Research Article

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Solution for a rotational pendulum system by the Rach–Adomian–Meyers decomposition method

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Abstract: In this article, we report for the first time the application of a novel and extremely valuable methodology called the Rach–Adomian–Meyers decomposition method (MDM) to obtain numerical solutions to the rotational pendulum equation. MDM is a tool for solving nonlinear differential equations that combines both series solutions and the Adomian decomposition method efficiently. We present a simple and highly accurate MDM-based algorithm and its numerical implementation via a one-step recurrence approach for obtaining periodic solutions to the rotational pendulum equation. Finally, numerical simulations are performed to demonstrate the efficiency and accuracy of the proposed technique for both large and small amplitudes of oscillation.

Keywords: rotational pendulum system, modified Adomian decomposition method, Adomian polynomials, nonlinear oscillators, approximate analytical method

MSC 2020: 34L30, 34C15, 65D15

1 Introduction

Several analytical and approximate methods for solving nonlinear oscillators have been developed in recent decades, including the parameter-expansion method [1], the harmonic balance method [2,3], the energy balance method [4,5], the Hamiltonian approach [6], the use of special functions [7–9], the max–min approach [10], the variational iteration method [11,12], homotopy perturbation [13,14], and so on. There are many studies related to our work, many of these techniques for solving nonlinear oscillatory problems can be seen in refs [15–17]. In addition, several methods related to problems involving fractional-order derivatives are also available [18–25]. The Adomian decomposition approach is seldom utilized to solve nonlinear problems related to nonlinear oscillators [26–32]. The Adomian decomposition method (ADM) provides the solution in a rapidly convergent series if the equation has a unique solution. The MDM [33] is a subset of the ADM, and an analytic approximation method, which always yields the Taylor expansion series, also known more simply as the power series, as the solution for any nonlinear differential equation that meets the prerequisites of the Cauchy–Kovalevskaya theorem of existence, uniqueness, and analyticity. The MDM was originally designed to facilitate computer programming of the decomposition approach permitting at will generation of high-order Adomian one-step methods, although it remains most valuable in generation of analytic approximate solutions. Furthermore, it has been shown that the modified decomposition method is equivalent to a decelerated ADM as well as an augmented power series method by combining the Adomian–Rach theorem of nonlinear transformation of series with the appropriate formulas of the Adomian polynomials [34,35]. The rotational pendulum equation was first reported in ref. [36], shortly then in ref. [37] the problem was approached using homotopy analysis method, and later the model was taken up for the case of large-amplitude oscillation in ref. [38]. Finally, solutions were obtained by Hamiltonian approach in ref. [39]. The rotational pendulum is distinguished from the simpler and more common pendulum in that its support rotates at a constant speed, whereas the simple pendulum’s support is always fixed. In this study, an accurate analytical approximate solution for a rotational pendulum system is obtained using an MDM-derived algorithm. Upon comparing the approximate solutions with the exact results, it has been proved that our approach is a useful and highly accurate methodology for the study of nonlinear dynamic systems.
The rest of the article is organized as follows. In Section 2, we state in a concise and complete manner, the method that we have abbreviated as MDM and some references are offered so that the interested reader can go deeper into the mathematical foundations, which are not part of the objectives of this work. In Section 3, we summarize the model described by the rotational pendulum equation, and we establish that the MDM can be used to solve it. In Section 4, we show, by means of five examples, the quality and accuracy of our method, comparing the obtained results with the exact solution that appears in the literature and which is given in terms of the complete elliptic integral of the first kind. Finally, in Section 5, we summarize our results and present our future directions for work using the same methods.

2 Brief description of the Rach–Adomian–Meyers decomposition method

Nonlinear differential equations play a very important role in the field of practically all scientific disciplines and arise to obtain mathematical models of real-life problems such as in three-layer beam theory, elastic stability, nuclear physics, image processing, fluid mechanics, nonlinear biological systems, astrophysics, among many other applications. More recently, applications have included nonlinear equations with fractional derivatives \cite{40,41}. In this article, we consider second-order nonlinear ordinary differential equations from the point of view of classical calculus of the type:

\[
\frac{du}{dt} + f(t, u(t)) - g(t) = 0, \quad t_0 \leq t \leq T, \tag{1}
\]

where the original nonlinear differential equation is written in the form of the residual error for convenience, i.e., the system input function \(g(t)\) is subtracted from the usual LHS. We mention in passing that the nonlinearity is simpler in form such that the independent and dependent variables are separable or factorable into functions depending only on one variable. The system input function and the system output function’s initial condition, or conditions, is, or are, specified as

\[
g(t) = \sum_{n=0}^{\infty} g_n(t - t_0)^n, \quad u(t_0) = C_0, \quad \frac{du}{dt}(t_0) = C_1. \tag{2}
\]

