Research Article

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An analysis of exponential kernel fractional difference operator for delta positivity

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Abstract: Positivity analysis for a fractional difference operator including an exponential formula in its kernel has been examined. A composition of two fractional difference operators of order \((v, \mu)\) in the sense of Liouville–Caputo type operators has been analysed in cases when \(v \neq \mu\) and \(v = \mu\). Due to the kernel of the fractional difference operator being convergent, there has been a restriction in the domain of the solution. Incidentally, a negative lower bounded condition has been carried out through analysing the positivity results. For a better understanding, an increasing function has been considered as a test for the main results.

Keywords: exponential kernel difference operators, positivity results, negative lower bounded

MSC 2020: 26A48, 26A51, 33B10, 39A12, 39B62

1 Introduction

In recent years, discrete fractional calculus has emerged as an important tool to analyse and model real-world phenomena. Fractional difference equations with various types of sums and differences can be used for a wide range of applications in science and engineering, which can be found in previous studies [1–4]. In particular, to note some of the applications of fractional sum and difference equations with Riemann–Liouville operators, one can refer to applications in logarithmic decay, material mechanics, such as fracture analysis, probability, physics, geology, mechanics, and chemistry [5–8]. In the meantime, the study of the existence of solutions to boundary value problem for fractional difference equations is one of the most important properties in applications. There are various methods and algorithms used by scholars [9–13].

The study of the positivity and monotonicity analyses to discrete operator problems for fractional difference equations are two of the most important properties in applications in the context of discrete fractional calculus. The \(\nu\)-monotonicity concept was first introduced by both mathematicians Atici and Uyanik in [14]. The \(\nu\)-monotonicity analysis in this article was on the Riemann–Liouville fractional difference operators on the time set \(\mathbb{N}_a = \{a, a+1, \ldots\}\). There are abundant research results regarding positivity and monotonicity analyses for various types of fractional difference operators. However, most of these results have considered the case of Riemann–Liouville fractional difference operators, see, e.g., [15–17] and the related references therein. Particularly, \(\nu\)-monotonicity to analyse Riemann–Liouville fractional difference operators open the door for many new results on positivity and monotonicity analyses for other types of fractional difference operators such as discrete Liouville–Caputo, Atangana–Baleanu and Caputo–Fabrizio operators on \(\mathbb{N}_a\) [18–21]. In addition, these results have been extended, and they have been analysed on the time set \(\mathbb{N}_a = \{a, a+h, a+2h, \ldots\}\) [22–24]. The extensions and generalisations of positivity and monotonicity analyses using fractional difference operators were presented in previous studies [25–30].

By inspiring and motivating the results of the study by Goodrich and Jornalagadda [31], in the current study, we will work on analysing the discrete Caputo–Fabrizio fractional \(C_{a,\nu}^{\mu}A_{\nu}^{\mu}\) of another discrete Caputo–Fabrizio \(C_{a,\nu}^{\mu}A_{\nu}^{\mu}\) in the sense of Liouville–Caputo such that \(\nu \neq \mu\). Specifically, we will analyse these composite operators when the orders are equal \(\nu = \mu\). In any cases, we use a set of conditions as lower bounds of the inequality. Incidentally, it is worth mentioning that our present work can be considered as an extension of ideas from the study by Goodrich and Jonnalagadda [31] to sequential fractional differences with delta operators. These have been investigated and analysed together with their domain of solutions in Theorems 3.1 and 3.2.

The article’s structure is arranged as follows: In Section 2, the definition of Caputo–Fabrizio fractional difference operator has been recalled and two essential lemmas are

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given with their proofs. In Section 3, we formulated the domain of solutions and we analysed the proposed model of sequential fractional operators. We discussed an example of an increasing function in Section 4 to validate the main theorems. Finally, concluding remarks and future considerations have been drawn in Section 5.

