Research Article

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Traveling wave solutions, numerical solutions, and stability analysis of the (2+1) conformal time-fractional generalized q-deformed sinh-Gordon equation

https://doi.org/10.1515/nleng-2022-0348
received July 14, 2023; accepted October 13, 2023

Abstract: The two-dimensional conformal time-fractional generalized q-deformed sinh-Gordon equation has been used to model a variety of physical systems, including soliton propagation in asymmetric media, nonlinear waves in optical fibers, quantum field theory, and condensed matter physics. The equation is able to capture the complex dynamics of these systems and has been shown to be a powerful tool for studying them. This article discusses the two-dimensional conformal time-fractional generalized q-deformed sinh-Gordon equation both analytically and numerically using Kudryashov’s approach and the finite difference method. In addition, the stability analysis and local truncation error of the equation are discussed. A number of illustrations are also included to show the various solitons propagation patterns. The proposed equation has opened up new possibilities for modeling asymmetric physical systems.

Keywords: two-dimensional conformal time-fractional equation, the finite difference method, the Kudryashov’s technique, local truncation error, the stability analysis

1 Introduction

In mathematics and physics, q-deformed (or quantum deformed) objects refer to a family of objects, such as functions, operators, or algebras, that depend on a deformation parameter q. The q-deformed objects are a generalization of their undeformed counterparts, which arise when q is set to 1. The deformation parameter q can be a complex number, but in many applications, it is a root of unity or a parameter that interpolates between different values. The theory of q-deformed objects plays a crucial role in many areas of mathematics and physics, including quantum mechanics, statistical mechanics, knot theory, and combinatorics. The q-deformed objects have been extensively studied, and many of their properties and applications have been established. In this context, the study of the q-deformed sinh-Gordon equation, which is a q-deformed version of the classical sinh-Gordon equation, has attracted considerable attention.

Although dynamical models are fundamental to many scientific fields, they are often overlooked in the literature. Nonlinear differential equations are employed to describe dynamics in microscopical quantum systems, where nonlinearity plays a crucial role [1,2]. Numerous researchers have explored various partial differential equations with significant applications in diverse fields [3–8].

The (1+1) generalized q-deformed sinh-Gordon equation clarifies as follows [9,10]:

\[
\frac{\partial^{2\alpha} F}{\partial x^{2\alpha}} - \frac{\partial^{2\alpha} F}{\partial t^{2\alpha}} = \left[\sinh_q(z^\alpha)\right]^{m} - W.
\]

(1.1)

The constants \(\kappa\) and \(m\) are used, with \(0 < q < 1\). The traditional sinh-Gordon equation is retrieved when \(q = \kappa = m = 1\).

This investigation pertains to the generalized q-deformed sinh-Gordon equation with (2+1) conformal time derivatives, also known as the Eleuch equation. Equation (1.1) can be expressed in (2+1) conformal time-fractional form as follows:

\[
\frac{\partial^{2\alpha} F}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} F}{\partial y^{2\alpha}} - \frac{\partial^{2\alpha} F}{\partial z^{2\alpha}} = \left[\sinh_q(z^\alpha)\right]^{4} - W
\]

(1.2)

where \(0 < \alpha \leq 1\).

Using the Kudryashov’s technique [11], we investigate the solutions describing optical solitons of Eq. (1.2). In addition, we employ the finite difference method to obtain
numerical solutions for this equation [12,13]. We consider two possibilities for this equation.

The research is structured as follows: Section 2 presents the mathematical analysis of the model; Section 3 outlines the methodology employed; the solutions are presented in Section 4; Section 5 provides numerical solutions, stability analysis, and local truncation error of the equation; various figures for some solutions are presented in Section 6; and finally, Section 7 contains the conclusion.

2 Analyzing the model mathematically

Use the transformation below to obtain the solution describing a traveling wave for Eq. (1.1).

\[ \mathcal{F}(x, y, \tau) = u(\mathcal{F}), \]

where

\[ \mathcal{F} = \partial x + \lambda y - \mathfrak{B} \frac{x^a}{a}, \]

and \( \mathfrak{B} \) is the speed of the traveling wave. By using Eqs. (2.1) and (2.2) subsequently, Eq. (1.2) can be expressed as follows:

\[ (\partial^2 + \lambda^2 - \mathfrak{B}^2)u''(\mathcal{F}) + \mathcal{W} - [\sinh (u(\mathcal{F}))^q]^q = 0. \]  (2.3)

Next, we will examine two scenarios for Eq. (2.3).

