Research Article

Feras Yousef*, Billel Semmar, and Kamal Al Nasr

Dynamics and simulations of discretized Caputo-conformable fractional-order Lotka–Volterra models

https://doi.org/10.1515/nleng-2022-0013
received January 22, 2022; accepted March 14, 2022

Abstract: In this article, a prey–predator system is considered in Caputo-conformable fractional-order derivatives. First, a discretization process, making use of the piecewise-constant approximation, is performed to secure discrete-time versions of the two fractional-order systems. Local dynamic behaviors of the two discretized fractional-order systems are investigated. Numerical simulations are executed to assert the outcome of the current work. Finally, a discussion is conducted to compare the impacts of the Caputo and conformable fractional derivatives on the discretized model.

Keywords: Caputo fractional derivative, conformable fractional derivative, prey–predator system, discretization, stability, bifurcations, chaos

1 Introduction

Dynamic analysis is widely used in engineering and science. Understanding the dynamics of prey–predator interactions is one of the most primary operations that shape the framework and function of ecological societies [1,2]. To describe the dynamic behavior between prey and predator, models can be a very useful and powerful tool. The oldest and most celebrated prey–predator model is the Lotka–Volterra model, independently introduced by Lotka [3] and Volterra [4]. This model is formulated in ordinary differential equations such that any small change of the model will lead to a qualitatively different type of behavior. The most important feature of this model is that it lumps prey birth and death rates into one logistic growth term, and it assumes that the predator birth rate remains a linear function of their per capita consumption [5].

Predation is normally quantified in terms of the functional and numerical responses, which are the effects of predation on the prey and predator growth rates, respectively [6]. The classical model of Lotka–Volterra under the linear functional response of predator reads as follows:

\[
\begin{align*}
\frac{dN}{dT} &= N(a - bN - cR), \\
\frac{dR}{dT} &= R(-d + eN),
\end{align*}
\]

where \( N \) and \( R \) are functions of time \( T \), representing the prey and predator population densities, respectively [7]. The parameters \( a, b, c, d, \) and \( e \) are considered as positive constants, representing the maximum birth rate per capita of prey species, the strength of intraspecific competition of the prey species, the strength of intraspecific between prey and predators, the death rate per capita of predator species, and the efficacy of converting ingested prey to new predators, respectively.

Recently, it has been shown that many mathematical models can be effectively reformulated via noninteger-order differential equations owing to the unsuitability and ability of the integer-order differential equations in formulating many phenomena [8]. The noninteger-order derivatives are oftentimes said to be fractional-order derivatives or artlessly fractional derivatives. It is well-known that the derivatives with integer order are in local nature, unlike the derivatives with fractional order [9]. It has been evinced that the nonlocality merit of the fractional derivatives is employed to make them a suitable tool for describing the dynamic behaviors of many life phenomena and dynamical models that have inherited the properties of memory [10,11]. Due to the memory effect,
the fractional models integrate all previous information from the past that makes it easier for the researchers to predict and translate the biological models more accurately [12]. Consequently, many existing differential equations describing various phenomena in engineering and science have been recasted by means of fractional derivatives [13–25], and their solutions and dynamic behaviors continue to be of widespread interest today in many other disciplines [26–35].

In the present work, we nondimensionalize the system (1) by performing the following change of variables.

\[
x(t) = \frac{ea}{d} N(T), \quad y(t) = \frac{c}{d} R(T), \quad t = \frac{cd}{a} T.
\]

Hence, we are lead to work with the following nondimensionalized system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= rx(t) - px^2(t) - qx(t)y(t), \\
\frac{dy(t)}{dt} &= x(t)y(t) - qy(t),
\end{align*}
\]

where \( r = \frac{a}{cd}, p = \frac{b}{ce}, \) and \( q = \frac{c}{d}. \)

Thus, the following fractional-order system is obtained by changing the first-order time derivatives by the derivatives of fractional order.

\[
\begin{align*}
D^\alpha x(t) &= rx(t) - px^2(t) - qx(t)y(t), \\
D^\alpha y(t) &= x(t)y(t) - qy(t),
\end{align*}
\]

where \( 0 < \alpha < 1 \) is the fractional-order derivative parameter and \( t > 0 \). It is easily verified that the equilibrium points of system (3) are the trivial state \( E_0 = (0, 0) \), the axial state \( E_1 = \left( \frac{c}{p}, 0 \right) \), and the steady state of coexistence \( E_2 = \left( q, \frac{r - pq}{q} \right) \) for \( r > pq \).