Because the solution and all of its derivatives are presumed to be analytic by the Cauchy–Kovalevskaya theorem for existence, uniqueness, and analyticity, we have

\[
u(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n, \tag{3}
\]

\[
\frac{du}{dt}(t) = \sum_{n=0}^{\infty} (n + 1)a_{n+1}(t - t_0)^n, \tag{4}
\]

\[
\frac{d^2u}{dt^2}(t) = \sum_{n=0}^{\infty} (n + 1)(n + 2)a_{n+2}(t - t_0)^n.
\]

By the Adomian–Rach nonlinear transformation of series \cite{34,35}, we have

\[
f(u(t)) = \sum_{m=0}^{\infty} A_m(t - t_0)^m, \text{ where the} \quad A_m = A_n(a_0, \ldots, a_n),
\]

are the classic Adomian polynomials in terms of the solution series coefficient \(a_n\) instead of the usual solution coefficients \(u_n(t)\) due to the Adomian–Rach theorem \cite{34,35}.

Substituting Eqs. (3) and (4) in Eq. (1), collecting terms of the same powers of \((t - t_0)^n\), considering that \(g(t)\) is a constant, and using the initial conditions we obtain the following algorithm for the solution coefficients

\[
\begin{align*}
  a_0 &= C_0, \\
  a_1 &= C_1, \\
  a_2 &= \frac{1}{2}g, \\
  a_{n+1} &= \frac{-A_n}{(n + 1)(n + 2)}, \quad n \geq 0.
\end{align*} \tag{6}
\]

Truncating the series (3), we obtain the solution to problem (1) with \(n\)th term approximation, i.e.,

\[
 u_n(t, t_0, C_0, C_1) = \sum_{m=0}^{n} a_m(t - t_0)^m, \quad n \geq 0, \tag{7}
\]

in the limit, it yields the exact solution, that is,

\[
\lim_{n \to \infty} u_n(t, t_0, C_0, C_1) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = u(t). \tag{8}
\]

We denote the \(n\)th-order numeric solution by \(u^{(n)}_k\), \(k = 1, 2, \ldots, N\), where we are considering the uniform partition \([t_0, t_1, \ldots, t_N]\) of the interval \([0, T]\). The one-step recurrence scheme corresponding to the numerical solution generated by \(u_{n+1}\) is given by

\[
u_0^{(n)} = C_0, \quad \dot{u}_0 = C_1, \quad \dot{u}_k^{(n)} = u_{n+1}(t_k, t_{k-1}, u_k^{(n)} - \dot{u}(t_k)) = u_{k+1}^{(n)} + \dot{u}_{k-1} + \sum_{m=2}^{n} a_m^{(n-1)}h^m, \tag{9}
\]

\[
\dot{u}_k = \frac{d}{dt}u_n(t, t_k, u_n(t_{k-1}, \dot{u}^{(n-1)}_{k-1})) \big|_{t=t_k}, \quad \dot{u}_{k-1} = \sum_{m=2}^{n} ma_m^{(n-1)}h^{m-1}, \quad k = 1, 2, \ldots, N; \quad h = t_k - t_{k-1}.
\]
where \( a_m^{(0)} = a_m \) and for \( k = 1, 2, \ldots, N \), \( a_m^{(k-1)} \), \( m = 2, 3 \ldots, n \) will be found recursively as in Eq. (6) with \( a_0^{(0)} = u_0^{(0)} \) and \( a_1^{(0)} = u_1^{(0)} \).

Due to the objectives of the present article, we have developed the algorithm derived from MDM for a second-order differential equation, but the method presented here and its algorithm can be applied to higher-order differential equations and even to systems of differential equations, as can be seen in ref. [42].

In the next section, we apply the MDM and its numerical version to find a periodic solution for a rotational pendulum system, where the obtained solution and the comparison with the exact ones will be shown through graphs in which the high accuracy of the method will be demonstrated.

