Research Article

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Novel soliton structures of truncated M-fractional (4+1)-dim Fokas wave model

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Abstract: In this research article, a nonlinear time–space fractional order (4+1)-dim Fokas wave equation that is crucial for examining the corporeal marvels of waves on and inside the surface of water is examined. For this purpose, a well-known analytical method is utilized, namely, the Sardar sub-equation (SSE) method along with a truncated M-fractional derivative. As a result, many new families of solitary wave solutions, such as kink-type solitons, singular and periodic solitons, dark and bright solitons, are established. By using the SSE method, the outcomes are portrayed in 3-dim, 2-dim, and contour plots for distinct parametric values. The obtained hyperbolic and trigonometric function-type results demonstrate the capability of recognizing the exact solutions of the other nonlinear evolution equations through the executed technique.

Keywords: truncated M-fractional derivative, (4+1)-dim Fokas wave model, Sardar sub-equation method, soliton solutions

1 Introduction

In the fields of mathematics and optical physics, the nonlinear evolution equation (NLEE) has received special attention from many researchers because of its wide scope of characteristics [1–8]. A soliton, or solitary wave, is created by eliminating nonlinear and dispersion effects from the medium so that it propagates at a constant velocity without any change in its shape [9–13]. Recently, extensive research by several scientists and mathematicians has established efficient approaches to finding exact solutions in the form of solitons and solitary waves of fractional NLEEs (F-NLEEs) [14–32].

In recent decade, Fokas derived the integer order (4+1)-dim wave equation [33] by extending the Lax pairs of the Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations. In nonlinear wave theory, the DS equation is proposed for the development of 3-dim waves on water of finite depth [34] and the KP equation is proposed to depict the internal waves and surface waves in channels with changing width and depth [35]. Thus, the Fokas model equation follows the physical behaviour of the DS and KP equations. The nonlinear Fokas wave model [36] is applied in many fields such as photonics, optics, plasmas, hydrodynamics, quantum mechanics, fluid mechanics, and solid-state physics. The neutral of the research article is to propose an effective technique, namely, the Sardar sub-equation (SSE) method [37–40] and attain the solitary wave solutions of the time–space fractional (4+1)-dim Fokas dynamical model along with truncated M-fractional derivative [41–43] read as follows:

\[ 4D_{M,x}^\alpha D_{M,y}^\beta p - D_{M,x}^\beta D_{M,y}^\alpha p + D_{M,x}^\alpha D_{M,y}^\beta p + 12D_{M,y}^\alpha p D_{M,y}^\beta p + 12p D_{M,y}^\alpha p D_{M,y}^\beta p - 6D_{M,y}^\alpha p D_{M,y}^\beta p = 0, \]  

where \( 0 < \alpha < 1 \). \( p \) is the horizontal velocity of wave and an unknown analytical function of \((x, y, z, w, t)\). Eq. (1) becomes an integer-order nonlinear Fokas wave model for \( \alpha = 1 \). We expect that in complex media, Eq. (1) will clarify the motion of waves. This will possibly catch the distortion or dispersion of the surface wave. Accordingly, various types of solutions to the fractional nonlinear Fokas model will accord us insights into the complex phenomena of the DS and KP equations. Given the significance of Eq. (1), it is possible to study the concept of complexifying time in the context of contemporary field theories by using integrable nonlinear equations in four spatial dimensions.
that involve complex time. Besides, in the light of the referenced realities about fractional derivatives and the significant utilization of the nonlinear Fokas model, it is important to investigate its novel soliton structures.

The key motivation of this current work is to apply the new concept of local fractional derivatives, the so-called truncated M-fractional derivative, to the time-space fractional order (4+1)-dim Fokas wave equation. To the best of our knowledge, the truncated M-fractional derivative has not been applied to the considered model in the literature. The wave solutions for many types of fractional-nonlinear partial differential equations (F-NLPDEs) have been extensively found in the literature using the truncated M-fractional derivative. The literature lacks some types of solutions to the (4+1)-dim Fokas wave equation, including elliptic functions, bright and dark solitons, combination bright-dark solitons, singular solitary waves, periodic solutions, especially solutions in general form, and many others. In a bid to fill this hole, we use the most appropriate approach to identify exact solutions that are not already available in the literature, as well as to examine several other solitontype solutions. Consequently, our proposed method along with local fractional derivative is capable of finding such solutions.