Several studies revealed that the discrete-time system exhibits much fruitful dynamic behaviors, such as bifurcations and chaos, than those of its continuous-time system counterpart. Consequently, in the sequel, we will explore the dynamic behaviors of the discretized fractional-order prey–predator system that includes both Caputo and conformable fractional derivatives.

2 The dynamics of the discretized Caputo fractional-order prey–predator system

In this section, we consider the Caputo fractional-order version of model (3) as follows:

\[
\begin{align*}
D^\alpha x(t) &= rx(t) - px^2(t) - qx(t)y(t), \\
D^\alpha y(t) &= x(t)y(t) - qy(t),
\end{align*}
\]

where \( D^\alpha \) is the fractional derivative of Caputo type and defined as follows:

\[
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha < 1.
\]

Our first aim is to discretize the model (4) making use of the piecewise-constant approximation [36] as follows:

\[
\begin{align*}
D^\alpha x(t) &= \frac{y\left( \frac{t}{h} \right)}{h} \left( x\left( \frac{t}{h} \right) - x\left( \frac{t}{h} - h \right) \right) \left( y\left( \frac{t}{h} \right) - y\left( \frac{t}{h} - h \right) \right), \\
D^\alpha y(t) &= x\left( \frac{t}{h} \right) \left( y\left( \frac{t}{h} \right) - y\left( \frac{t}{h} - h \right) \right) - q + y\left( \frac{t}{h} \right),
\end{align*}
\]

where \( \left[ \frac{t}{h} \right] \) denotes the integer part of \( t \in [nh, (n+1)h) \), \( n = 0, 1, \ldots \), and \( h > 0 \) is a discretization parameter.

The nth iterative solution of system (6) is given by:

\[
\begin{align*}
x_{n+1}(t) &= x_n(nh) + \frac{(t - nh)^\alpha}{a\Gamma(\alpha)}(x_n(nh)(r - px_n(nh)) - qy_n(nh)), \\
y_{n+1}(t) &= y_n(nh) + \frac{(t - nh)^\alpha}{a\Gamma(\alpha)}(y_n(nh)(-q + x_n(nh))).
\end{align*}
\]

If we let \( t \to (n+1)h \) in system (7), then we will achieve the discretized version of the model (4):

\[
\begin{align*}
x_{n+1} &= x_n + \frac{h^\alpha}{a\Gamma(\alpha)}(x_n(r - px_n - qy_n)), \\
y_{n+1} &= y_n + \frac{h^\alpha}{a\Gamma(\alpha)}(y_n(-q + x_n)).
\end{align*}
\]

We next investigate the local stability and bifurcation to the points of equilibrium for system (8).

**Theorem 2.1.** The point of equilibrium \( E_0 = (0, 0) \) is a

(i) *Saddle point when* \( 0 < h < \sqrt[2\alpha\Gamma(\alpha)]{\frac{2a\Gamma(\alpha)}{q}}; \)

(ii) *Source when* \( h > \sqrt[2\alpha\Gamma(\alpha)]{\frac{2a\Gamma(\alpha)}{q}}; \)

(iii) *Non-hyperbolic when* \( h = \sqrt[2\alpha\Gamma(\alpha)]{\frac{2a\Gamma(\alpha)}{q}}. \)

**Proof.** The Jacobian matrix computed at the point of equilibrium \( E_0 \) for the linearization of system (8) is given by

\[
J(E_0) = \begin{pmatrix}
1 + \frac{h^\alpha}{a\Gamma(\alpha)} & 0 \\
0 & 1 - \frac{h^\alpha}{a\Gamma(\alpha)}
\end{pmatrix}
\]
and has eigenvalues $\lambda_1 = 1 + \frac{h^a}{a\Gamma(a)} r$ and $\lambda_2 = 1 - \frac{h^a}{a\Gamma(a)} q$.
Since $r > 0$, then $|\lambda_1| > 1$. Now, since $q > 0$, we have the following cases:

(i) If $0 < h < \sqrt{\frac{2a\Gamma(a)}{q}}$, then $|\lambda_1| < 1$ and $E_0$ is a saddle point.

