Research Article

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Solution of two-dimensional fractional diffusion equation by a novel hybrid D(TQ) method

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Abstract: This work is an experiment to solve the fractional diffusion equation in two dimensions with a novel hybrid method. The method involves an amalgamation of the well-known differential transform method and the differential quadrature method. This work is not about the superiority of one method over the other, instead this is an idea that can be worked upon for possible greatness. Numerical examples are discussed with tables and figures.

Keywords: differential transform method, differential quadrature method, fractional diffusion equation

1 Introduction

The process of diffusion happens on its own, without stirring, shaking, or wafting. The phenomenon of osmosis and respiration happening in the living entities are the most prominent examples of diffusion. Mathematically this process had been modeled as a partial differential equation, and with time, it has evolved into a two-dimensional fractional differential equation, which is given as follows:

\[
\frac{\partial^\rho \phi(x, y, t)}{\partial t^\rho} = a(x, y, t) \frac{\partial^2 \phi(x, y, t)}{\partial x^2} + b(x, y, t) \frac{\partial^2 \phi(x, y, t)}{\partial y^2} + c(x, y, t)
\]

on a finite domain \(0 \leq x \leq B_1, 0 \leq y \leq B_2\), and \(0 < t \leq T\), \(0 < \rho \leq 1\) and \(a(x, y, t)\) and \(b(x, y, t)\) being the non zero diffusion coefficients. The function \(c(x, y, t)\) is used to represent sources and sinks along with the initial condition \(\phi(x, y, 0) = K(x, y)\), \(0 \leq x \leq B_1\), \(0 \leq y \leq B_2\) and the boundary conditions:

\[
\phi(0, y, t) = K_3(y, t), \quad \phi(B_1, y, t) = K_4(y, t), \quad 0 \leq y \leq B_2, \quad 0 < t \leq T,
\]

\[
\phi(x, 0, t) = K_5(x, t), \quad \phi(x, B_2, t) = K_6(x, t), \quad 0 \leq x \leq B_1, \quad 0 < t \leq T,
\]

where \(B_1, B_2, K_1(x, y), K_2(y, t), K_3(y, t), K_4(x, t)\) and \(K_5(x, t)\) are given and \(\frac{\partial^\rho \phi(x, y, t)}{\partial x^\rho}\) is the Caputo’s definition of the time fractional derivative.

This article is arranged as follows: Section 2 presents the brief introduction of some prerequisites, followed by the proposed methodology and the numerical experiments. The results have been presented using tables and graphs.

2 Prerequisites

In this section, some much required topics are discussed in brief. For a fractional derivative, there exist many definitions [1,2], and the most accepted Caputo’s definition is considered in this article. The semi-analytical and numerical methods, viz. differential transform method [3] and differential quadrature method [4], respectively, have been discussed for the two-dimensional variable \(\phi(x, y, t)\).

2.1 Caputo fractional derivative

Let \(\rho\) be a positive real number, \(-\infty < a < \infty\) and \(f\) be a function from \([a, b]\) to the set of real numbers. Then, the \(\rho\) ordered left Caputo derivative [2,5,6] is

\[
D^\rho f(x) = \frac{1}{\Gamma(n - \rho)} \int_a^x (x - s)^{n-\rho-1} f^n(s) ds,
\]

\(n - 1 < \rho < n\),
where \( n \) is an integer and \( \Gamma \) is the Euler Gamma \([7–9]\) function. For \( p = n \), the traditional definition of integer order derivative is obtained.

### 2.2 Differential transform method

In this method, a polynomial series solution \([3,10]\) is obtained by an iterative procedure. For an analytic function of two space and one time variable \( \phi(x, y, t) \), whose all the required derivatives are continuous, the differential transform of \( \phi(x, y, t) \) is given as follows:

\[
\Phi_k(x, y) = \frac{1}{\Gamma(p_k + 1)} [D_{tk}^k \phi(x, y, t)]_{t=t_0}, \quad k = 0, 1, 2, \ldots \tag{5}
\]

The inverse differential transform of \( \Phi_k(x, y) \) is given as follows:

\[
\phi(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(p_k + 1)} [D_{tk}^k \phi(x, y, t)]_{t=t_0}(t - t_0)^{p_k} \tag{6}
\]

Combining (5) and (6),

\[
\phi(x, y, t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(p_k + 1)} [D_{tk}^k \phi(x, y, t)]_{t=t_0}(t - t_0)^{p_k} \tag{7}
\]