### 3 Mathematical model for a rotational pendulum and the solution methodology derived from the MDM

The mathematical model of a simple pendulum of mass \( m \) and length \( l \) attached to a rotating rigid frame oscillating in the (constant) gravitational field of the Earth is described by the differential equation of second order with sinusoidal nonlinearities [37–39,43,44]:

\[
\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin(\theta) - \frac{1}{2} \Omega^2 \sin(2\theta) = 0, \tag{10}
\]

with initial conditions

\[
\theta(0) = A, \quad \theta'(0) = 0, \quad 0 < A < \pi, \tag{11}
\]

where \( \theta \) is the angle that the pendulum forms with the vertical direction in relation to the temporal coordinate \( t \), \( A \) is the initial amplitude of oscillation, \( g \) the acceleration of gravity, and \( \Omega^2 \) is the angular velocity of revolution.

By the Cauchy–Kovalevskaya theorem for existence, uniqueness, and analyticity, we have

\[
\theta(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{12}
\]

Using the Adomian–Rach theorem for the nonlinear transformation of series in the nonlinear part of Eq. (10), we have

\[
\frac{g}{l} \sin(\theta) - \frac{1}{2} \Omega^2 \sin(2\theta) = \sum_{n=0}^{\infty} A_n (a_0, a_1, \ldots, a_n) t^n, \tag{13}
\]

where the Adomian polynomials are tailored to the particular nonlinearity written as

\[
A_n = A_n \left[ \frac{g}{l} \sin(\theta) - \frac{1}{2} \Omega^2 \sin(2\theta) \right]. \tag{14}
\]

Now, calculating the first and second derivatives with respect to the independent variable \( t \), we obtain

\[
\frac{d\theta}{dt} = \frac{d^2}{dt^2} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (n+1) a_n t^n, \tag{15}\]

thus

\[
\frac{d^2\theta}{dt^2} = \frac{d^3}{dt^3} \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n. \tag{16}\]

Replacing Eq. (13) and Eq. (16) into Eq. (10), we have

\[
\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} t^n + \sum_{n=0}^{\infty} A_n(a_0, \ldots, a_n) t^n = 0. \tag{17}\]

Now, combining coefficients of like powers in the independent variable \( t \), we obtain

\[
\sum_{n=0}^{\infty} [(n+1)(n+2) a_{n+2} + A_n] t^n = 0. \tag{18}\]

If a power series equals zero, then each of its coefficients also equals zero, namely,

\[
[(n+1)(n+2) a_{n+2} + A_n] = 0, \tag{19}\]

from which we obtain

\[
a_{n+2} = \frac{-A_n}{(n+1)(n+2)}, \quad n \geq 0. \tag{20}\]

In addition, considering the initial conditions given by Eq. (11), we obtain

\[
\theta(0) = \left( \sum_{n=0}^{\infty} a_n t^n \right)|_{t=0} = A = a_0 \tag{21}\]

and

\[
\frac{d\theta}{dt}(0) = \left( \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n \right)|_{t=0} = a_1. \tag{22}\]

We obtain the approximate solution to order \( n \) as

\[
\varphi_n(t) = \theta_n(t, 0, A, 0) = \sum_{i=0}^{n} a_i t^i. \tag{23}\]

Now, following the algorithm given by Duan’s Corollary 3 of ref. [45] we calculate in a simple way the Adomian polynomials to a variable:

\[
A_0 = N(a_0), \quad A_n = \sum_{k=1}^{n} N^{(k)}(a_0) C_n^k, \quad n \geq 1. \tag{24}\]
where the superscript \((k)\) means derivative of order \(k\), \(N\) symbolizes the nonlinear term of the differential equation, and the coefficients \(C_n^k\) satisfy the recursive algorithm

\[
C_n^k = a_n, \quad n \geq 1, \\
C_n^k = \frac{1}{n} \sum_{j=0}^{n-k} (j+1)a_jC_{n-j-1}^k, \quad 2 \leq k \leq n.
\]

(25)

In the problem of the model given by Eq. (10) the nonlinear term turns out to be

\[
N(\theta) = \frac{g}{l} \sin(\theta) - \frac{1}{2} \Omega^2 \sin(2\theta),
\]