2 Definition of Caputo–Fabrizio and some lemmas

In the present section, we collect some basic results. Let’s start with the definition of discrete delta Caputo–Fabrizio fractional difference of Liouville–Caputo type of the first order [32]:

\[
(C^\mu_d^{\lambda} y)(\tau) = \frac{B(v)}{1 - 2v} \left[ \sum_{s=\tau}^{\tau+1} (A_d y)(s)(1 + \lambda)^{-s} \right],
\]

(2.1)

where \( \lambda = -\frac{v}{1-v} \) with \( v \in [0, 1) \). Also, the higher order is defined by [33]:

\[
(C^\mu_d^{\lambda} y)(\tau) = \frac{B(v - q)}{2q + 1 - 2v} \left[ \sum_{s=\tau}^{\tau+1} (A_d^q y)(s)(1 + \lambda)^{-s} \right],
\]

(2.2)

where \( \lambda_1 = -\frac{q-v}{q+1-v} \) with \( q \in (q - 1, q) \). Note that \((\Delta y)(\tau) = y(\tau + 1) - y(\tau)\) and \((\Delta^q y)(\tau) = \Delta(\Delta^{q-1} y)(\tau)\) for each \( \tau \in \mathbb{N}_{a+1} \), and \( B(v) > 0 \).

**Lemma 2.1.** Let \( \lambda_1 = -\frac{v}{1-v} \) and \( \lambda_2 = -\frac{\mu}{1-\mu} \) with \( 0 < v, \mu < \frac{1}{2} \), and \( 1 \leq \frac{v}{1-v} + \frac{\mu}{1-\mu} < \frac{3}{2} \) such that \( \mu \neq v \). Then, we have

\[
P(j) = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \left[ (1 + \lambda_1)^j - (1 + \lambda_2)^j \right] \geq 0,
\]

(2.3)

and

\[
Q(j) = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \left[ (1 + \lambda_1)^j - (1 + \lambda_2)^j \right] > 0,
\]

(2.4)

for each \( j \in \mathbb{N}_1 \).

**Proof.** To prove (2.3), we can proceed by induction. For the first step when \( j = 1 \), we have

\[
P(1) = \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_1) - (1 + \lambda_2) \right] = -1 - \lambda_2 - \lambda_2 \geq 0,
\]

because \( \frac{v}{1-v} + \frac{\mu}{1-\mu} \geq 1 \). Let us now assume that

\[
P(N) = \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_1)^N - (1 + \lambda_2)^N \right] \geq 0,
\]

(2.5)

for some \( N \in \mathbb{N}_1 \). Then, we have to show that \( P(N + 1) \geq 0 \). There are two cases to prove it:

**Case 1:** If \( v > \mu \), we see that \( 1 - \mu > 1 - v \Rightarrow \lambda_2 = -\frac{\mu}{1-\mu} > -\frac{v}{1-v} = \lambda_1 \). Therefore,

\[
P(N + 1) = \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_1)^{N+1} - (1 + \lambda_2)^{N+1} \right] 
\geq 0 \iff \lambda_1 (1 + \lambda_2)^{N+1} - \lambda_2 (1 + \lambda_1)^{N+1} \geq 0.
\]

Since \( v \in \left[ 0, \frac{1}{2} \right] \), we can show that \( 1 + \lambda_1 = \frac{1-2v}{1-v} > 0 \). Therefore, by considering (2.5), we have

\[
\lambda_1 (1 + \lambda_2)^{N+1} \geq \lambda_2 (1 + \lambda_1)^{N+1} \geq 0,
\]

which implies that \( P(N + 1) \geq 0 \) for \( v > \mu \).