This research article is structured as follows: in Section 2, mathematical analysis of Eq. (1) is developed. In Section 3, the geometric behavior of the outcomes is described. Finally, the conclusion is given in Section 4.

2 Mathematical analysis of Fokas wave model via SSE method

Let us assume \( p(x, y, z, w, t) = Q(\chi) \), where

\[
\chi = \frac{\Gamma(q + 1)}{a} y^a + \frac{\Gamma(q + 1)}{a} b^a + \frac{\Gamma(q + 1)}{a} - \beta_1 y^a + \frac{\Gamma(q + 1)}{a} - \beta_2^a - \beta_3 y^a + \frac{\Gamma(q + 1)}{a} - \beta_4 y^a,
\]

where \( y \) is the frequency and \( \beta_1, \beta_2, \beta_3, \beta_4 \) are the lengths of wave. Plugging the given transformation Eq. (2) in Eq. (1), we obtain

\[
\beta_2^{\beta_1^2 - \beta_2^2} - \beta_3^{Q''} (4\beta_1 y + 6\beta_2 b_3)Q - 6\beta_2 b_3 Q^2 = 0.
\]

According to SEE method, by balancing \( Q'' \) and \( Q^2 \), we obtain \( n = 2 \). Therefore, the general solution will take the form:

\[
Q = \Omega_0 + \Omega_1 M(\chi) + \Omega_2 M^2(\chi),
\]

where \( M(\chi) \) satisfies the following auxiliary ordinary differential equation:

\[
M'(\chi) = \sqrt{\rho + bM^2(\chi) + M(\chi)},
\]

After differentiating Eq. (4) twice, we attain

\[
Q''(\chi) = \Omega_1 \sqrt{\rho + bM^2(\chi) + M(\chi)} + 2\Omega_2 M(\chi) \sqrt{\rho + bM^2(\chi) + M(\chi)},
\]

\[
Q'''(\chi) = 6\Omega_2 M^4(\chi) + 2\Omega_2 M^3(\chi) + 4\Omega_2^2 M^2(\chi) + \Omega_2 M(\chi) + 2\rho \Omega_2.
\]

where \( \rho \) and \( b \) are the constants that will be determined later. Substituting Eq. (4) in Eq. (3) along with Eqs (5)–(7) and then equating the coefficients of \( M(\chi) \), \( i = 0, 1, 2, 3, 4 \) to zero, we obtain

\[
M^0(\chi) = -6\Omega_2^2 \beta_2 - 6\Omega_2^2 \beta_4,
\]

\[
M^1(\chi) = 4\beta_1 \beta_2 (\beta_1^2 - \beta_2^2) \Omega_1 + (4\beta_1 + 6\beta_2) \Omega_2 - 12\beta_2 \beta_3 \Omega_1,
\]

\[
M^2(\chi) = 4\beta_1 \beta_2 (\beta_1^2 - \beta_2^2) \Omega_2 + (4\beta_1 + 6\beta_2) \Omega_2 - 6\beta_2 \beta_3 (2\Omega_1 - \Omega_2),
\]

\[
M^3(\chi) = 2\beta_2 \beta_3 (\beta_1^2 - \beta_2^2) \Omega_1 - 12\beta_2 \beta_3 \Omega_2,
\]

\[
M^4(\chi) = 6\beta_2 \beta_3 (\beta_1^2 - \beta_2^2) \Omega_2 - 6\beta_2 \beta_3 \Omega_2^2.
\]

Solving the above algebraic system of equations using the Maple software, we obtain two various families.

