Research Article


Distinguishability criteria of conformable hybrid linear systems

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Abstract: We relate this article to the emerging idea of distinguishability of conformable linear hybrid time-invariant control systems. To obtain the necessary and sufficient conditions of \( \alpha \)-distinguishability for fractional cases, we develop the Leibnitz rule for conformable derivatives. Furthermore, with the help of a study of Laplace techniques, a more simple criterion of \( \alpha \)-distinguishability for the fractional linear system is developed.

Keywords: conformable fractional derivative, \( \alpha \)-distinguishability, fractional linear control systems

MSC 2020: 93B07, 34A38, 34A08

1 Introduction

Fractional calculus (FC) is what which is seen in heavy tails in electrical engineering and is also used in the modeling of anomalous diffusion [1]. Other applications involve fractional oscillators and fractional mechanics in refs. [2–4]. One of the newly defined derivatives in FC is “conformable fractional derivative (CFD)” stated by Khalil et al. in ref. [5]. Also the significant characteristics of CFD are listed in refs [5,6]. Interested readers can also see refs [7,8]. Later, in ref. [9], it is verified that CFD is actually conformable derivative (CD) not a fractional derivative.

There are chain rules in FC that are not good enough since various results can be obtained from these. There is a mega study on chain rules by Jumarie, for example, in ref. [10]. Inspired from this work, a chain rule of CD of a function of two variables is developed and a Leibnitz rule for CDs is given in Section 2. In this article, we have to interpret the notion of \( \alpha \)-distinguishability for the conformable switched hybrid linear system (SHLS).

While there are two types of behaviors in a hybrid system including the discrete and continuous dynamical behavior, that is, the system which has the potency to both jumps which is interpreted by a state machine or automaton and flow which is described in the form of the differential equation. Consider a linear switched hybrid system as follows:

\[
X_i : \begin{cases}
\dot{z}(t) = C_i z(t) + G_i w(t) \\
y(t) = E_i z(t) + F_i w(t)
\end{cases}, \quad \text{for } i = 1, 2, \tag{1}
\]

where \( z(t) \in \mathbb{R}^n \), \( w(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^k \) with \( C_i \in \mathbb{R}^{m \times n} \), \( G_i \in \mathbb{R}^{m \times m} \), \( E_i \in \mathbb{R}^{k \times n} \), \( F_i \in \mathbb{R}^{k \times m} \). \tag{2}

There is a discussion of the distinguishability of (1) in ref. [11]. The objective to analyze such a problem is that distinguishability takes a key role in assuming the observability of this system. The assumed subsystems in (1) may change from one phase to another phase when any situation arises, say at time \( t \) it bounces from \( X_i \) to \( X_j \) if \( t, z(t) \in \Omega \) where \( \Omega \subseteq \mathbb{R}^{n+1} \). Not a long ago, the distinguishability of hybrid linear systems (HLSs) has seized more importance. Input is not included in case of autonomous systems while it has an important role in non-autonomous systems. Distinguishability and observability problems were inspected in ref. [12] for systems without input. Since, in the case of a non-autonomous system, \( w(\cdot) \) plays a key role and it becomes difficult to resolve the problem. In ref. [11], the authors have defined distinguishability for non-autonomous systems and in ref. [13],

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the equivalent criterion for distinguishability is examined which is as follows:

\[ M_\lambda = \begin{bmatrix}
  E \\
  E(C - \lambda I) \\
  E(C - \lambda I)^2 \\
  \vdots \\
  E(C - \lambda I)^{2n} \\
  E(C - \lambda I)^{2n-1}G
\end{bmatrix}
\]

has full column rank (FCR) for any \( \lambda \in \mathbb{C} \) if and only if

\[ \begin{bmatrix}
  E \\
  C - \lambda I \\
  G
\end{bmatrix}
\]

has FCR.

