Research Article

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Construction of optical solitons of Radhakrishnan–Kundu–Lakshmanan equation in birefringent fibers

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Abstract: In this article, we are attracted to discover the multiple-optical solitons in birefringent fibers for Radhakrishnan–Kundu–Lakshmanan equation (RKLE) by applying the Sardar-subequation method (SSM) and the new extended hyperbolic function method (EHFM). We construct the solutions in the form of exponential, trigonometric, and hyperbolic functions solitons solutions like mixed complex solitons and multiple-optical solitons solutions. In addition, singular periodic wave solutions are constructed, and the restraint conditions for the presence of soliton solutions are also defined. Moreover, the physical interpretation of the obtained solutions is disclosed in forms of 3D and 2D plots for different suitable parameters. The attained results indicate that the implemented computational scheme is straight, proficient, and brief and can be applied in more complex phenomena with the associate of representative computations. We have obtained several sorts of new solutions.

Keywords: Sardar-subequation method, Radhakrishnan–Kundu–Lakshmanan equation, extended hyperbolic function method, optical solitons

1 Introduction

NLPDEs are utilized as a governing model to illuminate the complexity of the physical phenomena in science and engineering arenas like fluid mechanics, circuit analysis, solid state physics, plasma physics, chemical physics, geochemistry, quantum field theory, geology, optical fiber, and other physical sciences. It is essential to estimate the solutions of the leading NLPDEs to understand the behavior of complex physical phenomena. Usually, the solutions of the NLPDEs are classified into three sorts: analytic solutions, exact solutions, and numerical solutions. Seeking the exact solutions of NLPDEs has the prominence to deliberate the solidity of numerical solutions. Exact solutions to NLPDEs play a vital role in non-linear science, meanwhile they can offer much physical info and extra understanding of the physical structures of the problem and thus lead to advance applications. Wave phenomena in dissipation, dispersion, reaction, convection, and diffusion are very much important.

Furthermore, many typical researchers are paying more consideration to discover and increase the optical transmission contexts using optical fibers as a substitute of birefringent fibers. Also, the transmission of soliton through optical fibers directs next-generation technology, but there are some influences in birefringent fibers as appropriate that are used to yield the soliton propagations. So, the study of optical soliton is one of the most exciting and interesting regions of investigation in nonlinear optics. There are famous powerful and computational methods to discover the exact soliton solutions of the DEs [1–30] Thus, in this article we emphasize on creating the exact optical solitons solutions in various structures to the RKL equation [31–35] for birefringent fibers in media Kerr-law nonlinearity without 4WM terms, which is called the basic case of fiber nonlinearity. Many optical fibers that have been rather popular in recent times comply with this law nonlinearity. We apply two efficient computational approaches known as Sardar-subequation method [36] and new extended hyperbolic function
method [37–39] to discover the various sorts of soliton solutions. The restriction relationships are also witnessed during the mathematical analysis.

2 Governing model

The RKLE with Kerr law nonlinearity is given as follows [31–35]:

\[ i\Omega x + a\Omega_{xx} + B\Omega^3 = \alpha\Omega(|\Omega|^2\Omega)_x - i\delta\Omega_{xxx}. \] (1)

Here, \( \Omega(x, t) \) is the dependent variable of complex-valued wave profile that have \( x \) and \( t \) two independent variables that denote spatio-temporal component, respectively. On splitting Eq. (1) for birefringent fibers into two components without 4WM [40], we attain as

\[ i\gamma_Y + a_Y Y_{xx} + (\beta_Y |Y|^2 + \gamma_Y |\Delta|^2)Y = i\lambda_Y (|\Omega|^2\Omega)_x + \theta_Y (|\Delta|^2\Delta)_x - i\delta_1 Y_{xxx}, \] (2)

\[ i\Delta_Y + a_{\Delta} \Delta_{xx} + (\beta_{\Delta} |\Delta|^2 + \gamma_\Delta |Y|^2)\Delta = i\lambda_{\Delta} (|\Lambda|^2\Lambda)_x + \theta_\Delta (|\Delta|^2\Delta)_x - i\delta_2 \Delta_{xxx}. \] (3)