(26)

from where, considering that \(a_0 = A\) and \(a_1 = 0\), the Adomian polynomials \(A_n(a_0, \ldots, a_n)\) are calculated to be

\[
A_0 = \frac{g}{l} \sin(a_0) - \frac{1}{2} \Omega^2 \sin(2a_0), \\
A_1 = \frac{g}{l} \cos(a_0) - \frac{1}{2} \Omega^2 \cos(2a_0), \\
A_2 = -\frac{\sin(A)}{2l^2} (g - l\Omega^2 \cos(A))(g \cos(A) - l\Omega^2 \cos(2A)), \\
A_3 = -\frac{a_a g}{l} \sin(a_0) - \frac{a_a g}{6l} \cos(a_0) + \frac{a_a g}{l} \cos(a_0) \\
+ 2a_a \Omega^2 \sin(2a_0) - \frac{a_a g}{3} \Omega^2 \cos(2a_0) \\
- \frac{a_a g}{l} \cos(2a_0), \\
A_4 = \frac{\sin(A)}{96l^3} (g - l\Omega^2 \cos(A))(\sin(2A) + 11g\Omega^2 \cos(A) - 19g\Omega^2 \cos(3A) + 8l^2 \Omega^2 \cos(4A) - 4g^2 - 4l^2 \Omega^2), \\
A_5 = \frac{a_a g}{6l} \sin(a_0) + \frac{a_a g}{120l} \cos(a_0) \\
- \frac{a_a g}{2l} \cos(a_0) - \frac{a_a g}{2l} \cos(a_0) \\
- \frac{a_a g}{3} \Omega^2 \sin(2a_0) - \frac{a_a g}{15} \Omega^2 \cos(2a_0) \\
+ 2a_a \Omega^2 \cos(2a_0) - \frac{a_a g}{l} \sin(a_0) \\
- \frac{a_a g}{l} \sin(a_0) + \frac{a_a g}{l} \cos(a_0) \\
+ 2a_a \Omega^2 \sin(2a_0) + 2a_a \Omega^2 \cos(2a_0) \\
+ 2a_a \Omega^2 \cos(2a_0) - a_5 \Omega^2 \cos(2a_0),
\]

(27)

and so on.

The formula (25) does not need complicated derivatives, it only requires elementary arithmetic operations, which is conveniently convenient for symbolic implementation by any symbolic calculation software, such as MATHEMATICA. For readers interested in a MATHEMATICA code generating the first \(n + 1\) Adomian polynomials based on the algorithm given by Eq. (25), we recommend refs [46,47].

Using Eq. (20), the calculated solution coefficients are as follows:

\[
a_0 = A, \\
a_1 = 0, \\
a_2 = \frac{1}{2} \left( \frac{1}{2} \Omega^2 \sin(2A) - \frac{g}{l} \sin(A) \right), \\
a_3 = 0, \\
a_4 = \frac{1}{12} \left( \frac{1}{2} \Omega^2 \cos(2A) - \frac{g \cos(A)}{2l} \right) \left( \frac{1}{2} \Omega^2 \sin(2A) \\
- \frac{g \sin(A)}{l} \right), \\
a_5 = 0, \\
a_6 = \frac{\sin(A)}{2880} (\Omega^2 \cos(A) - g)(8g^2 \cos(2A) + 11g\Omega^2 \cos(A) - 19g\Omega^2 \cos(3A) + 8l^2 \Omega^2 \cos(4A) - 4g^2 - 4l^2 \Omega^2), \\
a_7 = 0, \\
a_8 = -\frac{\sin(A)}{161280} (c_1(30 \cos(A) - 34 \cos(3A))) \\
+ 4c_2(74 \cos(2A) + 79 \cos(4A) + 3) \\
+ 3c_3(122 \cos(A) + 113 \cos(3A) - 267 \cos(5A)) \\
+ c_4(711 \cos(2A) + 198 \cos(4A) + 791 \cos(6A) - 150) + 16c_5(15 \cos(A) + 15 \cos(3A) - 17(\cos(5A) + \cos(7A))))
\]

and so on.

### 3.1 Convergence analysis

We will now argue for the convergence of the method applied in the present study. The convergence of ADM and therefore of the MDM for solving nonlinear differential equations are presented in refs [48,49] and recently in ref. [50], the authors make an important study on the convergence of the method proposed here.