**Case 2:** If \( v < \mu \), we have \( 1 - v > 1 - \mu \Rightarrow \lambda_2 = -\frac{\mu}{1-\mu} > -\frac{v}{1-v} = \lambda_1 \). Therefore,

\[
P(N + 1) = \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_1)^{N+1} - (1 + \lambda_2)^{N+1} \right] 
\geq 0 \iff \lambda_1 (1 + \lambda_2)^{N+1} - \lambda_2 (1 + \lambda_1)^{N+1} \leq 0.
\]

Since \( \mu \in \left[ 0, \frac{1}{2} \right] \), we can show that \( 1 + \lambda_2 = \frac{1-2\mu}{1-\mu} > 0 \). Therefore, by considering (2.5), we have

\[
\lambda_2 (1 + \lambda_1)^{N+1} \geq \lambda_1 (1 + \lambda_2)^{N+1} \geq 0,
\]

which gives \( P(N + 1) \geq 0 \) for \( v < \mu \). Thus, \( P(j) \geq 0 \) for all \( v \neq \mu \) and \( j \in \mathbb{N}_1 \). Hence, first part of the lemma is proved.

To prove the second part of the lemma (i.e., (2.4)), we again use two possible cases: The first case when \( v > \mu \), we have \( \lambda_2 > \lambda_1 \) and \( 1 + \lambda_1 > 1 + \lambda_2 \). Therefore, have for each \( j \geq 1 \):

\[
Q(j) = \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_2)^j - (1 + \lambda_1)^j \right] 
> \frac{1}{\lambda_2 - \lambda_1} \left[ (1 + \lambda_1)^j - (1 + \lambda_2)^j \right] = 0.
\]

Similarly, we can prove that \( Q(j) > 0 \) for the case when \( v < \mu \). Thus, the proof of (2.4) is completed. Consequently, the proof of the lemma is done.

**Lemma 2.2.** Let \( \lambda = -\frac{v}{1-v} \) for \( v \in \left[ \frac{1}{2}, 1 \right] \). Then, we have

\[
J(j) = (1 + \lambda)^{\frac{1}{2}} (1 + \lambda)^{\frac{1}{2}} (1 + \lambda)^{\frac{1}{2}} \leq 0,
\]

(2.6)

for each \( j \in \mathbb{N}_1 \).
Proof. We prove the result by induction: For \( j = 1 \), we have
\[
J(1) = 2\lambda + 1 = \frac{1 - 3\nu}{1 - \nu} \leq 0,
\]
since \( \nu \in \left[ \frac{1}{2}, 1 \right] \) and \( 1 - \nu > 0 \). Suppose that \( J(N) \geq 0 \), that is,
\[
(1 + \lambda)^{N-1}[\lambda N + (1 + \lambda)] \leq 0,
\]
for some \( N \in \mathbb{N}_1 \). Then, to show that \( J(N + 1) \leq 0 \), we use
\[
(1 + \lambda)^{N}[(\lambda(N + 1) + (1 + \lambda)]
\]
where we have used that \( \lambda < 0 \), and as a result, we obtain \( J(N + 1) \leq 0 \) as required. Consequently, we obtain (2.6) is true for each \( j \in \mathbb{N}_1 \). Therefore, we have established the proof.

Lemma 2.3. For \( y \) defined as a function \( y : \mathbb{N}_a \rightarrow \mathbb{R} \), we have
\[
\Delta^{(CTA^a y)}(\tau) = \frac{B(\nu)}{1 - \nu} \left[ (\Delta y)(\tau) + \frac{\lambda}{1 + \lambda} \sum_{s=0}^{\tau} (\Delta y)(s)(1 + \lambda)^{\tau-s} \right],
\]
for \( \nu \in \left( 0, \frac{1}{2} \right] \) and \( \tau \in \mathbb{N}_{a+1} \).