### 2.3 Differential quadrature method

For any function \( \Psi = \Psi(x, y, t) \) having its field on rectangle \( 0 \leq x \leq a, 0 \leq y \leq b \), let the rectangle be further divided into a grid by taking \( M \) points on \( x \)-axis and \( N \) points on the \( y \)-axis. Then, the \( p \)-th derivative of the said function at a point \( x = x_i \) along any line \( y = y_j \) is

\[
\frac{\partial^p \Psi}{\partial x^p}(x_i, y_j, t) = \sum_{k=1}^{M} A_{ik}^{(p)} \Psi(x_k, y_j, t); \quad i = 1, 2, \ldots, M, \tag{8}
\]

and the \( q \)-th derivative at a point \( y = y_j \) along the line \( x = x_l \) is given as follows:

\[
\frac{\partial^q \Psi}{\partial y^q}(x_l, y_j, t) = \sum_{k=1}^{N} B_{jk}^{(q)} \Psi(x_l, y_k, t); \quad j = 1, 2, \ldots, N, \tag{9}
\]

where \( p \in \mathbb{Z}^+ \), \( 1 \leq i \leq M \), \( 1 \leq j \leq N \) and \( A_{ik}^{(p)} \) and \( B_{jk}^{(q)} \) are respective weight coefficients \([11–13]\) that give the approximate \( p \)-th and \( q \)-th derivative at the knots.

### 3 Proposed methodology

A hybrid method has an advantage of having the properties of all the methods it is composed of. Here, we propose an amalgamation of the DTM and DQM as follows.

Applying DTM to the Eq. (1), the following expression is obtained, where the \( A, B, C \) are the differential transforms (Table 1) of \( a, b, c \) occurring in the diffusion Eq. (1).

\[
\frac{\Gamma(kp + p + 1)}{\Gamma(kp + 1)} \Phi_{k+1} = \sum_{l=1}^{k} A(l) \frac{\partial^2 \Phi(k - l + 1)}{\partial x^2} + \sum_{l=1}^{k} B(l) \frac{\partial^2 \Phi(k - l + 1)}{\partial y^2} + C(k). \tag{10}
\]

The space differentials can be approximated by using DQM (8) and (9) and Eq. (10) can be written as follows:

\[
\Phi_{k+1}^{ij} = \frac{\Gamma(kp + 1)}{\Gamma(kp + \rho + 1)} \left[ \sum_{l=1}^{k} A(l) \sum_{m=1}^{M} a_{lm} \Phi_{m+1}^{ij-1} + \sum_{l=1}^{k} B(l) \sum_{m=1}^{N} b_{jm} \Phi_{m+1}^{ij-1} + C_{ij} \right]. \tag{11}
\]

Then, by using the inverse DTM, the approximate solution of the Eq. (1) can be written as follows,

\[
\phi(x, y, t) = \Phi_{ij}^{1} + \Phi_{ij}^{2} t^{p} + \Phi_{ij}^{3} t^{2p} + \ldots \tag{12}
\]

### Table 1: Table of differential transforms

<table>
<thead>
<tr>
<th>Relation in ( \phi(x, y, t) ), ( \psi(x, y, t) ) and ( \omega(x, y, t) )</th>
<th>Relation in ( \Phi(x, y), \Psi(x, y) ) and ( \Omega(x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi(x, y, t) = \psi(x, y, t) \pm \omega(x, y, t) )</td>
<td>( \Phi(x, y) = \Psi(x, y) \pm \Omega(x, y) )</td>
</tr>
<tr>
<td>( \phi(x, y, t) = c \psi(x, y, t) )</td>
<td>( \Phi(x, y) = c \Psi(x, y) )</td>
</tr>
<tr>
<td>( \phi(x, y, t) = \psi(x, y, t) \omega(x, y, t) )</td>
<td>( \Phi(x, y) = \psi(x, y, t) \Omega(x, y) )</td>
</tr>
<tr>
<td>( \phi(x, y, t) = f(x, y) \omega(x, y, t) )</td>
<td>( \Phi(x, y) = f(x, y) \Omega(x, y) )</td>
</tr>
<tr>
<td>( \phi(x, y, t) = (x - x_0)^m (y - y_0)^m (t - t_0)^{op} )</td>
<td>( \Phi(x, y) = (x - x_0)^m (y - y_0)^m \Omega(k - n) )</td>
</tr>
<tr>
<td>( \phi(x, y, t) = D_{tk}^p \psi(x, y, t) )</td>
<td>( \Phi(x, y) = \frac{\Gamma(kp + m - 1)}{\Gamma(kp + 1)} \Psi_{k+1}(x, y) )</td>
</tr>
</tbody>
</table>
4 Numerical experiment