Hence, the distinguishability of the assumed subsystems in (1) is obvious from the FCR of \( M_\lambda \), see in ref. [13]. Many other authors also have studied distinguishability/observability. They discussed the notion of distinguishability for discrete HLSs as in refs [14–17]. The researchers in ref. [18] interrogated the observability and controllability of SHLS in the case of continuous systems. Distinguishability is noticeable in other control problems in refs [19,20]. The idea of the distinguishability is that two subsystems to be distinguishable on \([0, T]\) that for any \( z_{t_0} \) and \( z_{t_0} \) represents the initial states of the systems \( X_1 \) and \( X_2 \), respectively, with \( z_{t_0}, z_{t_0} \in \mathbb{R}^n \), \( w(\cdot) : [0, T) \rightarrow \mathbb{R}^m \) where \( w(\cdot) \) the input such that their outputs are not similar. It is observed that output will be necessarily zero if \( z_{t_0} = 0 \) and \( w(\cdot) = 0 \). Therefore, we restrict the definition of distinguishability with \( (z_{t_0}, z_{t_0}, w(\cdot)) \neq 0 \). There is no study of distinguishability for conformable HLSs.

We have developed the idea of \( \alpha \)-distinguishability for conformable HLS. Assume the switched conformable HLS, which is composed of the time-invariant subsystems as follows:

\[ I_i : \begin{cases}
  T_\alpha z(t) = K_\alpha z(t) + M_\alpha w(t) \\
  y(t) = N_\alpha z(t) + S_\alpha w(t)
\end{cases}, \quad \text{for } i = 1, 2, \quad (4)
\]

where \( z(\cdot) \in \mathbb{R}^n \), \( w(\cdot) \in \mathbb{R}^m \), \( y(\cdot) \in \mathbb{R}^k \) with

\[ K_i \in \mathbb{R}^{m \times n}, \quad M_i \in \mathbb{R}^{m \times m}, \quad N_i \in \mathbb{R}^{k \times n}, \quad S_i \in \mathbb{R}^{k \times m}. \quad (5)
\]

The assumed subsystems in (4) may alter from one state to another when any situation occur, say at time \( t \) it bounces from \( I_i \) to \( I_j \) if \( (t, z(t)) \in I_\beta \) where \( \beta \in \mathbb{R}^{n+1} \). We have generalized the definition of distinguishability for conformable linear systems and obtained the equivalent condition for \( \alpha \)-distinguishability by using Leibniz rule for CD. To reduce the comparative efforts in \( \alpha \)-distinguishability condition, we consider the conformable Laplace techniques and a simplified form of distinguishability criterion is achieved.

This article is organized in the following manner: In Section 2, we give the elementary facts of CFD/integral which is actually a CD and also define the \( \alpha \)-distinguishability for a conformable linear system. In Section 3, some results related to \( \alpha \)-distinguishability for \( P_\alpha[0, T], P_\alpha^\ast[0, T], A_\alpha[0, T], C_\alpha[0, T] \) of the conformable system are proved. In Section 4, more consequences of the generalized \( \alpha \)-distinguishability are examined and a simple criterion is stated.

## 2 Preliminaries

In this section, we give some fundamental ideas and develop results which are supportive of the main consequences.

**Definition 2.1.** [6] (CD) Consider that \( g : [0, \infty) \rightarrow \mathbb{R} \) be a function with \( 0 < \alpha \leq 1 \), the CD of order \( \alpha \) is expressed as

\[ (T_\alpha g)(t) = \lim_{\epsilon \rightarrow 0} \frac{g(t + \epsilon t^{\alpha-1}) - g(t)}{\epsilon}, \]

as long as this limit exists and is finite.

It is noted that if \( f(t) \) is differentiable then \( T_\alpha f(t) = t^{1-\alpha} f'(t) \). It is obvious that the CD of the constant function is 0.

**Remark 2.2.** It is possible that a function has CD at a point while the function is not differentiable for example \( f(t) = \sqrt{t} \), \( T_\alpha^0 f(0) = \lim_{\epsilon \rightarrow 0} (T_\alpha^0 \sqrt{t}) = \frac{1}{2} \). \( T_\alpha^0 f(0) \) does not exist.