Proof. By taking \( \Delta \) with respect to \( \tau \) on both sides of the Eq. (2.1), we have that
\[
\Delta^{(CTA^a y)}(\tau) = \frac{B(\nu)}{1 - \nu} \left[ (\Delta y)(\tau) + \frac{\lambda}{1 + \lambda} \sum_{s=0}^{\tau} (\Delta y)(s)(1 + \lambda)^{\tau-s} \right] - \frac{\nu}{2}\left( (\Delta y)(\tau) + \frac{\lambda}{1 + \lambda} \sum_{s=0}^{\tau} (\Delta y)(s)(1 + \lambda)^{\tau-s} \right)
\]
Rearranging the summations and inside terms to obtain
\[
\Delta^{(CTA^a y)}(\tau) = \frac{B(\nu)}{1 - 2\nu} \left[ (1 + \lambda)(\Delta y)(\tau) + \frac{\nu}{2}\frac{\lambda}{1 + \lambda} \sum_{s=0}^{\tau-1} (\Delta y)(s)(1 + \lambda)^{\tau-s} \right],
\]
for each \( \tau \in \mathbb{N}_{a+1} \), and which completes the proof.

3 Positivity analyses

This section deals with two important results, which are the main results in this study. The first result for the compositions of two CF operators with two different orders \( \mu \neq \nu \) with \( \nu, \mu \in \mathbb{R} \), defined on the set \( \mathcal{D}_1 \), which is
\[
\mathcal{D}_1 = \left\{ (\nu, \mu); 0 < \nu, \mu < \frac{1}{2} \text{ and } 1 \leq \frac{\nu}{1 - \nu} + \frac{\mu}{1 - \mu} < \frac{3}{2} \text{ for } \mu \neq \nu \right\}.
\]
Furthermore, Figure 1 gives further details on the set \( \mathcal{D}_1 \) graphically.

On the other hand, the second result for the compositions of two CF operators with the same order, defined on the set \( \mathcal{D}_2 \), which is
\[
\mathcal{D}_2 = \left\{ (\nu, \mu); 0 < \nu, \mu < \frac{1}{2} \text{ and } 1 \leq \frac{\nu}{1 - \nu} + \frac{\mu}{1 - \mu} < \frac{3}{2} \text{ for } \mu = \nu \right\}.
\]
Looking at Figure 1, the dashed dot line in the diagonal refers to the set \( \mathcal{D}_2 \), which is not included from the set \( \mathcal{D}_1 \).

Remark 3.1. It is important to note that in the delta case, the domain of solutions has been restricted as appeared in (3.1) and (3.2). However, for the nabla case [31] these domains are wider, and they are as follows:
\[
\mathcal{D}_1 = \left\{ (\nu, \mu); 0 < \nu, \mu < 1 \text{ and } 1 \leq \nu + \mu < 1 \text{ for } \mu \neq \nu \right\}
\]
and
\[
\mathcal{D}_2 = \left\{ (\nu, \mu); 0 < \nu, \mu < 1 \text{ and } 1 \leq \nu + \mu < 2 \text{ for } \mu = \nu \right\}.
\]
Theorem 3.1. Let $\zeta \geq 0$, $\lambda_1 = -\frac{v}{1-v}$, and $\lambda_2 = -\frac{\mu}{1-\mu}$. If a function $y : \mathbb{N}_a \rightarrow \mathbb{R}$ satisfies

(i) $(\Delta y)(a) \geq 0$;
(ii) $(\Delta y)(\tau) \geq -\zeta B(v)B(\mu) (1 + \lambda_1)(\Delta y)(a)$;
(iii) $-\lambda_2 \left( (1 + \lambda_2)(\tau - 1) - (1 + \lambda_1)(\tau - 1) \right) \geq \zeta$,

for each $(v, \mu) \in \mathcal{D}_1$ and $\tau \in \mathbb{N}_{a+1} = \{a+1, a+2, ..., T\}$ with some $T \in \mathbb{N}_{a+1}$, then, $(\Delta y)(\tau - 1) \geq 0$ for every $\tau \in \mathbb{N}_{a+1}$.