• **Case one:** Assume \( \kappa = 1, s = 2, \mathit{W} = \frac{q}{2} \).

Thus, Eq. (2.3) can be expressed as follows:

\[ (\partial^2 + \lambda^2 - \mathfrak{B}^2)u''(\mathcal{F}) - (\sinh (u(\mathcal{F}))^q)^q - \frac{q}{2} = 0. \]  (2.8)

Simplification of Eq. (2.8) yields the following expression:

\[ (\partial^2 + \lambda^2 - \mathfrak{B}^2)u''(\mathcal{F}) - \frac{1}{2} \cosh (2u(\mathcal{F})) = 0. \]  (2.9)

Consider

\[ u(\mathcal{F}) = \frac{1}{2} \ln(v(\mathcal{F})). \]

Thus, Eq. (2.10) can be expressed as follows:

\[ (\partial^2 + \lambda^2 - \mathfrak{B}^2)v''(\mathcal{F}) + (2\partial^2 + \lambda^2 - \mathfrak{B}^2)v(\mathcal{F})v''(\mathcal{F}) + q^2v(\mathcal{F}) + v^2(\mathcal{F}) = 0. \]  (2.11)

Hence, we can use our method to solve Eq. (2.11) and obtain the solution to Eq. (1.2) for the second case using Eqs. (2.10) and (2.1).

3 The approach of the analytical method

Express the governing equation in the following form:

\[ G(\mathcal{F}, \mathcal{F}_x, \mathcal{F}_{xx}, \ldots) = 0. \]  (3.1)

We start with a polynomial \( \mathcal{F}(x, \tau) \) and its partial derivatives represented by \( G \). In order to transform Eq. (3.1) into an ordinary differential equation, we employ a traveling wave transformation given by Eq. (2.1). This results in the following expression:

\[ B(u, u'', \ldots) = 0. \]  (3.2)

The Kudryashov method can be broken down into the following series of steps:

**Step 1:** Let us assume that we can represent precise solutions to Eq. (3.2) in the following form:

\[ v(\mathcal{F}) = \sum_{i=0}^{N} M_i (\mathcal{F}(\mathcal{F}))^i. \]  (3.3)

The constants \( M_i \) (where \( i = 0, 1, 2, \ldots, N \) and \( M_N \neq 0 \) ) can be determined using appropriate techniques. The value of \( N \) can be determined by applying the homogeneous balancing principle.
Step 2: The function $\mathcal{F}(\mathcal{J})$ satisfies the following equation:

$$\frac{d\mathcal{F}}{d\mathcal{J}} = (\mathcal{F}(\mathcal{J}) - 1)\mathcal{F}(\mathcal{J}).$$

(3.4)

The solution to Eq. (3.4) can be expressed in the following manner:

$$\mathcal{F}(\mathcal{J}) = \frac{b}{b + d e^{\mathcal{J}}}.\quad (3.5)$$

Step 3: We can obtain a polynomial of $\mathcal{F}(\mathcal{J})$ by substituting Eq. (3.3) into Eq. (3.1), grouping together all terms with the same powers of $\mathcal{F}(\mathcal{J})$ and equating each corresponding coefficient to zero.

Step 4: The solutions to Eq. (3.2) can be obtained by utilizing the Mathematica software to solve the resulting system.

4 The mathematical solution of the model

In this section, we employ the Kudryashov method to obtain analytical solutions for two cases of Eq. (1.2).

• For case one, at $s = \kappa = 1$, and $\mathcal{W} = 0$:

By applying the balance principle to Eq. (2.7) between the terms $v^2$ and $V^3$, we obtain the relation $2N + 2 = 3N$, which yields $N = 2$. Thus, the solution to Eq. (2.7) can be expressed as follows:

$$v(\mathcal{J}) = \sum_{i=0}^{2} (M_i(\mathcal{F}(\mathcal{J})))^i.\quad (4.1)$$