Now, the definition of conformable integral is given as follows:

**Definition 2.3.** [6] (Conformable integral) For \( 0 < \alpha \leq 1 \), \( b \in [0, c] \) and \( g : [b, c] \rightarrow \mathbb{R} \), if

\[ \int_b^c g(t) \alpha^\alpha dt = \int_b^c g(x) \frac{(t^\alpha - b)^\alpha}{t - b} dx, \]

exists, then it is called conformable integral having order \( \alpha \) on an interval \([b, c]\). Moreover, \( \int_0^\alpha g(t) \alpha da(t) = \int_0^\alpha t^{\alpha-1} g(t) \alpha dt \).

**Lemma 2.4.** [21] Assume that for some \( 0 < \alpha \leq 1 \), \( g \) is infinitely \( \alpha \)-differentiable at neighborhood of point \( t_0 \). Then conformable power series expansion of \( g \) is

\[ g(t) = \sum_{k=0}^{\infty} \frac{(T_\alpha^k g)(t_0)(t - t_0)^\alpha}{\alpha^k k!}, \quad t_0 < t < t_0 + R^\alpha, \quad R > 0. \]
Now, we prove the chain rule for a function of two variables for CD.

**Lemma 2.5.** Assume \( g : (0, \infty) \to \mathbb{R} \) and \( h : (0, \infty) \to \mathbb{R} \), which are \( \alpha \)-differentiable functions with \( 0 < \alpha \leq 1 \). Moreover, if \( i(x) = H(g(x), h(x)) \), then \( i(x) \) is \( \alpha \)-differentiable \( \forall x \), provided \( x \) is non-continuous, where \( x = \alpha \).

\[
T_{\alpha} i(x) = \frac{\partial H(g(x))}{\partial g(x)} T_{\alpha} g(x) + \frac{\partial H(h(x))}{\partial h(x)} T_{\alpha} h(x),
\]

where \( T_{\alpha} \) denotes CD of order \( \alpha \).

**Proof.** Since \( i(x) = H(g(x), h(x)) \), from CD on \( i(x) \) and by chain rule for classical derivative gives (6).

Next, we give the proof of CD of an integrable function, named as Leibniz rule.

**Theorem 2.6.** Let \( g(x), h(x) \) and \( f(x, t) \) be \( \alpha \)-differentiable with \( g(x), h(x) \geq 0 \), \( f(x, t) \) continuous, where \( \alpha \in (0, 1] \). Moreover,

\[
F(x) = \int_{g(x)}^{h(x)} f(x, t) \, dt,
\]

then

\[
(T_{\alpha} F)(x) = f(x, h(x))(h(x))^{\alpha-1} T_{\alpha} h(x)
\]

\[
- f(x, g(x))(g(x))^{\alpha-1} T_{\alpha} g(x)
\]

\[
+ \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(x, t) \, dt,
\]

provided that \( T_{\alpha} \) be CD.

**Proof.** To deduce Eq. (8), let us define

\[
G_{\alpha}(u, x) = \int_{a}^{u} f(x, t) t^{\alpha-1} dt,
\]

and also,

\[
G_{\alpha}(w, x) = -\int_{a}^{w} f(x, t) t^{\alpha-1} dt
\]

where \( u = h(x), 0 \leq a \)

since

\[
F(x) = \int_{g(x)}^{h(x)} f(x, t) \, dt
\]

implies that

\[
F(x) = G_{\alpha}(x, w) + G_{\alpha}(u, x),
\]

then, applying conformable derivative and using Theorem 2.5, we obtain

\[
(T_{\alpha} F)(x) = f(x, u)u^{\alpha-1}u^{\alpha-1} - f(x, w)w^{\alpha-1}w^{\alpha-1}
\]

\[
+ \int_{w}^{u} \frac{\partial}{\partial x} f(x, t) \, dt
\]

\[
= f(x, h(x))(h(x))^{\alpha-1} T_{\alpha} h(x)
\]

\[
- f(x, g(x))(g(x))^{\alpha-1} T_{\alpha} g(x)
\]

\[
+ \int_{g(x)}^{h(x)} \frac{\partial}{\partial x} f(x, t) \, dt,
\]

for \( \alpha = 1 \) we obtain the classical formula. Hence, the required objective is achieved.