Here, \( Y(x, t) \) and \( \Delta(x, t) \) denote the complex functions and the wave profiles, while Eqs. (2) and (3) denote the governing model for soliton transmission over birefringent fibers without 4WM. In the aforementioned joined system, the coefficients \( \beta_{\gamma} \) and \( \gamma_\gamma \) correspond to self-phase modulation (SPM) and the cross-phase modulation (XPM) effect is rejected.

This article is structured as follows. The RKLE model is described in Section 2. The description of SSM is discussed in Section 3. Section 4 presents the solutions of RKLE via SSM. In Section 4, the new EHFM is described, and Section 5 consists of the solutions of RKLE by EHFM. In Section 6, results and discussion and finally conclusion of this article is given in Section 7.

2.1 Mathematical preliminaries

To solve the couple of Eqs. (2) and (3), we assume the traveling wave transformation (twt) by:

\[ Y(x, t) = \chi_Y(\eta)e^{i\eta}, \] (4)

\[ \Delta(x, t) = \chi_\Delta(\eta)e^{i\eta}, \] (5)

where

\[ \eta = B(x - vt), \quad \text{and} \quad Y = -kx + wt + \theta_0. \] (6)

Then, we get (4)–(6) into Eqs. (2) and (3), we get real and imaginary parts, respectively, of the form

\[ B^2(\alpha_j + 3k\delta_j)\chi''_j - (\omega + \alpha_jk^2 + \delta_jk^3)\chi_j + (\beta_j - kl_j)\chi_j^3 + y\chi_jk^2 - k\theta_j\chi_j^2 = 0, \] (7)

\[ B^2\delta_j\chi''_j - B(\nu + 2\alpha_j + 3k^2\delta_j)\chi_j - 3k\alpha_jk^2 - \theta_j\chi_j^2 = 0. \] (8)

Using the balancing rule gives (7) and (8) by using of \( \chi_j = \chi_j \) are as follows:

\[ B^2(\alpha_j + 3k\delta_j)\chi''_j - (\omega + \alpha_jk^2 + \delta_jk^3)\chi_j + (\beta_j - kl_j + \gamma_j - k\theta_j)\chi_j^3 = 0, \] (9)

\[ B^2\delta\chi''_j - (\nu + 2\alpha_j + 3k^2\delta_j)\chi_j - (\lambda_j + \theta_j)\chi_j^3 = 0. \] (10)

We obtained

\[ \omega = \frac{8k^3\delta_j^2 + 8k^3\alpha_j\delta_j + 3k^2\delta_j + 2k^2\delta_j^2 + v\alpha_j}{\delta_j}, \] (11)

\[ \beta_j = -\frac{2k\delta_j\alpha_j + 2k\delta_j^2 + \alpha_j\delta_j + \alpha_j\delta_j}{\delta_j}, \] (12)

which yields from

\[ \frac{a_j + 3k\delta_j}{\delta_j} = \frac{w + \alpha_jk^2 + \delta_jk^3}{\nu + 2\alpha_j + 3k^2\delta_j} = \frac{\beta_j - kl_j + \gamma_j - k\theta_j}{\lambda_j + \theta_j}. \] (13)

3 The Sardar-subequation method [36]

We consider

\[ H(z, \zeta, z_\xi, \zeta_\xi, z_\zeta, \zeta_\zeta, \ldots) = 0. \] (14)

We use next twt and obtain (14)

\[ z(x, t) = W(\eta)e^{i\gamma}, \quad \eta = B(x - ct), \]

\[ Y = -kx + wt + \theta_0, \] (15)

where \( v \) is nonzero constant.