Substituting Eq. (4.1) into Eq. (2.7) and equating the coefficients of similar powers of $\mathcal{F}(\mathcal{J})$ to zero, we obtain the following system:

$$2C_1M_0^2 + M_0q + M_0 = 0,$$

$$4C_1M_0M_0 + M_0q + 3M_1M_0^2 = 0,$$

$$4C_1M_0 + 2C_2M_0^2 - \lambda^2M_0^2 + M_0 - M_0^2\theta^2 + M_0^2\theta^2 + 3M_0M_0^2 + 3M_0M_0^2 = 0,$$

$$24C_1M_0M_1 + 4\lambda^2M_0^2 - 4\lambda^2M_0M_1 + 2M_0^2\theta - M_0M_0^2 + 2M_1^2\theta^2 + 4M_2M_0\theta^2 + M_1^2 + 6M_0M_2M_1 = 0,$$

$$2C_2M_0^2 - \lambda^2M_1^2 - 4\lambda^2M_0^2 + 8M_0M_2^2 - M_0^2\theta^2 + M_1^2\theta^2 + 4M_2^2\theta^2 - 8M_0M_2\theta^2 + 3M_0M_2^2 + 3M_0M_2^2 = 0,$$

$$88\lambda^2M_0^2 - 4\lambda^2M_0M_2 + 8M_0^2\theta^2 - 4M_0M_2\theta^2 - 8M_0^2\theta^2 + 4M_2M_0\theta^2 + 3M_0M_2^2 = 0,$$

$$-4\lambda^2M_0^2 - 4M_0^2\theta^2 + 4M_2^2\theta^2 + M_0^2 = 0.$$

Using the Mathematica program to solve the aforementioned set of equations yields the following solutions:

• Class 1:

$$M_0 = -\sqrt{q}, \quad M_1 = 4\sqrt{q}, \quad M_2 = -4\sqrt{q},$$

$$\nu\lambda = \sqrt{-\sqrt{q} - \theta^2 + 2\theta^2}, \quad C_1 = \sqrt{q}.\quad (4.2)$$

To obtain the solutions to Eq. (1.2) at $s = \kappa = 1$, and $\mathcal{W} = 0$, we can substitute Eq. (4.2) into Eq. (4.1) using Eqs. (2.6) and (2.1).

$$\mathcal{F}_{1,2}(x, y, \tau) = \ln\left(M_2\left(b + d \exp(\theta x + \lambda y - \theta \frac{\theta^2}{\theta^2})\right)^2 + \frac{M_0b}{b + d \exp(\theta x + \lambda y - \theta \frac{\theta^2}{\theta^2}) + M_0}\right).\quad (4.3)$$

• Class 2:

$$M_0 = \sqrt{q}, \quad M_1 = -4\sqrt{q}, \quad M_2 = 4\sqrt{q},$$

$$\nu\lambda = \sqrt{-\sqrt{q} - \theta^2 + 2\theta^2}, \quad C_1 = -\sqrt{q}.\quad (4.4)$$

By substituting Eq. (4.4) into Eq. (4.1) using Eqs. (2.6) and (2.1), we can obtain the solutions to Eq. (1.2) at $s = \kappa = 1$, and $\mathcal{W} = 0$.

$$\mathcal{F}_{3,4}(x, y, \tau) = \ln\left(M_2\left(b + d \exp(\theta x + \lambda y - \theta \frac{\theta^2}{\theta^2})\right)^2 + \frac{M_0b}{b + d \exp(\theta x + \lambda y - \theta \frac{\theta^2}{\theta^2}) + M_0}\right).\quad (4.5)$$

• For case two at $s = 2$, $\kappa = 1$, and $\mathcal{W} = -\frac{q}{\lambda^2}$:

By applying the balance principle to Eq. (2.11) between the terms $v^2$ and $v^3$, we obtain the relation $2N + 2 = 3N$, which yields $N = 2$. The solution to Eq. (2.11) can be obtained from Eq. (3.3) as follows:

$$v(\mathcal{J}) = \sum_{i=0}^{2} (M_i(\mathcal{F}(\mathcal{J})))^i.\quad (4.6)$$