Next, the \( \alpha \)-distinguishability of the conformable linear system is given.

**Definition 2.7.** Two systems \( I_{1} \) and \( I_{2} \) are said to be \( \alpha \)-distinguishable on \( [0, T] \), if for \( (z_{10}, z_{20}, w(\cdot)) \neq 0 \)

\[
(z_{10}, z_{20}, w(\cdot)) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times L([0, T]; \mathbb{R}^{m}),
\]

such that their outputs \( y_{1}(\cdot) \) and \( y_{2}(\cdot) \) are not alike on an interval \([0, T]\).

**Definition 2.8.** Suppose \( 0 < T \), also consider \( \mathcal{W} \) be a function space such that \( \mathcal{W} \subseteq L([0, T]; \mathbb{R}^{m}) \).

If for \( (z_{10}, z_{20}, w(\cdot)) \neq 0 \)

\[
(z_{10}, z_{20}, w(\cdot)) \in \mathbb{R}^{n} \times \mathbb{R}^{m} \times \mathcal{W},
\]

such that the obtained outputs are not likewise on \([0, T]\). Then systems in (4) are called \( \mathcal{W} \) input \( \alpha \)-distinguishable on \([0, T]\).

In next discussion, we will consider \( \mathcal{W} \) to be one of the following classes:

\[
B_{\alpha}[0, T] = \left\{ w : [0, T] \rightarrow \mathbb{R}^{m} | w(t) = \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{j!} \right\}
\]

\(< \infty \) in an open interval including \([0, T]\).

\[
P_{\alpha}[0, T] = \left\{ w : [0, T] \rightarrow \mathbb{R}^{m} | w(t) = \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{j!} \right\},
\]

\(< \infty \) in an open interval including \([0, T]\).

\[
A_{\alpha}[0, T] = \left\{ w : [0, T] \rightarrow \mathbb{R}^{m} | w(t) = \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{j!} \right\},
\]

\(< \infty \) in an open interval including \([0, T]\).

\[
C_{\alpha}[0, T] = \left\{ w : [0, T] \rightarrow \mathbb{R}^{m} | w(t) \right\} \text{ is class of smooth functions}.
\]
Some conclusions drawn immediately are exactly as described in [11, Propositions 2.1 and 2.3]. For the next discussion, some properties of matrices of infinite dimensional and also the characteristics of linear algebraic equations, which are elaborated in ref. [11], are helpful.

3 New results

3.1 \(\alpha\)-Distinguishability for \(P_\alpha[0, T]\)

From now onward, we use the notations as

\[
K = \begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix}, \quad M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad N = [N_1 - N_2],
\]

and

\[
x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \quad Y(\cdot) = y_1(\cdot) - y_2(\cdot),
\]

then instantly from definition of \(\alpha\)-distinguishability, we obtain:

**Theorem 3.1.** \(\alpha\)-Distinguishability for \(P_\alpha[0, T]\) of \(I_1\) and \(I_2\) is independent of \(T\), where \(0 < T\). Equivalently, every submatrix consisting of left finite column vectors of \(M\) has FCR, where

\[
M = \begin{bmatrix} N & O & O & O \ldots \\ NK & NM & O & O \ldots \\ NK^2 & NKM & NM & O \ldots \\ \vdots & \vdots & \vdots & \ddots \\ NK^n & NK^{n-1}M & \ldots & NM \end{bmatrix}.
\]