Utilizing (15) into (14) yields

\[ P(\chi, \chi', \chi'', \ldots) = 0, \] (16)

Consider (16) has the following solution:

\[ \chi(\eta) = \sum_{i=0}^{N} F_i \phi_i(\eta), \] (17)

where \( F_i(0 \leq i \leq N) \) and
\[(\Phi'(\eta))^2 = \varepsilon + a\Phi^2(\eta) + \Phi^4(\eta),\]  

where \(\varepsilon\) and \(a\) are constants and (18) presents the solution as follows:

**Case 1:** Let \(a > 0\) and \(\varepsilon = 0\). Then, we obtain
\[\Phi_1(\eta) = \pm \left(\frac{a}{2}\right) \tanh_{pq}(\sqrt{\alpha} \eta),\]
\[\Phi_2(\eta) = \pm \left(\frac{a}{2}\right) \coth_{pq}(\sqrt{\alpha} \eta),\]
where \(\tanh_{pq}(\eta) = \frac{2 \eta}{\cosh_{pq}(\eta)}\).

**Case 2:** Let \(a < 0\) and \(\varepsilon = 0\). Thus, we obtain
\[\Phi_3(\eta) = \pm \frac{\sqrt{a}}{2} \tanh_{pq}(\sqrt{2a} \eta) \pm \sqrt{pq} \text{sech}_{pq}(\sqrt{-2a} \eta),\]
\[\Phi_4(\eta) = \pm \frac{\sqrt{a}}{2} \coth_{pq}(\sqrt{2a} \eta) \pm \sqrt{pq} \text{csch}_{pq}(\sqrt{-2a} \eta),\]
where \(\tanh_{pq}(\eta) = \frac{2 \eta}{\cosh_{pq}(\eta)}\).

**Case 3:** Let \(a < 0\) and \(\varepsilon = \frac{a}{\eta}\). Therefore, we obtain
\[\Phi_5(\eta) = \pm \frac{\sqrt{a}}{2} \tanh_{pq}(\sqrt{\frac{a}{8}} \eta),\]
\[\Phi_6(\eta) = \pm \frac{\sqrt{a}}{2} \coth_{pq}(\sqrt{\frac{a}{8}} \eta),\]
where \(\tanh_{pq}(\eta) = \frac{2 \eta}{\cosh_{pq}(\eta)}\).

**Case 4:** Let \(a > 0\) and \(\varepsilon = \frac{a}{\eta}\). Therefore, we reach
\[\Phi_7(\eta) = \pm \frac{\sqrt{a}}{2} \tanh_{pq}(\sqrt{\frac{a}{2}} \eta),\]
\[\Phi_8(\eta) = \pm \frac{\sqrt{a}}{2} \coth_{pq}(\sqrt{\frac{a}{2}} \eta),\]
\[\Phi_9(\eta) = \pm \frac{\sqrt{a}}{2} \tanh_{pq}(\sqrt{\frac{a}{8}} \eta) \pm \sqrt{pq} \text{sech}_{pq}(\sqrt{\frac{a}{8}} \eta),\]
\[\Phi_{10}(\eta) = \pm \frac{\sqrt{a}}{2} \coth_{pq}(\sqrt{\frac{a}{8}} \eta) \pm \sqrt{pq} \text{csch}_{pq}(\sqrt{\frac{a}{8}} \eta),\]
where \(\tanh_{pq}(\eta) = \frac{2 \eta}{\cosh_{pq}(\eta)}\).