Substituting Eq. (4.6) into Eq. (2.11) and equating the coefficients of similar powers of $\mathcal{F}(\mathcal{J})$ to zero, we obtain the following system:
\[ M_0 q^2 + M_0^3 = 0, \]
\[-2\lambda^2 M_0 M_1 + M_0 q^2 - 2M_0 M_1 q^2 + 2M_0 M_2 q^2 + 3M_0^2 M_1 = 0, \]
\[-6\lambda^2 M_0 M_2 + M_0 q^2 + 6M_0 M_0 q^2 - 8M_0 M_0 q^2 \]
\[-6M_0 M_2 q^2 + 8M_0 M_2 q^2 + 3M_0 M_1 q^2 + 3M_0^2 M_2 = 0, \]
\[22\lambda^2 M_1^2 - 4\lambda^2 M_0 M_1 - 2\lambda^2 M_0 M_1 + 20\lambda^2 M_0 M_1 + 2M_1^2 q^2 \]
\[-4M_0 M_1 q^2 - 2M_0 M_1 q^2 + 20M_0 M_0 q^2 - 2M_1^2 q^2 + 4M_0 M_0 q^2 \]
\[+ 2M_0 M_2 q^2 - 20M_0 M_2 q^2 + M_1^3 + M_0 M_0 M_1 = 0, \]
\[-2\lambda^2 M_1^2 - 12\lambda^2 M_0 M_2 - 12\lambda^2 M_2 - 2M_1^2 q^2 - 12M_0 M_2 q^2 \]
\[+ 10M_0 M_2 q^2 + 2M_1^2 q^2 + 12M_0 M_2 q^2 - 10M_0 M_2 q^2 + 3M_0 M_2^2 \]
\[+ 3M_0^2 M_2 = 0, \]
\[44\lambda^2 M_1^2 - 8\lambda^2 M_0 M_2 + 4M_1^2 q^2 - 8M_0 M_2 q^2 \]
\[-4M_1^2 q^2 + 8M_0 M_0 q^2 + 3M_1^2 q^2 \]
\[-4\lambda^2 M_2^2 - 4M_1^2 q^2 + 4M_1^2 q^2 + M_1^3 = 0. \]

Solving the aforementioned set of equations using the Mathematica program yields the following set of solutions:

- **Class 1:**
  \[ M_0 = -iq, \quad M_1 = 4iq, \quad M_2 = -4iq, \]
  \[ \lambda = \sqrt{-iq - q^2 + \Omega^2}. \]  
  \( (4.7) \)

To obtain the solutions to Eq. (1.2) at \( s = 2, \kappa = 1 \), and \( \mathcal{W} = -\frac{q}{2} \), we can substitute Eq. (4.7) into Eq. (4.6) using Eqs. (2.10) and (2.1).

\[ \mathcal{F}_{1,2}(x, y, \tau) = \frac{1}{2} \ln \left( \frac{b}{b + d \exp(\partial x + \lambda y - \Omega^2/\Omega)} \right)^2 \]
\[ + \frac{M_0 b}{b + d \exp(\partial x + \lambda y - \Omega^2/\Omega)} + M_0. \]  
  \( (4.8) \)

- **Class 2:**
  \[ M_0 = iq, \quad M_1 = -4iq, \quad M_2 = 4iq, \]
  \[ \lambda = \pi \sqrt{-iq - q^2 + \Omega^2}. \]  
  \( (4.9) \)

Substituting (4.9) into (4.6), and utilizing (2.10) with (2.1), we can determine the solutions to Eq. (1.2) at \( s = 2, \kappa = 1 \), and \( \mathcal{W} = -\frac{q}{2} \). 

\[ \mathcal{F}_{3,4}(x, y, \tau) = \frac{1}{2} \ln \left( \frac{b}{b + d \exp(\partial x + \lambda y - \Omega^2/\Omega)} \right)^2 \]
\[ + \frac{M_0 b}{b + d \exp(\partial x + \lambda y - \Omega^2/\Omega)} + M_0. \]  
  \( (4.10) \)

## 5 The numerical solution of the model

In this section, we utilize approximations for the space \((x, y)\) and time \((\tau)\) derivatives, as mentioned in the studies by Raslan et al. [12] and EL-Danaf et al. [13]:

\[ \mathcal{F}_{xx} = \frac{U_{i+1,j,n} - 2U_{i,j,n} + U_{i-1,j,n}}{(\Delta x)^2}, \]
\[ \mathcal{F}_{yy} = \frac{U_{j+1,i,n} - 2U_{i,j,n} + U_{i-1,j,n}}{(\Delta y)^2}. \]  
  \( (5.1) \)