Family 1:

\[
\Omega_0 = 0, \quad \Omega_1 = 0, \quad \Omega_2 = \beta_1^2 - \beta_2^2,
\]

\[
y = \frac{-2b^2 \beta_1 \beta_4 + 2b^2 \beta_2^3 - 2b^2 \beta_4}{2\beta_1}.
\]

Case 1.1 If \( b > 0 \) and \( \rho = 0 \), then

\[
p_{1,1}^5(\chi) = (\beta_1^2 - \beta_2^2)(\pm \sqrt{b} \beta_1 \text{sech}_{\text{lin}}(\sqrt{b} \chi))^2,
\]

\[
p_{1,1}^6(\chi) = (\beta_1^2 - \beta_2^2)(\pm \sqrt{b} \beta_1 \text{csch}_{\text{lin}}(\sqrt{b} \chi))^2,
\]

where

\[
\text{sech}_{\text{lin}}(\chi) = \frac{2}{\text{le}^x + \text{me}^{-x}}, \quad \text{csch}_{\text{lin}}(\chi) = \frac{2}{\text{le}^x - \text{me}^{-x}}.
\]

Case 1.2 If \( b < 0 \) and \( \rho = 0 \), then

\[
p_{1,1}^5(\chi) = (\beta_1^2 - \beta_2^2)(\pm \sqrt{-\beta} \beta_1 \text{sech}_{\text{lin}}(\sqrt{-\beta} \chi))^2,
\]

\[
p_{1,1}^6(\chi) = (\beta_1^2 - \beta_2^2)(\pm \sqrt{-\beta} \beta_1 \text{csch}_{\text{lin}}(\sqrt{-\beta} \chi))^2,
\]

where

\[
\text{sec}_{\text{lin}}(\chi) = \frac{2}{\text{le}^x + \text{me}^{-x}}, \quad \text{csc}_{\text{lin}}(\chi) = \frac{2}{\text{le}^x - \text{me}^{-x}}.
\]
Case 1.3 If \( b < 0 \) and \( \rho = \frac{b^2}{4} \), then
\[
p_{1,15}(\chi) = (\beta_1^2 - \beta_2^2) \left[ \pm b \sqrt{-\frac{b}{2}} \tanh \left( \sqrt{-\frac{b}{2}} \chi \right) \right]^2,
\]
\[
p_{1,16}(\chi) = (\beta_1^2 - \beta_2^2) \left[ \pm b \sqrt{-\frac{b}{2}} \coth \left( \sqrt{-\frac{b}{2}} \chi \right) \right]^2,
\]
\[
p_{1,17}(\chi) = (\beta_1^2 - \beta_2^2) \times \left[ \pm b \sqrt{-\frac{b}{2}} (\tanh \left( \sqrt{-2b} \chi \right) \pm \sqrt{\text{im} \, \text{sech}_{\text{Im}} \left( \sqrt{-2b} \chi \right)} \right]^2 \times \left[ \pm b \sqrt{-\frac{b}{2}} (\coth \left( \sqrt{-2b} \chi \right) \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{-2b} \chi \right)} \right]^2,
\]
where
\[
\tanh_{\text{Im}}(\chi) = \frac{\text{le}^\chi - \text{me}^\chi}{\text{le}^\chi + \text{me}^\chi}, \quad \coth_{\text{Im}}(\chi) = \frac{\text{le}^\chi + \text{me}^\chi}{\text{le}^\chi - \text{me}^\chi}.
\]

Case 1.4 If \( b > 0 \) and \( \rho = \frac{b^2}{4} \), then
\[
p_{1,10}(\chi) = (\beta_1^2 - \beta_2^2) \left[ \pm \frac{b}{\sqrt{2}} \tan_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \right]^2,
\]
\[
p_{1,11}(\chi) = (\beta_1^2 - \beta_2^2) \left[ \pm \frac{b}{\sqrt{2}} \cot_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \right]^2,
\]
\[
p_{1,12}(\chi) = (\beta_1^2 - \beta_2^2) \times \left[ \pm \frac{b}{\sqrt{2}} \left( \tanh_{\text{Im}} \left( \sqrt{2b} \chi \right) \pm \sqrt{\text{im} \, \text{sec}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2 \times \left[ \pm \frac{b}{\sqrt{2}} \left( \coth_{\text{Im}} \left( \sqrt{2b} \chi \right) \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2,
\]
\[
p_{1,13}(\chi) = (\beta_1^2 - \beta_2^2) \times \left[ \pm \frac{b}{\sqrt{2}} \left( \tanh_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{sec}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2 \times \left[ \pm \frac{b}{\sqrt{2}} \left( \coth_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2,
\]
where
\[
\tanh_{\text{Im}}(\chi) = \frac{\text{le}^\chi - \text{me}^\chi}{\text{le}^\chi + \text{me}^\chi}, \quad \coth_{\text{Im}}(\chi) = \frac{\text{le}^\chi + \text{me}^\chi}{\text{le}^\chi - \text{me}^\chi}.
\]