3.1 Application of the SSM

We consider
\[\chi_i(\eta) = F_0 + F_i\Phi(\eta),\]  

where \(F_0\) and \(F_i\) are constants. Then, we get
\[F_0 = 0, \quad F_i = \frac{\sqrt{-2a_i - 6\delta_i}}{\sqrt{\alpha_i + 3\delta_i}} \left(\frac{\eta_j}{\alpha_i + 3\delta_i}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\},\]
\[B = \frac{\sqrt{\alpha_i + 3\delta_i}}{\sqrt{\alpha_i + 3\delta_i}}.\]

**Case 1:** Let \(a > 0\) and \(\varepsilon = 0\). Therefore, we reach
\[Y_{11}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{sech}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda},\]
\[Y_{11}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{csch}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda}.\]

**Case 2:** When \(a < 0\) and \(\varepsilon = 0\), then
\[Y_{21}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{sech}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda},\]
\[Y_{21}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{csch}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda}.\]

**Case 3:** When \(a > 0\) and \(\varepsilon = \frac{a}{\eta}\), then
\[Y_{31}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{sech}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda},\]
\[Y_{31}(x, t) = \frac{\sqrt{-2a_1 - 6\delta_1}}{\sqrt{\alpha_1 + 3\delta_1}} \left(\frac{\eta_j}{\alpha_1 + 3\delta_1}\right) \left\{\pm \sqrt{\eta_j - \eta_k - \eta_l}\right\} \times (\pm \sqrt{-pq} \text{csch}_{pq}(\sqrt{\alpha} B(x - vt))) \times e^{i\lambda}.
\[ Y^{1}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (29)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (30)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (31)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (32)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (33)

\[ \Lambda^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (34)

\[ \Lambda^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (35)

\[ \Lambda^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (36)

\[ \Lambda^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}. \] (37)

**Case 4:** When \( a > 0 \) and \( \epsilon = \frac{a^2}{\alpha} \), then

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (38)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (39)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (40)

\[ Y^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}. \] (41)

\[ \Lambda^{1}_{\pm}(x,t) = \frac{\sqrt{-2a_1 - 6b_1 \sqrt{w + k^2a_1 + k^2b_1}}}{\sqrt{a}} \left( \pm \frac{a}{2} \coth_{pq}(\sqrt{-2a}B(x-\nu t)) \right) \times e^{iy}, \] (42)
\[ \Lambda_{1,10}(x,t) = \frac{\sqrt{-2\alpha_2 - 6k\delta_2} \sqrt{w + k^2\alpha_2 + k^2\delta_2}}{\sqrt{\alpha_2 + 3k\delta_2}} \left( \pm \frac{a}{2} \tan_p \left( \frac{a}{2} B(x-\nu t) \right) \right) e^{\nu t}, \]

\[ \Lambda_{1,11}(x,t) = \frac{\sqrt{-2\alpha_2 - 6k\delta_2} \sqrt{w + k^2\alpha_2 + k^2\delta_2}}{\sqrt{\alpha_2 + 3k\delta_2}} \left( \pm \frac{a}{2} \cot_p \left( \frac{a}{2} B(x-\nu t) \right) \right) e^{\nu t}, \]

\[ \Lambda_{1,12}(x,t) = \frac{\sqrt{-2\alpha_2 - 6k\delta_2} \sqrt{w + k^2\alpha_2 + k^2\delta_2}}{\sqrt{\alpha_2 + 3k\delta_2}} \left( \pm \frac{a}{2} \tan_p \left( \frac{a}{2} B(x-\nu t) \right) \right) e^{\nu t}, \]

\[ \Lambda_{1,13}(x,t) = \frac{\sqrt{-2\alpha_2 - 6k\delta_2} \sqrt{w + k^2\alpha_2 + k^2\delta_2}}{\sqrt{\alpha_2 + 3k\delta_2}} \left( \pm \frac{a}{2} \cot_p \left( \frac{a}{2} B(x-\nu t) \right) \right) e^{\nu t}, \]

Then, we obtain

Set 1: Let \( \tau > 0 \) and \( \Theta > 0 \),

\[ \Phi(\eta) = \frac{-\tau}{\Theta} \csc(\sqrt{\tau}(\eta + \eta_0)). \]