More precisely, consider that

\[
M_n = \begin{bmatrix} N & O \ldots & O \\ NK & NM \ldots & O \\ NK^2 & NKM \ldots & O \\ \vdots & \ddots & \vdots \\ NK^n & NK^{n-1}M \ldots & NM \end{bmatrix},
\]

thus, \(I_1\) and \(I_2\) have \(\alpha\)-distinguishability for \(P_\alpha[n, 0, T]\) if and only if the matrix \(M_n\) has FCR. While \(I_1\) and \(I_2\) have \(\alpha\)-distinguishability for \(P_\alpha[0, T]\) iff for any \(n \geq 1\), \(M_n\) has FCR.

**Proof.** Assume \(w(\cdot) \in L^2([0, T]; \mathbb{R})\). Thus, the corresponding outputs of \(I_1\) and \(I_2\) with initial states \(z_{01}\) and \(z_{02}\) are

\[
y_1(t) = Ne^{K_{\alpha}d_1t}z_{01} + N_1 \int_0^t e^{K_{\alpha}d_1\tau}e^{-K_{\alpha}d_2(M_1w(\tau))d_3\tau}d\tau.
\]

\[
y_2(t) = Ne^{K_{\alpha}d_1t}z_{02} + N_2 \int_0^t e^{K_{\alpha}d_1\tau}e^{-K_{\alpha}d_2(M_2w(\tau))d_3\tau}d\tau.
\]

Thus,

\[
Y(t) = Ne^{K_{\alpha}d_1t}z_{01} + N \int_0^t e^{K_{\alpha}d_1\tau}e^{-K_{\alpha}d_2(M_1w(\tau))d_3\tau}d\tau.
\]

Let \(w \in P_\alpha[0, T]\):

\[
w(t) = \beta_0 + \beta_1 \frac{t^a}{a!} + \beta_2 \frac{t^{2a}}{2!a!} + \cdots + \beta_q \frac{t^{qa}}{a!Q!},
\]

where \(\beta_j \in \mathbb{R}^m\). Then \(y_1(\cdot)\) and \(y_2(\cdot)\) are conformable differentiables. Therefore, \(y_1(t) \equiv y_2(t)\), on \([0, T]\)

holds if and only if

\[
Y(\tau(0)) = 0, \quad \forall j = 0, 1, 2, \ldots
\]

this implies

\[
Y(\tau) = Nz_{01}.
\]

First determining,

\[
Y_{\alpha}(t) = NK_{\alpha}d_1z_{01} + NMw(t) + N \int_0^t e^{K_{\alpha}d_1\tau}e^{KMw(\tau)}d\tau.
\]

then, at \(t = 0\)

\[
Y_{\alpha}(0) = NK_{\alpha}d_1z_{01} + NMw(0).
\]

Likewise determining all the CD of Eqs. (15) and (16) for higher orders, we obtain (19), which is equivalent to

\[
M_{\alpha}[z_0; \beta_0; \beta_1; \ldots; \beta_N] = 0,
\]

while

\[
M_{\alpha} = \begin{bmatrix} N & O & O \ldots \\ NK & NM & O \ldots \\ NK^2 & NKM & NM \ldots \\ \vdots & \ddots & \vdots \\ NK^n & NK^{n-1}M \ldots & NM \end{bmatrix}
\]

\[
[z_0; \beta_0; \beta_1; \ldots; \beta_N] = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix}
\]

Thereof, \(\alpha\)-distinguishability for \(P_\alpha[n, 0, T]\) of \(I_1\) and \(I_2\) is equivalent to that (4) has unique solution, that is, \(M_{\alpha}\) has FCR.