(ii) If $h > \sqrt{\frac{2a\Gamma(a)}{q}}$, then $|\lambda_1| > 1$ and $E_0$ is a source.

(iii) If $h = \sqrt{\frac{2a\Gamma(a)}{q}}$, then $|\lambda_1| = 1$ and $E_0$ is a nonhyperbolic.

**Theorem 2.2.** The point of equilibrium $E_1 = \left( \frac{r}{p}, 0 \right)$ is a

(i) Sink when $\left( q - \frac{2a\Gamma(a)}{h^a} \right) p < r < \min\left( \frac{2a\Gamma(a)}{h^a}, pq \right)$;

(ii) Source when $r < \left( q - \frac{2a\Gamma(a)}{h^a} \right) p$ and $r > \max\left( \frac{2a\Gamma(a)}{h^a}, pq \right)$;

(iii) Non-hyperbolic when $r = \left( q - \frac{2a\Gamma(a)}{h^a} \right) p$ or $r = \frac{2a\Gamma(a)}{h^a}$ or $r = pq$;

(iv) Saddle point for the other values of parameters except those values in (i)–(iii).

**Proof.** The Jacobian matrix computed at the point of equilibrium $E_1$ of the linearization of system (8) is given by

$$J(E_1) = \begin{pmatrix} 1 - \frac{h^a}{a\Gamma(a)} r & - \frac{h^a}{a\Gamma(a)} q \frac{r}{p} \\ 0 & 1 + \frac{h^a}{a\Gamma(a)} \frac{r}{p} - q \end{pmatrix}$$

and has eigenvalues $\lambda_1 = 1 + \frac{h^a}{a\Gamma(a)} r$ and $\lambda_2 = 1 + \frac{h^a}{a\Gamma(a)} \left( \frac{r}{p} - q \right)$. Now, we have the following cases:

(i) If $\left( q - \frac{2a\Gamma(a)}{h^a} \right) p < r < \min\left( \frac{2a\Gamma(a)}{h^a}, pq \right)$, then $|\lambda_1| < 1$ and $|\lambda_2| < 1$. Hence, $E_1$ is a sink.

(ii) If $r < \left( q - \frac{2a\Gamma(a)}{h^a} \right) p$ and $r > \max\left( \frac{2a\Gamma(a)}{h^a}, pq \right)$, then $|\lambda_1| > 1$ and $|\lambda_2| > 1$. Hence, $E_1$ is a saddle point.

(iii) If $r = \left( q - \frac{2a\Gamma(a)}{h^a} \right) p$ or $r = \frac{2a\Gamma(a)}{h^a}$ or $r = pq$, then $|\lambda_1| = 1$ or $|\lambda_2| = 1$. Hence, $E_1$ is a nonhyperbolic.

(iv) For other values of parameters except those values in (i)–(iii), we have $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$. Hence, $E_1$ is a saddle point.

**Theorem 2.3.** The positive point of equilibrium $E_2 = \left( q, \frac{r - pq}{q} \right)$ is local asymptotically stable if and only if

$$\max\left( \frac{2ph^a}{\alpha^{(a)}}, \frac{4}{a} \right) + pq < r < \left( a\Gamma(a) \frac{h^a}{h^a} + q \right).$$

**Proof.** The Jacobian matrix computed at the point of equilibrium $E_2$ for the linearization of system (8) is given by

$$J(E_2) = \begin{pmatrix} 1 - \frac{pqh^a}{\alpha^{(a)}} - \frac{q^2h^a}{\alpha^{(a)}} & q(r - pq) \\ -\frac{hq^a}{\alpha^{(a)}} & 1 \end{pmatrix}.$$ 