To check the methodology, two problems solved in [16] are considered. Throughout this section, to show the precision of the proposed technique, the maximum absolute error between exact and approximate solutions is considered.

\[
\text{Error} = \max|\text{Exact value of } \phi(x, y, t) - \text{Approximate value of } \phi(x, y, t)|.
\]

4.1 Example 1

Consider Eq. (1) on the domain \([0, 1] \times [0, 1]\), with

\[
a = \frac{2 \pi^2 \rho}{\pi^2 \Gamma(1 - \rho)}, \quad b = \frac{\pi^2 \rho}{12 \pi^2 \Gamma(1 - \rho)},
\]

\[
c = \left[\frac{2 \pi^2 \rho}{\Gamma(1 - \rho)} + \frac{25}{12} \pi^2 \rho (t^2 + 1)\right] \sin \pi x \sin \pi y
\]

and with the initial condition \(\phi(x, y, 1) = \sin(\pi x) \sin(\pi y)\) and the boundary conditions \(\phi(0, y, t) = \phi(1, y, t) = \phi(x, 1, t) = \phi(x, 0, t) = 0 \forall x \in [0, 1], y \in [0, 1], t \in (0, 1]\). The exact solution [16] of Eq. (1), with these conditions is \(\phi(x, y, t) = (1 + t^2) \sin(\pi x) \sin(\pi y)\).

On taking the differential transform on both sides of Eq. (1),

\[
\Phi_{k+1} = \frac{\Gamma(k \rho + 1)}{\Gamma(k \rho + 1 + \rho)} \sum_{l=1}^{k} A(l) \frac{\partial^2 \Phi(0, l + 1)}{\partial x^2} + \sum_{l=1}^{k} B(l) \frac{\partial^2 \Phi(0, l + 1)}{\partial y^2} + C(k).
\]

The differential transforms of \(a, b, c\) are given as follows:

\[
A(l) = \frac{2}{\pi^2 \Gamma(1 - \rho)} \left(1 - \frac{2 - \rho}{\rho}\right)
\]

\[
B(l) = \frac{1}{12 \pi^2 \Gamma(1 - \rho)} \left(1 - \frac{2 - \rho}{\rho}\right)
\]

\[
C(k) = \frac{2}{\Gamma(3 - \rho)} \left(1 - \frac{2 - \rho}{\rho}\right) \sin \pi x \sin \pi y
\]

\[
+ \frac{25}{12 \Gamma(1 - \rho)} \left(1 - \frac{4 - \rho}{\rho}\right) \sin \pi x \sin \pi y
\]

\[
+ \left(1 - \frac{2 - \rho}{\rho}\right) \sin \pi x \sin \pi y.
\]

Thus, the recurrence relation in (11) can be written as follows:

\[
\Phi_{k+1} = \frac{\Gamma(k \rho + 1)}{\Gamma(k \rho + 1 + \rho)} \left[ \sum_{l=1}^{k} A(l) \frac{\partial^2 \Phi(0, l + 1)}{\partial x^2} + \sum_{l=1}^{k} B(l) \frac{\partial^2 \Phi(0, l + 1)}{\partial y^2} + C(k) \right]
\]

or

\[
\Phi_{k+1}^j = \frac{\Gamma(k \rho + 1)}{\Gamma(k \rho + 1 + \rho)} \left[ \sum_{l=1}^{k} A(l) \sum_{m=1}^{M} a_{lm} \Phi_{m}^j + \sum_{l=1}^{k} B(l) \sum_{n=1}^{N} b_{lm} \Phi_{m}^j + C(k) \right].
\]

Thus, \(\Phi_{ij}, \Phi_{ij}^j, \Phi_{ij}^j, \ldots\) can be calculated from the aforementioned recurrence formula, and then by using the inverse DTM, the approximate solution of this specific case of (1) can be written as follows:

\[
\phi(x, y, t) = \Phi_{ij}^j + \Phi_{ij}^{2j} + \Phi_{ij}^{3j} + \ldots
\]

It can be observed from Figures 1 and 2 that for \(\rho = 1\) and 0.4, the two solutions are agreeing and the error is zero. However, for different values of \(\rho\), Table 2 shows no monotonic nature of the error.