Family 2:
\[
\Omega_0 = \frac{2}{3} b (\beta_1^2 - \beta_2^2), \quad \Omega_1 = 0, \quad \Omega_2 = \beta_1^2 - \beta_2^2,
\]
\[
y = \frac{2 b \beta_1^2 \beta_2}{2 \beta_1}.
\]

Case 2.1 If \( b > 0 \) and \( \rho = 0 \), then
\[
p_{2,1}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{sech}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2, \quad p_{2,2}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2,
\]
where
\[
\text{sech}_{\text{Im}}(\chi) = \frac{2}{\text{le}^\chi + \text{me}^\chi}, \quad \text{csch}_{\text{Im}}(\chi) = \frac{2}{\text{le}^\chi - \text{me}^\chi}.
\]

Case 2.2 If \( b < 0 \) and \( \rho = 0 \), then
\[
p_{2,1}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{sec}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2, \quad p_{2,2}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{2b} \chi \right)} \right)^2,
\]
where
\[
\text{sec}_{\text{Im}}(\chi) = \frac{2}{\text{le}^\chi - \text{me}^\chi}, \quad \text{csch}_{\text{Im}}(\chi) = \frac{2i}{\text{le}^\chi + \text{me}^\chi}.
\]

Case 2.3 If \( b < 0 \) and \( \rho = \frac{b^2}{4} \), then
\[
p_{2,1}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{sec}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2 \times \left[ \pm \sqrt{\frac{b}{2}} \left( \tanh_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{sech}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2 \times \left[ \pm \sqrt{\frac{b}{2}} \left( \coth_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2,
\]
\[
p_{2,2}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) \left( \beta_1^2 - \beta_2^2 \right) \left( \pm \sqrt{\text{im} \, \text{sec}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2 \times \left[ \pm \sqrt{\frac{b}{2}} \left( \tanh_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{sech}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2 \times \left[ \pm \sqrt{\frac{b}{2}} \left( \coth_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right) \pm \sqrt{\text{im} \, \text{csch}_{\text{Im}} \left( \sqrt{\frac{b}{2}} \chi \right)} \right)^2,
\]
where
\[
\tanh_{\text{Im}}(\chi) = \frac{\text{le}^\chi - \text{me}^\chi}{\text{le}^\chi + \text{me}^\chi}, \quad \coth_{\text{Im}}(\chi) = \frac{\text{le}^\chi + \text{me}^\chi}{\text{le}^\chi - \text{me}^\chi}.
\]
where
\[ \tanh_{\text{lin}}(\chi) = \frac{\text{le}^\chi - \text{me}^{-\chi}}{\text{le}^\chi + \text{me}^{-\chi}}, \coth_{\text{lin}}(\chi) = \frac{\text{le}^\chi + \text{me}^{-\chi}}{\text{le}^\chi - \text{me}^{-\chi}}. \]

