frac{\partial^2 \theta_1}{\partial y_1^2}, \quad (11)$$

where b_1 is the slip factor, Gr, Nr, Pr are the parameters, and the boundary conditions become

$$w(y_1, 0) = 0, \theta_1(y_1, 0) = 0, \quad (12)$$

$$w(0, t_1) - b_1 \frac{\partial w(0, t_1)}{\partial y_1} = R_0 H_1(t_1) \cos(\omega t), \quad (13)$$

$$\theta_1(0, t_1) = 1, \left. \frac{\partial w(y_1, t_1)}{\partial t_1} \right|_{t_1=0} = 0, \quad (14)$$

$$w(y_1, t_1) = 0, \theta_1(y_1, t_1) = 0, \text{ as } y_1 \rightarrow \infty. \quad (15)$$

In this work, we use Caputo-Fabrizio derivative (CFD) of order $\alpha_1 \in (0, 1)$.

$$\begin{aligned} & (1 + \lambda_2 D_{t_1}^{\alpha_1}) \frac{\partial w(y_1, t_1)}{\partial t_1} \\ &= (1 + \lambda_r D_{t_1}^{\alpha_1}) \frac{\partial^2 w(y_1, t_1)}{\partial y_1^2} + \text{Gr} (1 + \lambda_2 D_{t_1}^{\alpha_1}) \theta_1(y_1, t_1), \end{aligned} \quad (16)$$

$$(1 + \lambda_2 D_{t_1}^{\alpha_1}) \tau_1(y_1, t_1) = \frac{\partial w(y_1, t_1)}{\partial y_1}, \quad (17)$$

$$\text{Pr}_{\text{eff}} D_{t_1}^{\alpha_1} \theta_1(y_1, t_1) = \frac{\partial^2 \theta_1(y_1, t_1)}{\partial y_1^2}, \quad (18)$$

where CFD is defined as:

$$D_{t_1}^{\alpha_1} w(y_1, t_1) = \frac{1}{1 - \alpha_1} \int_0^{t_1} e^{\left(\frac{-\alpha_1(t_1 - \tau_1)}{1 - \alpha_1}\right)} \dot{w}(y_1, \tau_1) d\tau_1; \quad 0 < \alpha_1 < 1, \quad (19)$$

$$\mathcal{L}\{D_{t_1}^{\alpha_1} w(y_1, t_1)\} = \frac{q_1 \mathcal{L}\{w(y_1, t_1)\} - w(y_1, 0)}{(1 - \alpha_1)q_1 + \alpha_1}. \quad (20)$$

2.2 Solution to the problem

We use the transformation, namely Laplace transform that is defined in the following procedure.

2.2.1 Temperature field

Using the Laplace transformation to Eqs. (18), (14)₁, (15)₂, and using the initial condition (12)₂ with Eq. (20), we obtain

$$\frac{y_1 \text{Pr}_{\text{eff}} q_1 \bar{\theta}_1(y_1, q_1)}{q_1 + y_1 \alpha_1} = \frac{\partial^2 \bar{\theta}_1(y_1, q_1)}{\partial y_1^2}, \quad (21)$$

$$\text{where } y_1 = \frac{1}{1 - \alpha_1}.$$

$$\bar{\theta}_1(0, q_1) = \frac{1}{q_1}, \quad \bar{\theta}_1(y_1, q_1) \rightarrow 0; \quad \text{as } y_1 \rightarrow \infty. \quad (22)$$

The solution of Eq. (21) subjected to conditions Eq. (22) is

$$\begin{aligned} \bar{\theta}_1(y_1, q_1) &= \frac{1}{q_1} \exp\left(-\frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{q_1 + \alpha_1 y_1}}\right) y_1 \\ &= \bar{\theta}_1(y_1, q_1; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1). \end{aligned} \quad (23)$$