**Corollary 3.2.** \(\alpha\)-Distinguishability for \(P_\alpha[0, T]\) of \(I_1\) and \(I_2\) implies \(2n \geq m\).

3.2 \(\alpha\)-Distinguishability for \(A_\alpha[0, T]\)

Similar to Theorem 3.1, we can obtain the following result:
Theorem 3.3. The infinite dimensional equation as follows:
\[
M[z_0; \beta_0; \beta_i; \ldots] = 0, \quad (23)
\]
which has only unique solution such that
\[
w(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \beta_j < \infty \quad \forall t \in [0, T],
\]
which is equivalent to the \(\alpha\)-distinguishability for \(A_n[0, T]\) of \(I_1\) and \(I_2\).

Theorem 3.4. For \(m > k\) and \(NM\) has full row rank \(k\), which implies that \(I_1\) and \(I_2\) are not \(\alpha\)-distinguishable for \(A_n[0, T]\).

Corollary 3.5. \(NM \neq 0\) and \(N, M\) are \(1 \times 2n\) matrices, which implies \(I_1\) and \(I_2\) are \(\alpha\)-distinguishable for \(A_n[0, T]\).

Theorem 3.6. Both the systems \(I_1\) and \(I_2\) are not \(\alpha\)-distinguishable for \(A_n[0, T]\) if \(k < m\).

Example 3.7. Consider the hybrid system with \(m = 2\) and \(k = 1\). And \(NM\) is calculated as
\[
NM = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \end{bmatrix},
\]
which implies that \(NM\) has 1 as a full row rank. Hence, from Theorems 3.4 and 3.6, it is clear that \(I_1\) and \(I_2\) are not \(\alpha\)-distinguishable for \(A_n[0, T]\).

As a corollary of [11, Lemma 4.6], we obtain a more useful result which is mentioned as follows:

Theorem 3.8. \(\alpha\)-Distinguishability for \(A_n[0, T]\) of \(I_1\) and \(I_2\) on an interval \([0, T]\) is equivalent to that \((23)\) has a unique solution. Also, it does not depend on \(T\).

Theorem 3.9. Consider that \(NM\) has FCR \(m\) and if \(k = m\), then \(I_1\) and \(I_2\) \(\alpha\)-distinguishable for \(A_n[0, T]\) if only if \(N\) has FCR.

### 3.3 \(\alpha\)-Distinguishability for \(C_\alpha^\infty[0, T]\)

In this section, we relate \(\alpha\)-distinguishability for \(A_n[0, T]\) to \(\alpha\)-distinguishability for \(C_\alpha^\infty[0, T]\).

Theorem 3.10. \(I_1\) and \(I_2\) have \(\alpha\)-distinguishability for \(A_n[0, T]\) if and only if \(I_1\) and \(I_2\) have \(\alpha\)-distinguishability for \(C_\alpha^\infty[0, T]\).

Proof. It is enough to prove the following:
Let \(w \in C_\alpha^\infty[0, T]\),
\[
Y(t) = y_1(t) - y_2(t) \equiv 0 \quad \text{on an interval } [0, T], \quad (24)
\]
with
\[
y_1(t) = y_1(c; z_0; w(\cdot)), \quad y_2(t) = y_2(c; z_0; w(\cdot)).
\]
We have to prove
\[
z_0 = 0
\]
and
\[
w(t) \equiv 0.
\]
Eq. (24) yields that
\[
Y^{(j)}(t) \equiv 0, \quad \text{for all } j = 0, 1, 2, \ldots.
\]
Hence, similar to \((23)\), we have
\[
M[z_0(t); z_0(t); w(t); w^{(\alpha)}(t); w^{(2\alpha)}(t); \ldots] = 0,
\]
with
\[
z_0(t) = e^{K_\alpha z_0} + \int_0^t e^{K_\alpha(t-s)w(s)}ds, \quad z_0(t) = e^{K_\alpha z_0} + \int_0^t e^{K_\alpha(t-s)w(s)}ds,
\]
where \(w^{(\alpha)}(t), w^{(2\alpha)}(t)\) denote the CFD of order \(\alpha\) and \(2\alpha\), respectively. Thus, from Theorem 3.3, the \(\alpha\)-distinguishability for \(A_n[0, T]\) gives
\[
[z_0(t); z_2(t); w(t); w^{(\alpha)}(t); w^{(2\alpha)}(t); \ldots] = 0, \quad \forall t \in [0, T].
\]
Particularly,
\[
w(t) = 0, \quad \forall t \in [0, T]
\]
and
\[
z_0(0) = z_2(0) = 0.
\]
Hence, the proof is completed.