Case 2.4 If \( b < 0 \) and \( \rho = \frac{b^4}{4} \), then
\[ p_{x,10}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) + (\beta_1^2 - \beta_2^2) \left[ \pm \sqrt{\frac{b}{2}} \tan_{\text{lin}} \left( \frac{b}{\sqrt{2} \chi} \right) \right]^2, \]
\[ p_{x,11}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) + (\beta_1^2 - \beta_2^2) \left[ \pm \sqrt{\frac{b}{2}} \cot_{\text{lin}} \left( \frac{b}{\sqrt{2} \chi} \right) \right]^2, \]
\[ p_{x,12}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) + (\beta_1^2 - \beta_2^2) \left[ \pm \sqrt{\frac{b}{2}} \coth_{\text{lin}} \left( \frac{\sqrt{b}}{\sqrt{2} \chi} \right) \right]^2 + \left[ \pm \sqrt{\frac{b}{2}} \sec_{\text{lin}} \left( \frac{\sqrt{b}}{\sqrt{2} \chi} \right) \right]^2, \]
\[ p_{x,13}(\chi) = \frac{2}{3} b (\beta_1^2 - \beta_2^2) + (\beta_1^2 - \beta_2^2) \left[ \pm \sqrt{\frac{b}{2}} \tan_{\text{lin}} \left( \frac{b}{\sqrt{2} \chi} \right) + \cot_{\text{lin}} \left( \frac{b}{\sqrt{2} \chi} \right) \right]^2, \]
where
\[ \tan_{\text{lin}}(\chi) = \frac{\text{le}^\chi - \text{me}^{-\chi}}{\text{le}^\chi + \text{me}^{-\chi}}, \cot_{\text{lin}}(\chi) = \frac{\text{le}^\chi + \text{me}^{-\chi}}{\text{le}^\chi - \text{me}^{-\chi}}. \]

3 Graphical discussion

A computational software Mathematica is used to envision the soliton’s way of behaving with the achieved results.

In Figures 1–4, (a) represents the 3-dim plot, (b) represents the contour plot, and (c) represents the 2-dim plot of the solution for different values of the parameters.

Figure 1 shows the solution of the Fokas wave model for Case 1.2. One can see that \(|p_{x,10}(\chi)|\) is a singular soliton for \( b = -1.5, l = 0.98, m = 0.95, a = 0.75, \) and \( \eta = 0.5 \).

Figure 2 shows the solution of the Fokas wave model for Case 1.4. One can see that \(|p_{x,11}(\chi)|\) is a singular soliton for \( b = 0.5, l = 0.98, m = 0.95, a = 0.25, \) and \( \eta = 0.75 \).

Figure 3 shows the solution of the Fokas wave model for Case 2.1. One can see that \(|p_{x,12}(\chi)|\) is a dark soliton for \( b = 1, l = 0.98, m = 0.95, a = 0.75, \) and \( \eta = 0.5 \).

Figure 4 shows the solution of the Fokas wave model for Case 2.3. One can see that \(|p_{x,13}(\chi)|\) is a bright soliton for \( b = -0.5, l = 0.98, m = 0.95, a = 0.5, \) and \( \eta = 1.25 \).

4 Conclusion

In this research article, the SSE method has been employed to attain exact and solitary wave solutions for the fractional order (4+1)-dim Fokas wave model by means of the M-truncated derivative. An assortment of various structures for these solutions was recovered in the form of trigonometric and hyperbolic functions. Our outcomes exhibited that the projected technique is effective, compact, and can be utilized on other high-dimensional F-NLEEs. All the solutions were confirmed with the help of the computational software Mathematica. Furthermore, by the choice of some appropriate parametric values, we visualized the graphical conduct of some gained outcomes in 3-dim, 2-dim, and counter plots. These plots show different kinds of bright, dark, and singular solitons. This research could be helpful for investigations of the other
Figure 2: $|p_{1,10}(\chi)|$ with $y = 0.5, z = 0.25, w = 0.5, \beta_1 = 1.5, \beta_2 = 1.25, \beta_3 = 0.75, \beta_4 = 2.5$ for (a) the 3-dim plot, (b) the contour plot, and (c) the 2-dim plot.

Figure 3: $|p_{1,25}(\chi)|$ with $y = 0.25, z = 0.5, w = 1.25, \beta_1 = 1.5, \beta_2 = 1.5, \beta_3 = 0.5, \beta_4 = 0.5$ for (a) the 3-dim plot, (b) the contour plot, and (c) the 2-dim plot.

Figure 4: $|p_{2,6}(\chi)|$ with $y = 0.25, z = 0.5, w = 0.75, \beta_1 = 1.25, \beta_2 = 1.5, \beta_3 = 1.75, \beta_4 = 2$ for (a) the 3-dim plot, (b) the contour plot, and (c) the 2-dim plot.
high-dimensional fractional equations in NL wave theory, such as those in plasma, photonics, quantum gases, optics, hydrodynamics, and solid-state physics.

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**Conflict of interest:** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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