Next, the trace and determinant of $J(E_2)$ are computed as follows:

$$\text{Tr}(J(E_2)) = 2 - \frac{pqh^a}{\alpha^{(a)}} \quad \text{and} \quad \text{Det}(J(E_2)) = 1 - \frac{pqh^a}{\alpha^{(a)}} + q(r - pq)\left( \frac{h^a}{\alpha^{(a)}} \right)^2.$$ \hfill (9)

According to the Jury conditions [37], both eigenvalues of the Jacobian matrix $J(E_2)$ have modulus less than 1 if

$$\max\left( \frac{2ph^a}{\alpha^{(a)}}, \frac{4}{a} \right) + pq, \frac{pq}{q} < r < \left( a\Gamma(a) \frac{h^a}{h^a} + q \right). \quad \Box$$

The following result is regarding the bifurcation of system (8). We refer the reader to [38] for more details about the major forms of bifurcations in two-dimensional maps.

**Theorem 2.4.** The positive point of equilibrium $E_2 = \left( q, \frac{r - pq}{q} \right)$ forfeits its stability through a

(i) Transcritical bifurcation if $r = pq$;

(ii) Flip bifurcation if $r = \frac{2ph^a}{\alpha^{(a)}}, \frac{4}{a} + pq$;

(iii) Neimark–Sacker bifurcation if $r = p\left( \frac{a\Gamma(a)}{h^a} + q \right)$.

**Proof.** From Eq. (9), we have

(i) $\text{Tr}(J(E_2)) = \text{Det}(J(E_2)) = 1$ when $r = pq$. Hence, the point of equilibrium $E_2$ forfeits its stability through a transcritical bifurcation when $r = pq$.

(ii) $-\text{Tr}(J(E_2)) = \text{Det}(J(E_2)) = 1$ when $r = \frac{2ph^a}{\alpha^{(a)}}, \frac{4}{a} + pq$.

Hence, the point of equilibrium $E_2$ forfeits its stability through a flip bifurcation when $r = \frac{2ph^a}{\alpha^{(a)}}, \frac{4}{a} + pq$.

(iii) $\text{Det}(J(E_2)) = 1$ when $r = p\left( \frac{a\Gamma(a)}{h^a} + q \right)$. Hence, the point of equilibrium $E_2$ forfeits its stability through a Neimark–Sacker bifurcation if $r = p\left( \frac{a\Gamma(a)}{h^a} + q \right)$. \hfill \Box
3 The dynamics of the discretized conformable fractional-order prey–predator system

Several authors prefer to study the fractional-order models by using Caputo derivatives because the definition of Caputo-fractional derivatives shows the property of linearity and yields zero the output for a fractional derivative to a constant. However, this definition fails to show the usual properties of the derivative such as the chain rule, product rule, and quotient rule. More recently, Khalil et al. [39] proposed a novel local derivative operator recognized as a conformable fractional derivative, which is essentially an extension of the well-known limit-based derivative, and this definition satisfies all the usual properties of the integer-order derivatives [40]. We refer the reader to ref. [41] for a comparative study of all definitions of fractional-order derivatives.

The conformable fractional-order version of system (3) is presented as follows:

\[
\begin{align*}
D^\alpha_{t_0} x(t) &= -rx(t) - px^\alpha(t) - qx(t)y(t), \\
D^\alpha_{t_0} y(t) &= -x(t)y(t) - gy(t),
\end{align*}
\]

where \(D^\alpha_{t_0}\) is the fractional derivative of conformable-type and defined for a function \(f : [a, \infty) \to \mathbb{R}, a \geq 0:\)

**Figure 1:** Stable dynamical behavior of system (8) subject to the initial condition \((x(0), y(0)) = (0.25, 0.2)\) for the parameters \(p = 0.5, q = 1, h = 0.15, \) and \(r = 1:\) (a) \(\alpha = 0.95\), (b) \(\alpha = 0.75\), (c) \(\alpha = 0.6\), and (d) \(\alpha = 0.5\).
From the aforementioned definition, it has been evinced in ref. [40] the following necessary fact:

\[ D_{\alpha}^a f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t - a)^{1-a}) - f(t)}{\epsilon}, \quad 0 < \alpha < 1. \]  