Let \(X = \mathcal{C}[0, T]\) be a Banach space and in \(X\) we will consider the norm

\[
\|\theta\| = \max_{0 \leq t \leq T} |\theta(t)|, \quad \theta \in X.
\]

(29)

Considering that \(a_i = 0\) we can rewrite the equality (12) in a more convenient way

\[
\theta = a_0 + M(\theta),
\]

(30)

where...
$M(\theta) = M \left( \sum_{i=0}^{\infty} a_i t^i \right) = \sum_{i=0}^{\infty} \frac{-A_i}{(i+1)(i+2)} t^{i+2}$.\quad (31)

$M$ is a nonlinear operator acting on the space $X$. Consequently, the succession of functions given in Eq. (23), can be rewritten as

$$\varphi_n(t) = a_0 + \sum_{i=0}^{n-2} \frac{-A_i}{(i+1)(i+2)} t^{i+2}.\quad (32)$$

Now, considering the action of the nonlinear operator $M$, we can see that Eq. (32) is conveniently rewritten as

$$\varphi_n = a_0 + M(\varphi_{n-1}), \quad n \geq 1.\quad (33)$$

It is easy to prove that for each $0 \leq \beta < 1$, the sequence $\{\varphi_n\}$ satisfies

$$|\varphi_{n+1} - \varphi_n| \leq \beta^n |a_0|.\quad (34)$$

We will now prove that the sequence $\{\varphi_n\}$ of the $n$th-partial sums converges to the exact solution $\theta$.

The next theorem provides a prerequisite for the convergence of the sequence $\{\varphi_n\}$, and the results are standard and can be seen in ref. [50].

**Theorem 1.** Let $M(\theta)$ be the non-linear operator expressed by (31), which satisfies the Lipschitz condition $\|M(\varphi) - M(\xi)\| \leq \beta \|\varphi - \xi\|$, for all $\varphi, \xi \in C[0, T]$, $T$ is a fixed real number, with Lipschitz constant, $0 \leq \beta < 1$. Then the sequence $\varphi_n = a_0 + M(\varphi_{n-1})$ converges to $\theta$.

**Proof.** We will show that $\{\varphi_n\}$ is a Cauchy sequence in the Banach space $X = C[0, T]$.

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**Figure 1:** (a) Comparison of the MDM approximation $\theta_{M}(t, 0, \pi/8, 0)$ (dotted line), $\theta_{M}(t, 0, \pi/8, 0)$ (dash-dotted line), $\theta_{M}(t, 0, \pi/8, 0)$ (dashed line) and the exact solution (solid line). (b) The exact solution (solid line), the 20th-order MDM numeric solution $\theta_{M}^{(20)}$ with step-size $h = 0.25$ (dashed line), and the 25th-order MDM numeric solution $\theta_{M}^{(25)}$ with step-size $h = 0.25$ (dots). (c) Absolute error when we compare $\theta_{M}^{(25)}$ with the exact solution (dots) and the 25th-order ADM numeric solution (squares), respectively. (d) Phase plane diagram, exact (solid line), and $\theta_{M}^{(25)}$ approximation (dashed line) for Example 1.
For every $n, m \in \mathbb{N}$, $n \geq m$ we have

$$
\|\varphi_n - \varphi_m\| = \|(\varphi_n - \varphi_{n-1}) + (\varphi_{n-1} - \varphi_{n-2}) + \cdots + (\varphi_{m+1} - \varphi_m)\|
\leq \|(\varphi_n - \varphi_{n-1})\| + \|(\varphi_{n-1} - \varphi_{n-2})\| + \cdots + \|(\varphi_{m+1} - \varphi_m)\|
\leq \beta^{n-1}\|a_0\| + \beta^{n-2}\|a_0\| + \cdots + \beta^m\|a_0\| 
\leq \beta^m(1 + \beta + \beta^2 + \cdots + \beta^{n-m})\|a_0\|
\leq \beta^m\left(1 - \beta^{n-m}\right)\|a_0\|.
\tag{35}
$$

Considering that $a_0 \in \mathbb{R}$ and since $0 \leq \beta < 1$, we have $1 - \beta^{n-m} < 1$. It readily follows that

$$
\|\varphi_n - \varphi_m\| \leq \frac{\beta^m}{1 - \beta}\|a_0\|.
\tag{36}
$$

Taking the limit as $m \to \infty$, we obtain

$$
\|\varphi_n - \varphi_m\| \to 0.
$$

Hence, $\{\varphi_n\}$ is the Cauchy sequence in the Banach space $X$. Hence, there exists $\varphi$ in $X$ such that $\lim_{n \to \infty} \varphi_n = \varphi$. Note that $\varphi$ is the exact solution of Eq. (10)

$$
\theta = \sum_{t=0}^{\infty} a_t t = \lim_{n \to \infty} \varphi_n = \varphi.
\tag{37}
$$