Based on the properties of the conformal fractional derivative, if \( 0 < \alpha \leq 1 \) and \( \mathcal{F} \) is \( \alpha \)-differentiable at a point \( \tau > 0 \), then

\[ \frac{\partial^{\alpha} \mathcal{F}}{\partial \tau^{\alpha}} = \tau^{1-\alpha} \frac{\partial \mathcal{F}}{\partial \tau}. \]  
  \( (5.2) \)

The approximation for the time derivative with respect to \( \tau \) is:

\[ \mathcal{F}_{\tau} = \frac{U_{i,j,n+1} - U_{i,j,n}}{\Delta \tau}. \]  
  \( (5.3) \)

Assuming that \( \mathcal{F} \) is the exact solution at the grid point \((x_i, y_j, \tau_n)\), and that \( U_{i,j,n} \) is the corresponding numerical solution, we can obtain a system of difference equations by substituting Eqs. (5.1) and (5.2) into Eq. (1.2) as follows:

\[ \begin{align*}
U_{i+1,j,n} - 2U_{i,j,n} + U_{i-1,j,n} \\
+ \frac{U_{j+1,i,n} - 2U_{i,j,n} + U_{i-1,j,n}}{\Delta \tau} \\
- (1 - \alpha)\tau^{1-2\alpha}U_{i,j,n+1} - U_{i,j,n} \\
- \tau^{2-2\alpha}U_{i,j,n+1} - 2U_{i,j,n} + U_{i,j,n-1} - G_{i,j,n} = 0,
\end{align*} \]  
  \( (5.4) \)

where \( G_{i,j,n} = [\sinh(\mathcal{F})]^\alpha - \mathcal{W} \).
5.1 Local truncation error

Let us now introduce the local truncation error of our scheme.

**Theorem 1.** The local truncation error of the finite difference scheme given by Eq. (5.4) is

\[ O((\Delta x)^2 + (\Delta y)^2 + (\Delta \tau)^2). \]

**Proof.** By employing Taylor’s expansion in Eq. (5.4), we can investigate the local truncation error in two-dimensional space and time as follows:

\[ T_{i,j,n} = \frac{1}{12} h^2 \frac{\partial^4 U}{\partial x^4} + \frac{1}{12} h^2 \frac{\partial^4 U}{\partial y^4} + \frac{1}{12} q^2 \frac{\partial^4 U}{\partial \tau^4} + \ldots. \]

Hence, \( T_{i,j,n} \to 0 \) as \((\Delta x)^2, (\Delta y)^2, (\Delta \tau)^2 \to 0\). Consequently, the local truncation error of the finite difference scheme given by Eq. (5.4) can be expressed as follows:

\[ O((\Delta x)^2 + (\Delta y)^2 + (\Delta \tau)^2). \]

5.2 Stability of the finite difference scheme

In this subsection, we examine the stability of the difference scheme.

**Theorem 2.** The difference scheme given by Eq. (5.4) is stable if \( G_{i,j,n} = MU_{i,j,n} \).

**Proof.** The stability analysis of the scheme yields the following:

To begin, we express the scheme in matrix form as follows:

\[
\begin{bmatrix}
\frac{1}{(\Delta x)^2} & 0 & 0 & 0 \\
0 & \frac{1}{(\Delta y)^2} & 0 & 0 \\
0 & 0 & (1 - a)(n(\Delta \tau))^{1-2a} & -n(\Delta \tau)^{2-2a} \\
0 & 0 & -(n(\Delta \tau))^{1-2a} & (1 - a)(n(\Delta \tau)^{2-2a})
\end{bmatrix}
\begin{bmatrix}
U_{i+1,j,n} \\
U_{i,j+1,n} \\
U_{i,j,n+1} \\
U_{i,j,n-1}
\end{bmatrix}
= M
\begin{bmatrix}
U_{i,j,n} \\
U_{i,j,n} \\
U_{i,j,n} \\
U_{i,j,n}
\end{bmatrix}
\]

Next, we determine the eigenvalues of the matrix:

\[
\Omega^4 + \left( \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} - (1 - a)(n(\Delta \tau))^{1-2a} - \frac{1}{(\Delta \tau)^2} \right) \Omega^2 - \frac{1}{(\Delta x)^2} - \frac{1}{(\Delta y)^2} + (1 - a)(n(\Delta \tau))^{1-2a} + (n(\Delta \tau)^{2-2a} = 0.
\]

We solve Eq. (5.6) for \( \Omega \) as follows:

\[
\Omega = \frac{1}{\sqrt{\Delta x}} \left[ \sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} - (1 - a)(n(\Delta \tau))^{1-2a} - (n(\Delta \tau)^{2-2a}} \right].
\]

To verify the stability of the scheme, we examine the sign of the real part of \( \Omega \). If the real part of \( \Omega \) is negative, the scheme is stable.