(11)

From the aforementioned definition, it has been evinced in ref. [40] the following necessary fact:

\[ D_{\alpha}^a f(t) = (t - a)^{1-a} f'(t). \]  

(12)

In the following, we will adopt piecewise-constant approximation to discretize the model (10).

\[
\begin{align*}
D_{\alpha}^a x(t) &= x(t) \left( r - px(t) - qy \left( \left\lfloor \frac{t}{h} \right\rfloor h \right) \right), \\
D_{\alpha}^a y(t) &= y(t) \left( -q + x \left( \left\lfloor \frac{t}{h} \right\rfloor h \right) \right). 
\end{align*}
\]  

(13)

Applying the rule in Eq. (12) to the first equation in the system (13), for \( h > 0 \) and \( t \in [nh, (n + 1)h), n = 0, 1, \ldots, \) gives the following Bernoulli differential equation:

\[ (t - nh)^{1-a} \frac{dx(t)}{dt} + (qy(nh) - r)x(t) = -px^2(t). \]  

(14)

We obtain by simplifying this equation

\[ -x'(t) + \frac{(r - qy(nh))}{x(t)(t - nh)^{1-a}} = \frac{p}{(t - nh)^{1-a}}. \]  

(15)

Figure 2: Stable dynamical behavior of system (22) subject to the initial condition \((x(0), y(0)) = (0.25, 0.2)\) for the parameters \( p = 0.5, q = 1, h = 0.15, \) and \( r = 1: \) (a) \( \alpha = 0.95, \) (b) \( \alpha = 0.75, \) (c) \( \alpha = 0.6, \) and (d) \( \alpha = 0.5. \)
Multiplying Eq. (15) by $e^{(r-qy(nh))\frac{(t-nh)^\alpha}{t}}$, we have
\[
\frac{d}{dt}\left(\frac{1}{x(t)} e^{(r-qy(nh))\frac{(t-nh)^\alpha}{t}}\right) = \frac{p}{(t-nh)^{1-\alpha}} e^{(r-qy(nh))\frac{(t-nh)^\alpha}{t}}, \quad t \in [nh, (n+1)h).
\] (16)

Integrating with respect to $t$ on $[nh, t)$ both sides of Eq. (16), we obtain
\[
\frac{1}{x(t)} e^{(r-qy(nh))\frac{(t-nh)^\alpha}{t}} = \frac{1}{x(nh)} + \frac{p}{r-qy(nh)} \left[ e^{(r-qy(nh))\frac{(t-nh)^\alpha}{t}} - 1 \right].
\] (17)

Ultimately, let $t \rightarrow (n+1)h$ in Eq. (17) and replacing $x(nh)$ with $x_n$ yields
\[
x_{n+1} = \frac{x_n(r-qy_n)}{px_n + (r-qy_n-px_n)e^{(r-qy_n)\frac{(t-nh)^\alpha}{t}}}.
\] (18)

In a similar way, from the second equation in the model (13), we obtain
\[
\frac{dy(t)}{y(t)} = \frac{(x(nh)-q)}{(t-nh)^{1-\alpha}} dt.
\] (19)

Integrating with respect to $t$ on $[nh, t)$ both sides of Eq. (19), we obtain
\[
\ln y(t) - \ln y(nh) = \frac{(x(nh)-q)(t-nh)^\alpha}{\alpha}, \quad t \in [nh, (n+1)h).
\] (20)
For $t \to (n + 1)h$ in Eq. (20) and replacing $y(nh)$ with $y_n$ provides
\begin{equation}
    y_{n+1} = y_n e^{(x_n - q)y_n}. \tag{21}
\end{equation}

Consequently, the discretized version of the model (13) is derived as follows:
\begin{equation}
    \begin{aligned}
    x_{n+1} &= \frac{x_n(r - qy_n)}{px_n + (r - qy_n - px_n)e^{-(r - qy_n)x_n}}, \\
    y_{n+1} &= y_n e^{(x_n - q)y_n}. \tag{22}
    \end{aligned}
\end{equation}

We next investigate the local stability and bifurcation to the points of equilibrium for system (22).