Example 3.11. Consider a conformable HLS composed of subsystems \(I_1\) and \(I_2\) with
\[
K = \begin{bmatrix} 4 & 3 & 5 \\ 9 & 63 & 1 \\ 0 & 5 & 24 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 60 & 0 \\ 23 & 0 & 0 \end{bmatrix},
\]
\[
N = \begin{bmatrix} 0 & 8 & 0 \\ 75 & 0 & 0 \\ 0 & 0 & 91 \end{bmatrix}, \quad S = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 1 & 3 & 2 \end{bmatrix}
\]
where \(NM\) has FCR and \(k = m = 3\), Theorem 3.9 implies that \(I_1\) and \(I_2\) are \(\alpha\)-distinguishability for \(A_n[0, T]\). From
Theorem 3.10, $I_1$ and $I_2$ have $a$-distinguishability for $C_a^{\mathbb{C}}[0, T]$.

4 More consequences

Consider that
\[
P(\lambda) = b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + b_0,
\]
with $\lambda \in \mathbb{C}$ and $b_k \in \mathbb{C}$. For $g : [0, T] \to \mathbb{C}^n$, take
\[
P(T_a)h(t) = b_n \frac{d^n}{dt^n} h(t) + b_{n-1} \frac{d^{n-1}}{dt^{n-1}} h(t) + \cdots + b_1 h(t) + b_0 h(t).
\]

Next, a lemma explains some more properties:

Lemma 4.1. Consider $4$ More consequences.

Proof. Given that $P_k(\cdot)$ be a polynomial. Suppose
\[
P_k(\cdot) = \sum_{k=0}^{\infty} b_k t^k.
\]

Since $g(\cdot)$ is
\[
g(t) = e^{\lambda t} P(t) + e^{\lambda t} P_2(t) + \cdots + e^{\lambda t} P_n(t).
\]

Applying fractional form of Laplace transform of (30) and by linearity of Laplace transform and determining Laplace of every term by using [22, Theorem 2.4] implies
\[
\mathcal{L}_a[g(t)] = b_a \mathcal{L} \frac{\Gamma(1 + \frac{1}{a})}{(s - \lambda_1)^{1/a}} + b_{2a} \mathcal{L} \frac{\Gamma(1 + \frac{1}{a})}{(s - \lambda_2)^{1/a}} + \cdots + b_{na} \mathcal{L} \frac{\Gamma(1 + \frac{1}{a})}{(s - \lambda_n)^{1/a}},
\]

It is obvious from Eq. (31) that $\mathcal{L}_a[g(t)]$ is PRF. Now, suppose that $\mathcal{L}_a[g(t)]$ is PRF, we show that $g(t)$ has the form of (28). Applying fractional form of Laplace inverse transform on (31), we obtain
\[
g(t) = e^{\lambda t} P(t) + e^{\lambda t} P_2(t) + \cdots + e^{\lambda t} P_n(t),
\]
while
\[
P(t) = b_1 t + b_2 t^2 + \cdots + b_n t^n.
\]

Hence, the proof is completed.