For this particular case, the real part of \( \Omega \) is negative for all values of \( \Delta x, \Delta y, n, \) and \( a \). Thus, we can conclude that the scheme is stable.

**Theorem 3.** The difference scheme (5.4) is stable if

\[
\frac{(1 - a)(n(\Delta \tau))^{1-2a} + n(\Delta \tau)^{2-2a}}{\Delta \tau} < \frac{(\Delta x)^2}{2} + \frac{(\Delta y)^2}{2}.
\]

**Proof.** The stability of the scheme is determined by the eigenvalues of the amplification matrix, which can be defined as follows:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-(1 - a)(n(\Delta \tau))^{1-2a} & 1 & -n(\Delta \tau)^{2-2a} & 1 \\
-\frac{2}{(\Delta x)^2} & \frac{2}{(\Delta y)^2} & 0 & 1
\end{bmatrix}
\]

The eigenvalues of \( A \) can be obtained as follows:

\[
\Omega_1 = 1, \quad \Omega_2 = 1, \quad \Omega_3 = \frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}, \quad \Omega_4 = \frac{2}{(\Delta x)^2} - \frac{2}{(\Delta y)^2}.
\]

The scheme is stable if all the eigenvalues of \( |A| < 1 \). This is true if

\[
\frac{(1 - a)(n(\Delta \tau))^{1-2a} + n(\Delta \tau)^{2-2a}}{\Delta \tau} < \frac{(\Delta x)^2}{2} + \frac{(\Delta y)^2}{2}.
\]
5.3 The numerical outcomes

In the following section, we present some of the results obtained through numerical analysis of the (2+1) conformal time-fractional generalized $q$-deformed sinh-Gordon equation in general. Moreover, we provide the numerical solution for two specific scenarios of the generalized $q$-deformed sinh-

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<th>Analytical solution</th>
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Figure 1: Structure of Eq. (4.5) at $b = 0.4$, $d = 0.38$, $q = 0.7$, $\vartheta = 0.3$, and $\mathfrak{B} = 0.1$. 
Gordon equation, for which we have already explored the analytical solution.

**Case one:** $s = \kappa = 1$, $\mathcal{W} = 0$:

We compare the results obtained through numerical analysis and the solution obtained through analytical method given by Eq. (4.5) for Eq. (1.1) at $\Delta x = \Delta y = 1$, $\Delta \tau = 0.01$, $\alpha = 1$, $b = 0.4$, $d = 0.38$, $q = 0.7$, $\mathfrak{B} = 0.1$, $\vartheta = 0.3$, and $y = 1$. The comparison is presented in Table 1 and Figure 3.

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<th>$\alpha$</th>
<th>Numerical solution</th>
<th>Analytical solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>1.36052</td>
<td>1.36581</td>
<td>$2.05514 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.8</td>
<td>1.35960</td>
<td>1.36274</td>
<td>$5.31933 \times 10^{-3}$</td>
</tr>
<tr>
<td>0.9</td>
<td>1.35923</td>
<td>1.36107</td>
<td>$9.08451 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

**Case two:** $s = 2$, $\kappa = 1$, and $\mathcal{W} = -\frac{q}{2}$.

Table 2 and Figure 4 depict a comparison made between the results obtained through numerical analysis and the solution obtained through analytical method given by Eq. (4.8) for Eq. (1.1) at $\Delta x = \Delta y = 1$, $\Delta \tau = 0.01$, $\alpha = 1$, $b = 0.4$, $\mathfrak{B} = 0.1$, $d = 0.02$, $q = 0.7$, $\vartheta = 0.7$, and $y = 5$.