Eq. (23) can be written as

$$\bar{\theta}_1(y_1, q_1; a, b) = \frac{1}{q_1} \exp\left(-\frac{\sqrt{aq_1}}{\sqrt{q_1 + b}}\right) y_1, \quad (24)$$

using the following formula:

$$\begin{aligned} \psi_1(y, q; a, b, c) &= \mathcal{L}^{-1}\{y, q; a, b, c\} \\ &= \exp\left(ct - \frac{\sqrt{aq}}{\sqrt{c+b}}\right) - 1 \\ &\quad - \frac{2a}{\pi} \int_0^\infty \frac{\sin(yx)}{x(ac + (c+b)x^2)} \exp\left(-\frac{btx^2}{(a+x^2)}\right) dx. \end{aligned}$$

We obtain the inverse Laplace transform of Eq. (24)

as

$$\theta_1(y_1, t_1; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1) = 1 - \frac{2\text{Pr}_{\text{eff}} y_1}{\pi} \int_0^\infty \frac{\sin(y_1 x)}{x(\text{Pr}_{\text{eff}} y_1 + x^2)} \exp\left(-\frac{\alpha_1 y_1 t x^2}{\text{Pr}_{\text{eff}} y_1 + x^2}\right) dx, \quad 0 < \alpha_1 < 1 \quad (25)$$

and

$$\mathcal{L}^{-1}\{\bar{\theta}_1(y, q; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1)\} = \theta_1(y, t; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1). \quad (26)$$

For ordinary case, put $\alpha_1 = 1$.

The expression for temperature equivalent to the ordinary case is found based on the property of the CFD, namely,

$$\begin{aligned} \theta_1(y_1, t_1) &= \lim_{\alpha_1 \rightarrow 0} \bar{\theta}_1(y_1, t_1; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1) \\ &= \lim_{y_1 \rightarrow \infty} \bar{\theta}_1(y_1, t_1; \text{Pr}_{\text{eff}} y_1, \alpha_1 y_1) \\ &= 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin(yx)}{x} \exp\left(-\frac{tx^2}{\text{Pr}_{\text{eff}}}\right) dx, \end{aligned} \quad (27)$$

using the formula

$$\int_0^\infty \frac{\sin(yx)}{x} \exp(-ax^2) dx = \frac{\pi}{2} \text{erf}\left(\frac{b}{2\sqrt{a}}\right), \quad (28)$$

we obtain

$$\theta_1(y_1, t_1) = 1 - \text{erf}\left(\frac{y_1 \sqrt{\text{Pr}}}{2\sqrt{t_1}}\right) = \text{erfc}\left(\frac{y_1 \sqrt{\text{Pr}}}{2\sqrt{t_1}}\right), \quad (29)$$

where $\text{erf}(x)$ is the error and $\text{erfc}(x)$ is the error function of complementary.

2.2.2 Velocity distribution

Applying the Laplace transform to Eqs. (16), (13), (15)₁, and using the initial condition (12)₁, (14)₂ using Eq. (20), we obtain the following transformed problem:

$$\begin{aligned} &\left(1 + \frac{\lambda_2 q_1}{(1 - \alpha_1)q_1 + \alpha_1}\right) q \bar{w}(y_1, q_1) \\ &= \left(1 + \frac{\lambda_1 q_1}{(1 - \alpha_1)q_1 + \alpha_1}\right) \frac{\partial^2 \bar{w}(y_1, q_1)}{\partial y_1^2} \\ &\quad + \frac{\text{Gr}\left(1 + \frac{\lambda_2 q_1}{(1 - \alpha_1)q_1 + \alpha_1}\right)}{q_1} \\ &\quad \times \exp\left(-\frac{\sqrt{\text{Pr}_{\text{eff}}}}{\sqrt{q_1 + \alpha_1 y_1}}\right) y_1, \end{aligned} \quad (30)$$

with the boundary conditions

$$\bar{w}(0, q_1) - b_1 \frac{\partial \bar{w}(0, q_1)}{\partial y_1} = \frac{q_1}{q_1^2 + \omega^2}, \quad (31)$$