We also solved this problem by DTM only. Its approximate series solution is obtained as follows:

\[
\phi(x, y, t) = U_1 + U_2 t^p + U_3 t^{2p} + \ldots
\]

where

\[
U_1 = \sin(\pi x) \sin(\pi y)
\]

\[
U_2 = A \sin(\pi x) \sin(\pi y),
\]

\[
U_3 = B \sin(\pi x) \sin(\pi y),
\]

where

\[
A = \frac{\Gamma(\rho + 1)}{\Gamma(2 \rho + 1)} \left[ \frac{-26(1 - \frac{2 - \rho}{\rho})}{\Gamma(3 - \rho)} - \frac{26(1 - \frac{2 - \rho}{\rho})}{\Gamma(1 - \rho)} \right]
\]

\[
+ \frac{26(1 - \frac{2 - \rho}{\rho})}{\Gamma(3 - \rho)} + \frac{26(1 - \frac{2 - \rho}{\rho})}{\Gamma(1 - \rho)}
\]

\[
+ \frac{25}{12 \Gamma(1 - \rho)} \left(1 - \frac{4 - \rho}{\rho}\right) + \frac{25}{12 \Gamma(1 - \rho)}
\]

\[
+ \frac{25}{12 \Gamma(1 - \rho)} \left(1 - \frac{4 - \rho}{\rho}\right) + \frac{25}{12 \Gamma(1 - \rho)}
\]

\[
+ \frac{25}{12 \Gamma(1 - \rho)} \left(1 - \frac{4 - \rho}{\rho}\right) + \frac{25}{12 \Gamma(1 - \rho)}
\]

and so on ...
As we increase the number of terms of the series solution and decrease the time variable, the solutions with DTM (Table 3) as well as with D(TQ)M (Table 4) are getting better. A comparison of the exact solution and approximate solution using the hybrid method and using only DTM, at all the mesh points for a $5 \times 5$ mesh, is mentioned in the table of Figure 3.

### Table 2: Error using D(TQ)M for $M = N = 21$, $t = 0.1$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Maximum absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0100</td>
</tr>
<tr>
<td>0.25</td>
<td>0.0077</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0148</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0212</td>
</tr>
<tr>
<td>0.75</td>
<td>0.0100</td>
</tr>
<tr>
<td>1</td>
<td>0.0900</td>
</tr>
</tbody>
</table>

4.2 Example 2

Consider Eq. (1) on the domain $[0, \pi] \times [0, \pi]$ with

$$a = 1, \ b = 1, \ c = 2 \left[ \frac{t^{2-\rho}}{1(3-\rho)} + t^{2} \right] \sin x \sin y$$
and with the initial condition \( \phi(x, y, 1) = 0 \) and the boundary conditions \( \phi(0, y, t) = \phi(x, \pi, t) = \phi(x, 0, t) = \phi(x, \pi, t) = 0 \) \( \forall x \in [0, \pi], y \in [0, \pi], t \in (0, 0.5]. \) The exact solution [16] of Eq. (1) with these conditions is: \( \phi(x, y, t) = t^2 \sin(x) \sin(y). \) On taking the differential transform on both sides of Eq. (1):

\[
\frac{\Gamma(kp + \rho + 1)}{\Gamma(kp + 1)} \Phi_{k+1} = \sum_{l=1}^{k} A(l) \frac{\partial^2 \Phi(k - l + 1)}{\partial x^2} + \sum_{l=1}^{k} B(l) \frac{\partial^2 \Phi(k - l + 1)}{\partial y^2} + C(k).
\]

The recurrence relation in (11) can be written as follows:

\[
\Phi_{k+1} = \frac{\Gamma(kp + 1)}{\Gamma(kp + \rho + 1)} \sum_{m=1}^{M} a_{m}\Phi_{k+1}^{m} + \sum_{m=1}^{N} b_{m}\Phi_{m}^{k} + C_{k},
\]

where

\[
C_{k} = \frac{\delta k - (\frac{2 - \rho}{\rho})}{\Gamma(3 - \rho)} \sin x \sin y
\]

Thus, \( \Phi_{k+1}, \Phi_{k}, \Phi_{m}, \ldots \) can be calculated from the aforementioned recurrence formula, and then by using the inverse DTM, the approximate solution of this specific case of Eq. (1) can be written as follows:

\[
\phi(x, y, t) = \Phi_{k+1} + \Phi_{k} t^{\rho} + \Phi_{m} t^{2\rho} + \ldots
\]

We also solved this problem by DTM only. The series solution for some values of \( \rho \) is given as follows:

When \( \rho = 1, \) the approximate solution is \( \phi(x, y, t) = t \sin x \sin y. \)