**Theorem 3.1.** The point of equilibrium $E_0 = (0, 0)$ is a saddle point.

**Proof.** The Jacobian matrix computed at the point of equilibrium $E_0$ for the linearization of system (22) is given by
\begin{equation}
    J(E_0) = \begin{pmatrix}
    e^{x_0} & 0 \\
    0 & e^{y_0}
    \end{pmatrix}
\end{equation}
and has eigenvalues $\lambda_1 = e^{x_0}$ and $\lambda_2 = e^{-y_0}$. Thus, $E_0$ is a saddle point since $|\lambda_1| > 1$ and $|\lambda_2| < 1$. \hfill \square
Theorem 3.2. The point of equilibrium $E_1 = \left( \frac{r}{p}, 0 \right)$ is a
(i) Saddle point if $r > pq$;
(ii) Sink if $r < pq$;
(iii) Non-hyperbolic if $r = pq$.

Proof. The Jacobian matrix computed at the point of equilibrium $E_1$ for the linearization of system (22) is given by

$$J(E_1) = \begin{pmatrix}
    e^{r \frac{q^\alpha}{p}} \quad \frac{q}{p} \left( -1 + e^{r \frac{q^\alpha}{p}} \right) \\
    0 & e^{(r-q) \frac{q^\alpha}{p}}
\end{pmatrix},$$

and has eigenvalues $\lambda_1 = e^{r \frac{q^\alpha}{p}}$, which satisfy $|\lambda_1| < 1$, and $\lambda_2 = e^{(r-q) \frac{q^\alpha}{p}}$. Now, we have the following cases:

(i) If $r > pq$, then $|\lambda_1| > 1$ and $E_1$ is a saddle point.
(ii) If $r < pq$, then $|\lambda_1| < 1$ and $E_1$ is a sink.
(iii) If $r = pq$, then $|\lambda_1| = 1$ and $E_1$ is a nonhyperbolic. $\square$

Theorem 3.3. The positive point of equilibrium $E_2 = \left( q, \frac{r-pq}{q} \right)$ is local asymptotically stable if and only if $r < p\left( \frac{q}{h^\alpha} + \frac{q}{h^\alpha} \right)$.

Proof. The Jacobian matrix computed at the point of equilibrium $E_2$ for the linearization of system (22) is given by

$$J(E_2) = \begin{pmatrix}
    q & 1 \\
    0 & \frac{r}{h^\alpha}
\end{pmatrix},$$

and has eigenvalues $\lambda_1 = q$, which satisfy $|\lambda_1| < 1$, and $\lambda_2 = \frac{r}{h^\alpha}$. Now, we have the following cases:

(i) If $r > pq$, then $|\lambda_1| > 1$ and $E_1$ is a saddle point.
(ii) If $r < pq$, then $|\lambda_1| < 1$ and $E_1$ is a sink.
(iii) If $r = pq$, then $|\lambda_1| = 1$ and $E_1$ is a nonhyperbolic. $\square$
The trace and determinant of $J(E_2)$ are computed as follows:

$$
\begin{aligned}
\text{Tr}(J(E_2)) &= 1 + e^{-\frac{rpq\alpha}{p}} \frac{q}{p}(1 + e^{-\frac{rpq\alpha}{p}}) \\
\text{Det}(J(E_2)) &= \frac{pa^e r \cdot h^a(r - pq)(-1 + e^{-\frac{rpq\alpha}{p}})}{p a}.
\end{aligned}
$$

According to the Jury conditions, both eigenvalues of the Jacobian matrix $J(E_2)$ have modulus less than 1 if and only if $r < p\left(\frac{\alpha}{\alpha + q}\right)$.

The following result is regarding the bifurcation for model (22).

**Theorem 3.4.** The positive point of equilibrium $E_2 = \left(q, \frac{rpq\alpha}{q}\right)$ forfeits its stability through a Neimark–Sacker bifurcation if $r = p\left(\frac{\alpha}{\alpha + q}\right)$.