$$\bar{w}(y_1, q_1) \rightarrow 0, \text{ as } y_1 \rightarrow \infty.$$

The solution of Eq. (30), along with the boundary conditions in Eq. (31), is

$$\begin{aligned} \bar{w}(y_1, q_1) = & \frac{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)} + b\sqrt{(1 - \alpha_1)q_1^2 + q_1\alpha_1 + \lambda_2 q_1^2}} \times \frac{q_1}{q_1^2 + \omega^2} \\ & \times \exp\left(-y_1 \frac{\sqrt{q_1(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_2 q_1^2)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)}}\right) \\ & + \left(\frac{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)} + b\sqrt{(1 - \alpha_1)q_1^2 + q_1\alpha_1 + \lambda_2 q_1^2}} \right) \exp\left(-y_1 \frac{\sqrt{q_1(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_2 q_1^2)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)}}\right) \\ & \times \left[\frac{\text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)(q_1 + \alpha_1 y_1)}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)} \right. \\ & \quad \left. + \frac{b\sqrt{\text{Pr}_{\text{eff}} \gamma_1 q_1}(q_1 + \alpha_1 y_1) \text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)} \right] \\ & - \frac{\text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)(q_1 + \alpha_1 y_1) \exp\left(-\frac{\sqrt{\text{Pr}_{\text{eff}} \gamma_1 q_1}}{\sqrt{q_1 + \alpha_1 y_1}}\right) y_1}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)}. \end{aligned} \quad (32)$$

2.3 Shear stress

The shear stress is attained by using the following relation:

$$\bar{\tau}_1(y_1, q_1) = \frac{q_1(1 - \alpha_1) + \alpha_1}{q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1} \left(\frac{\partial \bar{w}(y_1, q_1)}{\partial y_1} \right),$$

we obtain

3 Results and discussion

We analyzed the time-dependent Oldroyd-B fluid flow at the oscillatory vertical plate under the slip effects. The results of the dimensionless system are obtained through an analytical technique such as Laplace transform, its inversion, and semi-analytical solution of the shear stress, velocity, and temperature. The numerical scheme of

$$\begin{aligned} \bar{\tau}_1(y_1, q_1) = & \frac{q_1(1 - \alpha_1) + \alpha_1}{(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)} \times \frac{q_1}{q_1^2 + \omega^2} \times \left(\frac{\sqrt{(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_2 q_1^2)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)} + b\sqrt{(1 - \alpha_1)q_1^2 + q_1\alpha_1 + \lambda_2 q_1^2}} \right) \\ & \times \exp\left(-y_1 \frac{\sqrt{q_1(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_2 q_1^2)}}{\sqrt{\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1}}\right) - \left(\frac{\sqrt{(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_2 q_1^2)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)} + b\sqrt{(1 - \alpha_1)q_1^2 + q_1\alpha_1 + \lambda_2 q_1^2}} \right) \\ & \times \exp\left(-y_1 \frac{\sqrt{q_1(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_2 q_1^2)}}{\sqrt{\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1}}\right) \\ & + \frac{\text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)(q_1 + \alpha_1 y_1)}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)} \\ & + \left(\frac{\sqrt{(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_2 q_1^2)}}{\sqrt{(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1)} + b\sqrt{(1 - \alpha_1)q_1^2 + q_1\alpha_1 + \lambda_2 q_1^2}} \right) \\ & \times \frac{b\sqrt{\text{Pr}_{\text{eff}} \gamma_1 q_1}(q_1 + \alpha_1 y_1) \text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)} \end{aligned} \quad (33)$$

$$\times \exp\left(-y_1 \frac{\sqrt{q_1(\alpha_1 + (1 - \alpha_1)q_1 + \lambda_2 q_1^2)}}{\sqrt{\alpha_1 + (1 - \alpha_1)q_1 + \lambda_r q_1}}\right) + \frac{\text{Gr}(q_1(1 - \alpha_1) + \alpha_1 + \lambda_2 q_1)(q_1 + \alpha_1 y_1)}{(\text{Pr}_{\text{eff}} \gamma_1 q_1^2)(\alpha_1 q_1 + (1 - \alpha_1)q_1^2 + \lambda_r q_1^2) - (q_1^2 + \alpha_1 q_1 y_1)(q_1^3(1 - \alpha_1) + \alpha_1 q_1^2 + \lambda_2 q_1^3)}$$

$$\times \left(\frac{\sqrt{\text{Pr}_{\text{eff}} \gamma_1 q_1}}{\sqrt{q_1 + \alpha_1 y_1}}\right) \times \exp\left(-\frac{\sqrt{\text{Pr}_{\text{eff}} \gamma_1 q_1}}{\sqrt{q_1 + \alpha_1 y_1}}\right) y_1.$$

Figure 2: The velocity profile with different dimensionless Prandtl numbers at different times.