Set 2: When \( \tau < 0 \) and \( \Theta > 0 \),

\[ \Phi(\eta) = \frac{-\tau}{\Theta} \sec(\sqrt{-\tau}(\eta + \eta_0)). \]

Set 3: When \( \tau > 0 \) and \( \Theta < 0 \),

\[ \Phi(\eta) = \frac{-\tau}{\Theta} \csc(\sqrt{\tau}(\eta + \eta_0)). \]

Set 4: When \( \tau < 0 \) and \( \Theta > 0 \),

\[ \Phi(\eta) = \frac{-\tau}{\Theta} \sec(\sqrt{-\tau}(\eta + \eta_0)). \]

Set 5: When \( \tau > 0 \) and \( \Theta = 0 \),

\[ \Phi(\eta) = \exp(\sqrt{\tau}(\eta + \eta_0)). \]

Set 6: When \( \tau < 0 \) and \( \Theta = 0 \),

\[ \Phi(\eta) = \cos(\sqrt{-\tau}(\eta + \eta_0)) + i \sin(\sqrt{-\tau}(\eta + \eta_0)). \]

Set 7: When \( \tau = 0 \) and \( \Theta > 0 \),

\[ \Phi(\eta) = \pm \frac{1}{\sqrt{\Theta}(\eta + \eta_0)}. \]

Set 8: When \( \tau = 0 \) and \( \Theta < 0 \),

\[ \Phi(\eta) = \pm \frac{1}{\sqrt{-\Theta}(\eta + \eta_0)}. \]

**Form 2:** We consider

\[ \frac{d\Phi}{d\eta} = \tau + \Theta \Phi^2, \tau, \Theta \in R. \]

Substituting (48) into (9) along with (58) with value of \( N \), makes a set of equations with the values of \( F_i \) \( (i = 1, 2, 3, \ldots N) \).

Set 1: Let \( \tau \Theta > 0 \),

\[ \Phi(\eta) = \text{sgn}(\tau) \frac{\tau}{\Theta} \tan(\sqrt{\tau\Theta}(\eta + \eta_0)). \]

Set 2: When \( \tau \Theta > 0 \),

\[ \Phi(\eta) = -\text{sgn}(\tau) \frac{\tau}{\Theta} \cot(\sqrt{-\tau\Theta}(\eta + \eta_0)). \]

Set 3: When \( \tau \Theta < 0 \),

\[ \Phi(\eta) = \text{sgn}(\tau) \frac{\tau}{\Theta} \tanh(\sqrt{-\tau\Theta}(\eta + \eta_0)). \]

Set 4: When \( \tau \Theta < 0 \),

\[ \Phi(\eta) = \text{sgn}(\tau) \frac{\tau}{\Theta} \coth(\sqrt{-\tau\Theta}(\eta + \eta_0)). \]

4 New extended hyperbolic function method

The phases of the new EHFM [28–30] are taken as follows:

**Form 1:** Let PDE as given in (1) with the wave transformation in (4)–(6) using wave transformation ODE is obtained as in (9). We consider

\[ \chi(\eta) = \sum_{i=0}^{N} F_i \Phi(\eta) \]

and

\[ \frac{d\Phi}{d\eta} = \Phi \tau + \Theta \Phi^2, \tau, \Theta \in R. \]
Set 5: When $\tau = 0$ and $\Theta > 0$,
$$\Phi(\eta) = -\frac{1}{\Theta(\eta + \eta_0)}.$$  \hfill (63)

Set 6: When $\tau \in \mathbb{R}$ and $\Theta = 0$,
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (64)

5 Application of the new EHFM

Form 1: We apply the new EHFM for the solutions of RKLE. Applying the balancing method in (9) gives $N = 1$, so (48) yields
$$\chi(\eta) = F_0 + F_1 \Phi(\eta),$$  \hfill (65)
where $F_0$ and $F_2$ are constants. Substituting (65) into (9) and equating the coefficients polynomials of $\Phi(\eta)$ to zero, we reach a set of equations in $F_0$, $F_1$, $\tau$, $\Theta$ and $B$.