Proof. Due to the Eq. (2.1) and Eq. (2.8), we obtain

$$
\begin{align*}
(\Delta y)(\tau) &= \frac{B(v)B(\mu)}{1 - 2v} \left( \sum_{s=a+1}^{r-1} \left( (\Delta y)(s)(1 + \lambda_1)^{r-s} \right) \right) \\
&= \frac{B(v)B(\mu)}{1 - 2v} \left( \sum_{s=a+1}^{r-1} \left( (\Delta y)(s)(1 + \lambda_1)^{r-s} \right) \right) \\
&= \frac{B(v)B(\mu)}{1 - 2v} \left( \sum_{s=a+1}^{r-1} \left( (\Delta y)(s)(1 + \lambda_1)^{r-s} \right) \right) \\
&= \frac{B(v)B(\mu)}{1 - 2v} \left( A_1 + A_2 \right).
\end{align*}
$$

(3.3)

We proceed by computing $A_1$ and $A_2$ and we obtain that

$$
A_1 = \sum_{s=a+1}^{r} (\Delta y)(s)(1 + \lambda_1)^{r-s} = (1 + \lambda_1)(\Delta y)(\tau - 1) + (1 + \lambda_1) \sum_{s=a+1}^{r-2} (\Delta y)(s)(1 + \lambda_1)^{r-s-1},
$$

(3.4)

and

$$
A_2 = \frac{\lambda_2}{1 + \lambda_2} \sum_{s=a+1}^{r-1} \left( (\Delta y)(r)(1 + \lambda_2)^{r-s} \right) \left( \sum_{r=a+1}^{r} \left( \frac{1 + \lambda_2}{1 + \lambda_1} \right) \right).
$$

(3.5)

This equation is well defined when $(v, \mu) \notin \mathcal{D}_2$. By making use of (3.4)–(3.5) and (ii) in (3.3) with a little simplification, we have

$$
(\Delta y)(\tau - 1) \geq -\zeta (\Delta y)(a) - \frac{(1 + \lambda_2)(\tau - 1) - (1 + \lambda_1)(\tau - 1)}{\lambda_2 - \lambda_1} (\Delta y)(a)
$$

$$
= \frac{\sum_{s=a+1}^{r-2} (\Delta y)(s)(1 + \lambda_2)^{r-s-1}}{\lambda_2 - \lambda_1} - \frac{\sum_{r=a+1}^{r-2} (\Delta y)(r)(1 + \lambda_2)^{r-s-1} - (1 + \lambda_1)(r-1)}{\lambda_2 - \lambda_1}.
$$

(3.6)

It is worth mentioning that

$$
\frac{\lambda_2(1 + \lambda_2)^{r-s-1} - (1 + \lambda_1)(r-1)}{\lambda_2 - \lambda_1} \geq 0 \quad \text{and}
$$

$$
-\lambda_2 \left( (1 + \lambda_2)^{r-s-1} - (1 + \lambda_1)^{r-s-1} \right) > 0,
$$

in view of (2.3) and (2.4), respectively. So, the inequality (3.6) is well defined.

If we use $\tau = a + 2$ into (3.6), we obtain

$$
(\Delta y)(a + 1) \geq \sum_{s=a+1}^{r+1} (\Delta y)(s) \frac{(1 + \lambda_2)^{(b-s+1)} - (1 + \lambda_1)^{(b-s+1)}}{\lambda_2 - \lambda_1}
$$

$$
= 0.
$$

Let us try $\tau = a + 3$ into (3.6), we see that

$$
(\Delta y)(a + 2) \geq \sum_{s=a+1}^{r+2} (\Delta y)(s) \frac{(1 + \lambda_2)^{(b-s+2)} - (1 + \lambda_1)^{(b-s+2)}}{\lambda_2 - \lambda_1}
$$

$$
= (\Delta y)(a + 1) \frac{(1 + \lambda_2)^{(b-s+2)} - (1 + \lambda_1)^{(b-s+2)}}{\lambda_2 - \lambda_1} \geq 0.
$$

By repeating this action together with the help of (2.3), we attain that $(\Delta y)(\tau - 1) \geq 0$ for each $\tau \in \mathbb{N}_{a+2}$. Also, from the assumption (i), we know that $(\Delta y)(a) \geq 0$. Consequently, we obtain $(\Delta y)(\tau - 1) \geq 0$ for every $\tau \in \mathbb{N}_{a+1}$ as desired. $\square$