**Figure 7:** Maximum Lyapunov exponents corresponding to: (a) Figure 5(a) and (b) Figure 6(a).

**Figure 8:** Chaotic attractor for the parameters $p = 0.5$, $q = 1$, $\alpha = 0.95$, $h = 0.15$, and $r = 4$: (a) system (8) and (b) system (22).
Proof. From Eq. (23), we have $\text{Det}(J(E_2)) = 1$ when $r = p\left(\frac{\alpha}{h^r} + q\right)$. Hence, the point of equilibrium $E_2$ forfeits its stability through a Neimark–Sacker bifurcation if $r = p\left(\frac{\alpha}{h^r} + q\right)$. □

4 Numerical simulations

Theoretical studies cannot be verified without numerical investigation of the obtained results. In the present study, numerical computations have been accomplished through the use of MATLAB-R2020a software. Let $p = 0.5$ and $q = 1$ be fixed and vary $\alpha$, $h$, and $r$. Suppose that the initial state of systems (8) and (22) is $(0.25, 0.2)$. Figures 1 and 2 demonstrate the local stable dynamic behaviors for the two-dimensional discrete systems (8) and (22), respectively, at the positive point of equilibrium $E_2$ with lessening the fractional-order parameter $\alpha$. We note that lessening the fractional-order parameter $\alpha$ and fixing the discretization parameter $h$ lead to destabilize the two-dimensional discrete systems and chaotic behavior occurs. Figures 3 and 4 show the local stable dynamic behaviors for the two-dimensional discrete systems (8) and (22), respectively, at the positive point of equilibrium $E_2$ with rising the discretization parameter $h$. We note that rising the discretization parameter $h$ along with fixed fractional-
order parameter $\alpha$ leads to destabilize the two-dimensional discrete systems and chaotic behavior occurs.

The bifurcation diagrams in Figures 5 and 6 demonstrate that rising the values of $r$ may destabilize the point of equilibrium $E_2$ through a Neimark–Sacker bifurcation. The maximum Lyapunov exponents corresponding to Figures 5(a) and 6(a) are given in Figure 7. The chaotic attractor for system (8) and system (22) is presented in Figure 8. The chaotic behavior exists when lessening the fractional-order parameter $\alpha$ and rising the discretization parameter $h$.

5 Discussion and concluding remarks

In the present work, fractional-order Lotka–Volterra models for two fractional-order derivatives types are considered. Discretization process by means of the piecewise-constant approximation is applied, and discrete versions of these systems are obtained. The local stability of the points of equilibrium of these discrete systems is investigated. Moreover, the necessary and sufficient asymptotically stable condition for the point of equilibrium $E_2$ of these discrete systems is obtained. Numerical simulations are presented to bolster the analytical results. To further confirm the chaos, we plot the time series of the systems (8) and (22), see Figure 9. It is clear that when the bifurcation parameter $r$ is increasing, it leads to chaotic behavior in the systems. The findings of the current work can be summarized in the following points.

- It can be seen that when lessening the fractional-order parameter $\alpha$ and fixing the discretization parameter $h$ (or rising the discretization parameter $h$ and fixing the fractional-order parameter $\alpha$), the discrete Caputo-conformable model is destabilized and chaotic behavior occurs (Figure 8). Thus, the discretized system is stabilized only for relatively large fractional order $\alpha$ ($\alpha$ tends to one) and for relatively small discretization parameter $h$ ($h$ tends to zero).
- It can be deduced that when the Caputo-derivative acts on the fractional-order Lotka–Volterra model, the discrete Caputo-system exhibits richer dynamic behaviors than the discrete conformable-model.
- It can be observed that the discrete conformable-system forfeits its stability faster than the discrete Caputo-model (Figures 1–6).
- By using bifurcation theory, we showed that the discretized Caputo-system undergoes transcritical, flip, and Neimark–Sacker bifurcation at the positive point of equilibrium $E_2$. While, the discrete conformable-system undergoes only a Neimark–Sacker bifurcation.

On conclusion, these fractional derivatives act to some extent the same function in importing some of the inherited properties of the time fractional to the time-integer Lotka–Volterra prey–predator models.

Funding information: The authors state no funding involved.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors declare that they have no known conflict of interest that could appear to influence the work presented in this article.

References