On resolving the set of equations, we attain
$$F_0 = 0, \quad F_1 = \frac{B \sqrt{\Theta} \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + \gamma_1 - k\delta_1 - k\delta_2}}, \quad \Theta = \Theta.$$  \hfill (66)

Set 1: When $\tau > 0$ and $\Theta > 0$,
$$\Lambda_1(x, t) = 0,$$
$$\Phi(\eta) = -\frac{1}{\Theta(\eta + \eta_0)},$$  \hfill (70)

Set 3: When $\tau > 0$ and $\Theta < 0$,
$$\Lambda_3(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (71)

Set 4: When $\tau < 0$ and $\Theta < 0$,
$$\Lambda_4(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (72)

Set 5: When $\tau = 0$ and $\Theta > 0$,
$$\Lambda_5(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (73)

Set 2: When $\tau < 0$ and $\Theta > 0$,
$$\Lambda_2(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (74)

Set 6: When $\tau = 0$ and $\Theta < 0$,
$$\Lambda_6(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (75)

Set 7: When $\tau > 0$ and $\Theta < 0$,
$$\Lambda_7(x, t) = 0,$$
$$\Phi(\eta) = \tau(\eta + \eta_0).$$  \hfill (76)
Set 6: When \( \tau = 0 \) and \( \Theta < 0 \),

\[
\begin{align*}
\gamma_6(x, t) &= \frac{B \sqrt{\Theta} \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \frac{t}{\sqrt{-\Theta(\eta + \eta_0)}} \right) \times e^{iY}, \\
\lambda_6(x, t) &= \frac{B \sqrt{\Theta} \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \frac{t}{\sqrt{-\Theta(\eta + \eta_0)}} \right) \times e^{iY},
\end{align*}
\]

(77)

(78)

where \( \eta = B(x - vt) \), and \( Y = -kx + wt + \theta_0 \).

Form 2:

Operating balancing rule in (9), gives \( N = 1 \), so (48) reduces to

\[
\chi_j(\eta) = F_0 + F_1\Phi(\eta),
\]

(79)

where \( F_0 \) and \( F_1 \) are constants. Replacing (79) into (9) and equating the coefficients of polynomials of \( \Phi(\eta) \) to zero, we obtain a set of equations in \( F_0, F_1, \tau, \Theta, \) and \( k \).

On resolving the set of equations, we reach

\[
\begin{align*}
F_0 &= 0, \quad F_1 = \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}}, \\
\tau &= \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta(a_1 + 3k\delta_1)}, \quad \Theta = \Theta.
\end{align*}
\]

(80)

Set 1: When \( \tau \Theta > 0 \),

\[
\begin{align*}
\gamma_1(x, t) &= \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \Gamma \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta^2(a_1 + 3k\delta_1)} \right) + \tan \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta(a_1 + 3k\delta_1)}(\eta + \eta_0) \right) \right) \times e^{iY}, \\
\lambda_1(x, t) &= \frac{B\Theta \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \Gamma \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta^2(a_2 + 3k\delta_2)} \right) + \tan \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta(a_2 + 3k\delta_2)}(\eta + \eta_0) \right) \right) \times e^{iY}.
\end{align*}
\]

(81)

(82)

Set 2: When \( \tau \Theta > 0 \),

\[
\begin{align*}
\gamma_2(x, t) &= \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \Gamma \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta^2(a_1 + 3k\delta_1)} \right) + \cot \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta(a_1 + 3k\delta_1)}(\eta + \eta_0) \right) \right) \times e^{iY}, \\
\lambda_2(x, t) &= \frac{B\Theta \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \Gamma \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta^2(a_2 + 3k\delta_2)} \right) + \cot \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta(a_2 + 3k\delta_2)}(\eta + \eta_0) \right) \right) \times e^{iY}.
\end{align*}
\]