**Theorem 2.** Assume that $\theta(t)$ is the exact solution of the operator Eq. (10). Let $\{\varphi_n\}$ be the sequence of approximate series solutions defined by $\varphi_n = \sum_{t=0}^{n} a_t t$. Then there holds...
\[
\max_{0 \leq t \leq T} \left| \theta - \frac{1}{1 - \beta} \sum_{k=0}^{m} a_k t^k \right| \leq \frac{\beta^m}{1 - \beta} \|a_0\|. \tag{38}
\]

**Proof.** For any \(n \geq m\) and using the relation (36), we obtain
\[
||\varphi_n - \varphi_m|| \leq \frac{\beta^m}{1 - \beta} \|a_0\|. \tag{39}
\]
Since \(\lim_{n \to \infty} \varphi_n = \theta\) fixing \(m\) and letting \(n \to \infty\) in the above estimation, we obtain
\[
||\theta - \varphi_m|| \leq \frac{\beta^m}{1 - \beta} \|a_0\|. \tag{40}
\]

The exact solution of the problem expressed in terms of elliptical integrals is given in ref. [37] and is
\[
\theta(t) = A \cos(\omega t), \quad \text{where} \quad \omega(A) = 2n \left[ \int_0^{\pi/2} \left( \frac{\mu}{\sqrt{1 - \mu^2 \sin^2(y)}} \right. \right.
\]
\[
\left. \left( \frac{\mu}{\sqrt{1 - \mu^2 \sin^2(y)}} \right)^2 \right] \, dy \right]^{-1}, \quad \mu = \sin(A/2). \tag{42}
\]

Now we can use the general \((n + 1)\)-term approximation \(\theta_{n+1}(t, 0, A)\) to generate the \(n\)-th order MDM numeric solutions. By \(\theta_k^{(n)}\) we denote the \(n\)-th order MDM numeric solution. The recurring one-step scheme for the solution of Eq. (10) with the initial conditions Eq. (11) is given as follows:
\[
\theta_0^{(n)} = A, \quad \dot{\theta}_0 = 0, \tag{43}
\]

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**Figure 3:** (a) Comparison of the MDM approximation \(\theta_{20}(t, 0, \pi/2, 0)\) (dash-dotted line), \(\theta_{25}(t, 0, \pi/2, 0)\) (dashed line), \(\theta_{20}(t, 0, \pi/2, 0)\) (dotted line), and the exact solution (solid line). (b) The exact solution (solid line), the 20th-order MDM numeric solution \(\theta_{20}^{(n)}\) with step-size \(h = 0.3\) (dashed line) and the 25th-order MDM numeric solution \(\theta_{25}^{(m)}\) with step-size \(h = 0.3\) (dots). (c) Absolute error when we compare \(\theta_{25}^{(m)}\) with the exact solution (dots) and the 25th-order ADM numeric solution (squares), respectively. (d) Phase plane diagram, exact (solid line) and \(\theta_{25}^{(m)}\) approximation (dashed line) for Example 3.
\[ \Theta_k^{(0)} = \theta_{n+1}(t_k, t_{k-1}, \Theta_{k-1}^{(0)}, \dot{\Theta}_{k-1}) \\
= \theta_{n+1}(t_k, t_{k-1}, \Theta_{k-1}^{(0)}), \dot{\Theta}_{k-1}) \\
= \theta_{n+1}(t_k, t_{k-1}) + \sum_{m=2}^{n} a_{m}^{(k-1)} h_k^{m}, \quad k = 1, 2, \ldots, N. \tag{44} \]

\[ \dot{\Theta}_k = \frac{d}{dt} \Theta_n(t, t_{k-1}, \Theta_k^{(0)}, \dot{\Theta}_{k-1})|_{t=t_k} \\
= \dot{\Theta}_{k-1} + \sum_{m=1}^{n} ma_{m}^{(k-1)} h_k^{m-1}, \quad k = 1, 2, \ldots, N. \tag{45} \]

In Eqs. (44) and (45), we are considering the partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) of \([0, T]\), where the \( k \)-th step-size is \( h_k = t_k - t_{k-1}, \ a_m^{(0)} \) is exactly the \( a_m \) in Eqs. (20), (21), and (22), and for \( k = 2, \ldots, N, \ a_m^{(k-1)} \), \( m = 2, 3, \ldots, n \) are determined by an algorithm analogous to that offered by Eqs. (20), (21), and (22) with \( a_0^{(k-1)} = \Theta_k^{(0)} \) and \( a_0^{(k-1)} = \dot{\Theta}_{k-1} \). The convergence of our algorithm is due to the polynomial nature of Eq. (32) for each fixed value of the initial amplitude \( A \) and by virtue of Theorem 1.