Figure 3: The velocity for the variants of a fractional factor for different times.

Figure 4: The velocity profile for the variants of relaxation time at different times.

Stehfest's and Tzou's algorithms is used to achieve shear stress and velocity results. The effects of different fractional as well as physical parameters are plotted by using Mathcad software. The results of the graphs have clearly described that the velocity declines with the upsurge in the Prandtl number and the Grashof number. Velocity increases with the time magnitude. The results are displayed in Figures 1(b) and 2, respectively. Figure 3 depicts the changes in fractional parameter at several values of the time. The influence of relaxation time on fluid velocity is represented in Figure 4; it is pointed out that the fluid velocity increases as we increase the time. The velocity behavior for deviation of slip parameter is shown in Figure 5 with different estimations of time. It is pointed out from Figures 6 and 7 that for different values of the

fractional parameter and Prandtl number, the behavior is the same for the temperature field at different time estimations. Also, shear stress increases with the Grashof number but declines with the growth of retardation. These results are shown in Figures 8–10. Equivalence relation is described in Figure 11 as well as in Table 1, to check for the validation of the results by using the numerical inversion of Laplace transforms, namely, Stehfest's and Tzou's algorithms for velocity and shear stress. Table 1 shows the values of two different algorithms, namely, Stehfest's and Tzou's, to find the inverse Laplace for the velocity field and shear stress solution. Numerically, it is clear that the values taken from these two different algorithms are approximated to each other.

Figure 5: The velocity profile for the variation of the slip parameter at different times.

Figure 6: The fluid temperature for the variants of the fractional factor.

Figure 7: The fluid temperature for the variation of the Prandtl number.

Figure 8: The shear stress for the variation of the Grashof number.

Figure 9: The shear stress for various estimations of retardation time.

Figure 10: The shear stress profile for variation of slip parameter.

Figure 11: The validation of obtained results for velocity and stress function, with $\alpha_1 = 0.2$ and $t_1 = 0.5$.

Table 1: Comparison of velocity and shear stress with two different “numerical algorithms, *i.e.*, Stehfest’s and Tzou’s” at $\alpha_1 = 0.2$ and $t_1 = 0.5$

y_1	Velocity (Stehfest’s)	Velocity (Tzou’s)	y_1	Shear stress (Stehfest’s)	Shear stress (Tzou’s)
0	0.947	0.935	0	0.602	0.599
0.2	0.852	0.853	0.2	0.505	0.505
0.4	0.743	0.743	0.4	0.413	0.413
0.6	0.629	0.629	0.6	0.329	0.33
0.8	0.52	0.52	0.8	0.258	0.258
1.0	0.421	0.421	1.0	0.199	0.199
1.2	0.335	0.335	1.2	0.151	0.151
1.4	0.263	0.264	1.4	0.115	0.115
1.6	0.206	0.206	1.6	0.087	0.087
1.8	0.16	0.16	1.8	0.067	0.067
2.0	0.126	0.126	2.0	0.052	0.052

4 Conclusion

The analytical and semi-analytical solutions for fractional Oldroyd-B fluid with wall slip conditions are found by the latest and technical approach of the fractional derivatives such as the Caputo and the Fabrizio. Numerical inversion procedures named “Stehfest’s and Tzou’s” have been used for finding the inverse Laplace transformation for the non-dimensional problem. The following points are concluded from the present work.

- Velocity is increased by decreasing the estimations of relaxation time λ as well as slip parameter.
- The magnitude of velocity is increased for significant estimations of the time.
- Temperature is increasing for a large estimation of the fractional factor.
- Shear stress and Grashof number have the same behavior of enhancing.

- Shear stress is decreased as the retardation time increased.
- Our solutions achieved by the use of inversion algorithms, *i.e.*, Stehfest's and Tzou's, are equivalent.

Acknowledgements: All the authors are grateful to the reviewers for their fruitful suggestions. They are also thankful to Professor Muhammad Azam, Department of English, Govt College, Jauharabad, for improving the paper grammatically.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors declare no conflict of interest.

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