(83)

(84)

Set 3: When \( \tau \Theta < 0 \),

\[
\begin{align*}
\gamma_3(x, t) &= \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \Gamma \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta^2(a_1 + 3k\delta_1)} \right) - \cot \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta(a_1 + 3k\delta_1)}(\eta + \eta_0) \right) \right) \times e^{iY}, \\
\lambda_3(x, t) &= \frac{B\Theta \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \Gamma \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta^2(a_2 + 3k\delta_2)} \right) - \cot \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta(a_2 + 3k\delta_2)}(\eta + \eta_0) \right) \right) \times e^{iY}.
\end{align*}
\]

(85)

(86)

Set 4: When \( \tau \Theta < 0 \),

\[
\begin{align*}
\gamma_4(x, t) &= \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \Gamma \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta^2(a_1 + 3k\delta_1)} \right) \right) \times e^{iY}, \\
\lambda_4(x, t) &= \frac{B\Theta \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \Gamma \left( \frac{w + k^2a_2 + k^3\delta_2}{2B^2\Theta^2(a_2 + 3k\delta_2)} \right) \right) \times e^{iY}.
\end{align*}
\]

(87)

(88)

Set 5: When \( \tau = 0 \) and \( \Theta > 0 \),

\[
\begin{align*}
\gamma_5(x, t) &= \frac{B\Theta \sqrt{-2a_1 - 6k\delta_1}}{\sqrt{\beta_1 + y_1 - k\theta_1 - k\lambda_1}} \left( \frac{1}{\Theta(\eta + \eta_0)} \right) \times e^{iY}, \\
\lambda_5(x, t) &= \frac{B\Theta \sqrt{-2a_2 - 6k\delta_2}}{\sqrt{\beta_2 + y_2 - k\theta_2 - k\lambda_2}} \left( \frac{1}{\Theta(\eta + \eta_0)} \right) \times e^{iY}.
\end{align*}
\]

(89)

(90)

where \( \Gamma = \text{sgn} \left( \frac{w + k^2a_1 + k^3\delta_1}{2B^2\Theta(a_1 + 3k\delta_1)} \right) \), \( \eta = B(x - vt) \), and \( Y = -kx + wt + \theta_0 \).
6 Results and discussion

We demonstrate the simulations by the following figures. For graphical representation for (1), the absolute behavior of (20) with the suitable values of parameters $a = 1.75, v = 0.65, p = 0.98, q = 0.95, A = 2.7, B = 0.6, \alpha_1 = 0.75, \beta_1 = 0.5k = 0.8, \theta_1 = 5, \lambda_5 = 1.6, w = 2, \gamma_1 = 1.4$, (A-1) 2D representation of (20) with $t = 1$.

Figure 1: (A) 3D graph of (20) with $a = 1.75, v = 0.65, p = 0.98, q = 0.95, A = 2.7, B = 0.6, \alpha_1 = 0.75, \beta_1 = 0.5k = 0.8, \theta_1 = 5, \lambda_5 = 1.6, w = 2, \gamma_1 = 1.4$. (B-1) 2D representation of (20) with $t = 1$.

Figure 2: (B) 3D graph of (28) with $-a = 1.7, v = 0.66, p = 0.98, q = 0.95, A = 2.9, B = 0.5, \alpha_1 = 0.77, \beta_1 = 0.6k = 0.9, \theta_1 = 4, \lambda_5 = 1.7$, $w = 2.1, \gamma_1 = 1.3$. (B-1) 2D representation of (28) with $t = 1$.
Figure 3: (C) 3D graph of (34) with $-a = 1.65$, $v = 0.6$, $p = 0.98$, $q = 0.95$, $A = 2.5$, $B = 0.9$, $a_1 = 0.72$, $\beta_1 = 0.7k = 0.9$, $\theta_1 = 4.1$, $\lambda_1 = 1.7$, $w = 2.2$, $\gamma_1 = 1.2$. (C-1) 2D representation of (34) with $t = 1$.