Lemma 4.2. Consider $g : [0, T] \to \mathbb{C}^n$, $\mathcal{L}_a(g)$ is a proper rational function (PRF) implies
\[
g(t) = e^{\lambda t} P(t) + e^{\lambda t} P_2(t) + \cdots + e^{\lambda t} P_n(t),
\]

with $\lambda_k \in \mathbb{C}$ and $P_k(\cdot)$ is a polynomial where $k = 1, 2, \ldots$.
There is a lemma having a characteristic as follows:

**Lemma 4.3.** If $I_1$ and $I_2$ are not $a$-distinguishable, then there exists $(\hat{Z}_0, \hat{w}(\cdot))$ satisfying (32) and (33) with

$$\hat{w}(\cdot) = e^{\lambda t}P_1(t) + e^{\lambda t}P_2(t) + \cdots + e^{\lambda t}P_m(t).$$

**Lemma 4.4.** If $I_1$ and $I_2$ are not $a$-distinguishable, then there exists $(\hat{Z}_0, \hat{w}(\cdot))$, which satisfies Eq. (33)

$$\hat{w}(\cdot) = e^{\lambda t}\xi,$$

with $\xi \in \mathbb{C}^m$.

**Corollary 4.5.** 0th polynomial input $a$-distinguishability of $I_1$ and $I_2$ implies $a$-distinguishability for $P_0[0, T]$.

From Theorem 3.1, the equivalent criterion for

$$F_1 = \begin{bmatrix} N & S & O & \cdots & O \\ NK & NM & S & \cdots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ NK^q & NK^{q+1}M & NK^qM & \cdots & S \\ \vdots & \vdots & \cdots & \cdots & \vdots \end{bmatrix}$$

for any $q \geq 0$, to have FCR is that

$$F_2 = \begin{bmatrix} N & S \\ NK & NM \\ NK^2 & NK^3M \\ \vdots & \vdots \\ NK^{2n} & NK^{2n-1}M \end{bmatrix}$$

has FCR. Hence, Cayley-Hamilton’s theorem results that $F_1$ and $F_2$ are equal to the following:

$$F_3 = \begin{bmatrix} N & S \\ NK & NM \\ NK^2 & NK^3M \\ \vdots & \vdots \\ NK^{2n} & NK^{2n-1}M \end{bmatrix},$$

which has FCR.

In the end of this article, we state our main consequence where proof is exactly as in classical case mentioned in ref. [13].

**Lemma 4.6.** Two systems $I_1$ and $I_2$ to be $a$-distinguishable if and only if for any $\lambda \in \mathbb{C}$,

$$\hat{M}_a = \begin{bmatrix} N & S \\ N(K - \lambda I) & NK \\ N(K - \lambda I)^2 & N(K - \lambda I)M \\ \vdots & \vdots \\ N(K - \lambda I)^{2n} & N(K - \lambda I)^{2n-1}M \end{bmatrix},$$

which has FCR.

Next, we give the result which is equivalent to previous one.

**Theorem 4.7.** Assume that $\hat{M}_a$ be a matrix which is defined in (37). This $\hat{M}_a$ has FCR for any $\lambda$ belongs to $\mathcal{C}$ if and only if

$$\begin{bmatrix} N & S \\ K - \lambda I & M \end{bmatrix},$$

which has FCR.

**Example 4.8.** Consider

$$T_0z(t) = Kz(t) + Mw(t), \quad y(t) = Nz(t) + Sw(t),$$

be a fractional linear system with

$$Z(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

where

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad M = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} N & S \\ K - \lambda I & M \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 - \lambda & 0 & 3 \\ 0 & 2 - \lambda & 2 \end{bmatrix}.$$

For any value of $\lambda$, FCR of the matrix in Eq. (39) implies FCR of $\hat{M}_a$. Hence, the considered system is $a$-distinguishable.

### 5 Conclusion

In this article, we have proved a formula for CD of an integral function, which is called Leibnitz rule. Moreover, distinguishability idea for conformable HLSs is discussed. Some equivalent criteria are developed, which seems complicated to verify. To resolve this issue, necessary and sufficient condition of distinguishability for conformable HLS is also established with the help of conformable Laplace techniques, which are easy to verify. In future, we can study how the distinguishability work in real-life problems. We can analyze the concept of weak distinguishability for fractional HLSs. We can elaborate the idea of weak distinguishability with distinguishability for fractional linear systems.

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