We compare the results obtained through numerical analysis and the solution obtained through analytical method.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Numerical solution</th>
<th>Analytical solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.678325</td>
<td>0.677788</td>
<td>$6.10149 \times 10^{-4}$</td>
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<tr>
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<td>0.678105</td>
<td>$3.64791 \times 10^{-4}$</td>
</tr>
<tr>
<td>0.9</td>
<td>0.678467</td>
<td>0.678277</td>
<td>$2.15636 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
method given by Eq. (4.5) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.38, q = 0.7, \vartheta = 0.3$, and $\Psi = 1$. The comparison is presented in Table 3.

Table 4 displays a comparison made between the results obtained through numerical analysis and the solution obtained through analytical method given by Eq. (4.8) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.02, q = 0.7, \vartheta = 0.7$, and $\Psi = 0.2$. In addition, we provide a comparison between the numerical results of Eq. (1.1) and its analytical solution given by Eq. (4.5) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.38, q = 0.7$, and $\vartheta = 0.3$, with $\alpha = 1$, in Figure 3. Finally, Figure 4 displays a comparison between the numerical findings of Eq. (1.1) and the analytical solution of Eq. (4.8) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.02, q = 0.7$, and $\vartheta = 0.7$.

6 Illustrations with graphics

In order to provide a better understanding of the solutions presented, we present some two-dimensional and three-dimensional figures. Specifically, Figures 1–4 display both analytical and numerical solutions. Figure 1 illustrates the graph of Eq. (4.5) at $b = 0.4, d = 0.38, q = 0.7, \vartheta = 0.3$, and $\Psi = 0.1$ obtained using our method. Furthermore, Figure 2 depicts the graph of Eq. (4.8) at $b = 0.4, d = 0.02, q = 0.7, \vartheta = 0.7$, and $\Psi = 0.2$. In addition, we provide a comparison between the numerical results of Eq. (1.1) and its analytical solution given by Eq. (4.5) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.38, q = 0.7$, and $\vartheta = 0.3$, with $\alpha = 1$, in Figure 3. Finally, Figure 4 displays a comparison between the numerical results of Eq. (1.1) with the analytical solution of Eq. (4.8) at $\Delta x = \Delta y = 1, \Delta \tau = 0.01, \alpha = 1, b = 0.4, \Psi = 0.1, d = 0.02, q = 0.7$, and $\vartheta = 0.7$.

7 Discussion

The results obtained through the Kudryashov method and the numerical analysis provide insights into the behavior of the solution of the $q$-deformed sinh-Gordon equation.
The numerical results provide a good approximation of the analytical solution, with minor errors observed in comparing the two solutions. The stability analysis shows that the numerical scheme used is stable for the chosen parameters, and the evaluation of the local truncation error indicates that the difference scheme is convergent.

Furthermore, the results obtained in this study can be used to describe physical systems that have lost their symmetry. The q-deformed sinh-Gordon equation is a valuable tool for modeling such systems, and the numerical and analytical solutions obtained in this study can be used to predict the behavior of these systems. In addition, our study presents a clear strategy for solving other models using the Kudryashov method and the finite difference method, which can be used to explore alternative techniques for solving different systems.

In addition, our study contributes to the field of mathematical modeling and provides a foundation for further research in this area. The numerical and analytical solutions presented in this study can be used to explore the behavior of physical systems that have lost their symmetry, and our results can be applied to a wide range of practical applications.

8 Conclusion

In conclusion, we have demonstrated how to apply the Kudryashov method to investigate the two-dimensional conformal time-fractional generalized q-deformed sinh-Gordon equation. We have also conducted a comprehensive numerical analysis of this model using the finite difference method, including a stability analysis and an evaluation of the local truncation error for the difference scheme. Furthermore, we have compared the analytical and numerical solutions and found that our results represent a significant contribution to the field. In the future, we plan to apply this method to solve additional models and explore alternative techniques for solving different systems. The proposed equation has opened up new avenues for describing physical systems that have lost their symmetry.

Acknowledgements: The author thanks the editor-in-chief of the journal and all those in charge of it.

Funding information: There is no funding.

Author contributions: If we look at the contribution of each author in this article, we will find one author works in this article.

Conflict of interest: The author declares no competing interests.

Ethics approval and consent to participate: The author confirms that all the results they obtained are new and there is no conflict of interest with anyone.

Consent for publication: The author agrees to publish.

Data availability statement: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References