When \( \rho = 2/3, \)

\[
\phi(x, y, t) = \left[ \frac{2^{k+1/3}}{r(1/3)} - \frac{2^{k+1/3}}{r(1/3)} + \frac{2^{k+1/3}}{r(1/3)} \right] \sin x \sin y.
\]

When \( \rho = 1/2, \)

\[
\phi(x, y, t) = \left[ \frac{2^{k+1/2}}{r(1/2)} - \frac{2^{k+1/2}}{r(1/2)} + \frac{2^{k+1/2}}{r(1/2)} \right] \sin x \sin y.
\]

When \( \rho = 2/5, \)

\[
\phi(x, y, t) = \left[ \frac{2^{k+1/5}}{r(1/5)} - \frac{2^{k+1/5}}{r(1/5)} + \frac{2^{k+1/5}}{r(1/5)} \right] \sin x \sin y.
\]

The comparison of errors due to DTM and D(TQ)M can be observed in Tables 5 and 6, whereas Tables 6 and 7 compare the errors due to D(TQ)M at \( t = 0.5 \) and \( t = 0.01, \) respectively. A comparison of the exact solution and approximate solution using the D(TQ)M and using only DTM, at all the mesh points for a \( 5 \times 5 \) mesh, is shown in Figure 4.

It is observed that the series obtained due to DTM is absolutely convergent for various values of \( \rho. \) However,
the theoretical analysis of the convergence of D(TQ)M is to be attempted as a future work.

5 Results

From the numerical examples, it can be observed that DTM is an efficient method for solving two-dimensional

Table 5: The error for different values of $\rho$ and for different number of non zero terms ($K$) in the series solution of Example 4.2 by the DTM (at $t = 0.5$) on $15 \times 15$ mesh

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$K = 3$</th>
<th>$K = 4$</th>
<th>$K = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2500</td>
<td>0.2500</td>
<td>0.2500</td>
</tr>
<tr>
<td>2/3</td>
<td>0.0707</td>
<td>0.0998</td>
<td>0.0880</td>
</tr>
<tr>
<td>1/2</td>
<td>0.0129</td>
<td>0.1307</td>
<td>0.0543</td>
</tr>
<tr>
<td>2/5</td>
<td>0.0587</td>
<td>0.2015</td>
<td>0.1963</td>
</tr>
</tbody>
</table>

Table 6: The error for different values of $\rho$ and for different number of non zero terms($K$) in the series solution by the D(TQ)M (at $t = 0.5$, $M = N = 15$) in Example 4.2

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$K = 3$</th>
<th>$K = 4$</th>
<th>$K = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7500</td>
<td>0.68762</td>
<td>0.67520</td>
</tr>
<tr>
<td>2/3</td>
<td>0.03872</td>
<td>0.35016</td>
<td>0.55380</td>
</tr>
<tr>
<td>1/2</td>
<td>0.07775</td>
<td>0.85338</td>
<td>1.5244</td>
</tr>
<tr>
<td>2/5</td>
<td>0.18509</td>
<td>1.31846</td>
<td>2.5550</td>
</tr>
</tbody>
</table>

Table 7: The error for different values of $\rho$ and for different number of non zero terms ($K$) in the series solution by the D(TQ)M (at $t = 0.01$, $M = N = 15$) in Example 4.2

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$K = 3$</th>
<th>$K = 4$</th>
<th>$K = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.99990</td>
<td>0.99989</td>
<td>0.99989</td>
</tr>
<tr>
<td>2/3</td>
<td>0.93084</td>
<td>0.93093</td>
<td>0.93092</td>
</tr>
<tr>
<td>1/2</td>
<td>0.83427</td>
<td>0.83476</td>
<td>0.83478</td>
</tr>
<tr>
<td>2/5</td>
<td>0.73638</td>
<td>0.73260</td>
<td>0.73353</td>
</tr>
</tbody>
</table>
fractional diffusion equation. The hybrid method, approximates the solution for small values of time as presented in Tables 3 and 4. In the first example, the solution obtained from the hybrid method is same as exact solution for $\rho = 1$. Moreover, as the time is getting smaller, the error is also reducing, for $t < 1$.

In second example, as the value of $\rho$ is decreasing, the errors are also decreasing in case of the D(TQ)M at time $t = 0.01$. However, this pattern is not there for the greater value of time $t$ as well as for the DTM.

Thus, this work is an experiment of combining a semi analytical method with a numerical method. The experiment can be called successful as the results are near to the exact solution for small values of time. However, the method can be explored for its efficiency for large value of time, along with its theoretical analysis.

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**References**