### 4 Numerical applications

In this section, we apply the Rach–Adomian–Meyers decomposition method (MDM) and its numerical version to obtain numeric solutions to the nonlinear oscillators governed by Eq. (10) with different values of parameters \( l \), \( \Omega^2 \), and amplitude \( A \). The approximate solutions are shown graphically.

**Example 1.** We will consider the parameters \( A = \pi / 8 \), \( g = 9.81 \), \( l = 10 \), and \( \Omega^2 = 1/120 \). The numerical results are plotted in Figure 4.
by the MDM for different orders of approximation, the absolute error by the best approximation, that is, by $\theta_k^{(25)}$ versus the comparison with the absolute error obtained by ADM, and the phase plane diagram are illustrated in Figure 1.

**Example 2.** We will consider the parameters $A = \pi / 4$, $g = 9.81$, $l = 12.2$, and $\Omega^2 = 1/100$. The numerical results by the MDM for different orders of approximation, the absolute error by the best approximation, that is, by $\theta_k^{(25)}$ versus the comparison with the absolute error obtained by ADM and the phase plane diagram are illustrated in Figure 2.

**Example 3.** We will consider the parameters $A = \pi / 2$, $g = 9.81$, $l = 14$, and $\Omega^2 = 1/100$. The numerical results by the MDM for different orders of approximation, the absolute error by the best approximation, that is, by $\theta_k^{(25)}$ versus the comparison with the absolute error obtained by ADM and the phase plane diagram are illustrated in Figure 3.

**Example 4.** We will consider the parameters $A = 3\pi / 4$, $g = 9.81$, $l = 15$, and $\Omega^2 = 1/90$. The numerical results by the MDM for different orders of approximation, the absolute error by the best approximation, that is, by $\theta_k^{(30)}$ versus the comparison with the absolute error obtained by ADM, and the phase plane diagram are illustrated in Figure 4.

**Example 5.** We will consider the parameters $A = 14\pi / 15$, $g = 9.81$, $l = 140$, and $\Omega^2 = 1/80$. The numerical results by the MDM for different orders of approximation, the absolute error by the best approximation, that is, by $\theta_k^{(35)}$ versus the comparison with the absolute error

![Figure 5](image-url)
obtained by ADM and the phase plane diagram are illustrated in Figure 5. For this same case, because it is an example in which the initial amplitude of oscillation is relatively large, we have compared our results with those recently reported in ref. [44] and the 35th-order MDM numeric approximation produces errors of not more than 1.08\%, which clearly reveal the high accuracy of the calculations of MDM. This is shown in Table 1.

The comparison of the results between MDM, ADM, and the exact solutions are shown in Figures 1–5. As shown in the graphs, MDM is very close and converges to the exact solution with absolute error less than the ADM. It shows that the new approach is more accurate than the standard ADM. In the calculation of our solutions, the fast algorithms for generation of the Adomian polynomials proposed in ref. [45] guarantee the efficiency of our proposal.

Although the solution to the rotational pendulum equation can be determined exactly, the expression is given implicitly by means of an elliptic integral in Eq. (42). It is not a beneficial strategy for understanding the physics of nonlinear response because it produces only numerical solutions.

Finally, we can observe from Figures 4 and 5 that $\theta_k^{30}$ and $\theta_k^{35}$ turn out to be acceptable approximations to the exact solution for cases in which the initial amplitude is close to $\pi$; this is valuable since at $A = \pi$ the physical system has a saddle point or point of instability [37]. Consequently, the alternative offered here provides analytical approximation solutions that aid in the comprehension of the nonlinear system role for specific physical characteristics and are available for both small and large values of the amplitudes of oscillation.

<table>
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<th>Exact solution $\theta_k(t)$</th>
<th>35th-order MDM numeric solution</th>
<th>Relative errors (%)</th>
<th>$\theta(t)$ approx of ref. [44]</th>
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References