Figure 4: (D) 3D graph of (41) with $a = 1.55$, $v = 1.55$, $p = 0.98$, $q = 0.95$, $A = 2.3$, $B = 0.6$, $a_1 = 0.76$, $\beta_1 = 0.6k = 0.9$, $\theta_1 = 3$, $\lambda_1 = 1.5$, $w = 2$, $\gamma_1 = 1.5$. (D-1) 2D representation of (41) with $t = 1$.

Figure 5: (E) 3D graph of (67) with $a_1 = 1.75$, $v = 0.65$, $p = 0.98$, $q = 0.95$, $A = 2.7$, $B = 0.6$, $a_1 = 0.75$, $\beta_1 = 0.5k = 0.8$, $\theta_1 = 5$, $\lambda_1 = 1.6$, $w = 2$, $\gamma_1 = 1.1$. (E-1) 2D representation of (66) with $t = 1$. 
Figure 6: (F) 3D graph of (71) with \( a = 0.3, v = -0.65, p = 0.98, q = 0.95, A = 3, B = 0.6, \alpha_i = 0.75, \beta_i = 0.7, \theta_1 = -0.4, \lambda_i = 1.7, w = 2.4, \gamma_1 = 1.7 \). (F-1) 2D representation of (71) with \( t = 1 \).

Figure 7: (G) 3D graph of (81) with \( a = 0.75, v = 0.65, p = 0.98, q = 0.95, A = 2.7, B = 2.6, \alpha_i = 1.75, \beta_i = 0.5, \theta_1 = 5, \lambda_i = 1.6, w = 1.22, \gamma_1 = 1.2 \). (G-1) 2D representation of (81) with \( t = 1 \).

Figure 8: (H) 3D graph of (83) with \( a = 2.25, v = 1.21, p = 0.98, q = 0.95, A = 2.9, B = 0.9, \alpha_i = 1.75, \beta_i = 0.5, \theta_1 = 0.82, \theta_i = 3.3, \lambda_i = 1.8, w = 2, \gamma_1 = 1.3 \). (H-1) 2D representation of (83) with \( t = 1 \).
\[ \theta_1 = -0.4, \lambda_1 = 1.7, w = 2.4, \gamma_1 = 1.2, \text{ and } t = 1 \] are shown in Figure 6, the absolute behavior of (81) with the suitable values of parameters \( a = 0.75, \nu = 0.65, \rho = 0.98, \eta = 0.95, A = 2.7, B = 2.6, \alpha_1 = 1.75, \beta_1 = 0.55k = 2.8, \theta_1 = 5, \lambda_1 = 1.6, w = 1.22, \gamma_1 = 1.1, \text{ and } t = 1 \) are shown in Figure 7, the absolute behavior of (83) with the suitable values of parameters \( a = 2.25, \nu = 1.21, p = 0.98, q = 0.95, A = 2.9, B = 0.9, \alpha_1 = 1.75, \beta_1 = 0.55k = 0.82, \theta_1 = 3.3, \lambda_1 = 1.8, w = 2, \gamma_1 = 1.2, \text{ and } t = 1 \) are shown in Figure 8.

### 7 Conclusion

In this article, we positively explained the dynamics behavior of optical solitons of the RKL equation without 4WM in birefringent fibers using the Sardar-subequation method and the extended hyperbolic function method. These methods are very strong for solving NLEEs. The acquired solutions are in the form of hyperbolic, trigonometric, and exponential functions along with exact optical solitons solutions. As well, we achieved the singular, bright, periodic, and dark wave solutions. These novel families of solutions exposed the authority, efficiency, and productivity of these methods. This article provides the application of optical fibers. Furthermore, these novel solutions have many applications in physics and other branches of physical sciences. We will report these results in future research studies.

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