As we mentioned earlier, our second result is defined on the set $\mathcal{D}_2$ and it is not included from the set $\mathcal{D}_1$. Furthermore, we have demonstrated this set in detail in Figure 2.
Theorem 3.2. Let \( \frac{1}{3} \leq v < \frac{1}{2} \), \( \lambda = -\frac{v}{1-v} \), and \( \zeta \geq 0 \). Assume that the function \( y \) defined on \( \mathbb{N}_a \) satisfies

(i) \( (Ay)(a) \geq 0; \)

(ii) \( (\mathcal{CF}^v_A \mathcal{CF}^v_A y)(\tau) \geq -\zeta \frac{B^2(v)}{(1-v)(1-2v)} (Ay)(a); \)

(iii) \( -\lambda(\tau - a)(1 + \lambda)^{r-2-a} \geq \zeta, \)

for each \( \tau \in \mathbb{N}^T_{a+1} \) with some \( T \in \mathbb{N}_{a+1} \). Then, \( (Ay)(\tau - 1) \geq 0 \) for every \( \tau \in \mathbb{N}^T_{a+1} \).

Proof. By following (2.8) and (2.1), we have

\[
\begin{align*}
(\mathcal{CF}^v_A \mathcal{CF}^v_A y)(\tau) &= \frac{B^2(v)}{1-2v} \sum_{s=a+1}^{\tau} (\mathcal{CF}^v_A \mathcal{CF}^v_A y)(s)(1 + \lambda)^{r-s} \\
&= \frac{B^2(v)}{1-2v} \sum_{s=a+1}^{\tau} \left[ (Ay)(s)(1 + \lambda)^{r-s} + \frac{\lambda}{1 + \lambda} \sum_{r=a}^{s-1} (Ay)(r)(1 + \lambda)^{r-s} \right] \\
&= \frac{B^2(v)}{(1-2v)(1-v)} \left[ \sum_{s=a+1}^{\tau} (Ay)(s)(1 + \lambda)^{r-s} + \frac{\lambda}{1 + \lambda} \sum_{s=a+1}^{\tau} \sum_{r=a}^{s-1} (Ay)(r)(1 + \lambda)^{r-s} \right] \\
&= \frac{B^2(v)}{(1-2v)(1-v)} \left[ A_3 + A_4 \right].
\end{align*}
\]

Calculating \( A_3 \) and \( A_4 \) successfully, we have

\[
\begin{align*}
A_3 &= \sum_{s=a+1}^{\tau-1} (Ay)(s)(1 + \lambda)^{r-s} \\
&= (1 + \lambda)(Ay)(\tau - 1) \\
&+ (1 + \lambda) \sum_{s=a+1}^{\tau-2} (Ay)(s)(1 + \lambda)^{r-s-1} \\
&= \lambda \sum_{s=a+1}^{\tau-1} \sum_{r=a}^{s-1} (Ay)(r)(1 + \lambda)^{r-s-1}
\end{align*}
\]

and

\[
A_4 = \lambda \sum_{s=a+1}^{\tau-1} \sum_{r=a}^{s-1} (Ay)(r)(1 + \lambda)^{r-s-1} \\
= \lambda \sum_{s=a}^{\tau-2} \left( \sum_{r=a}^{s-1} (Ay)(r)(1 + \lambda)^{r-s} \right) \\
= \lambda \sum_{s=a}^{\tau-2} (Ay)(r)(1 + \lambda)^{r-s} \tau - r - 1 \\
= \lambda(\tau - a)(1 + \lambda)^{r-2-a}(Ay)(a) \\
&+ \sum_{r=a+1}^{\tau-2} (Ay)(r)(1 + \lambda)^{r-1}(\tau - r - 1). \tag{3.9}
\]

By making use of (3.8)–(3.9) and the assumption (ii) into (3.7) with simplifying the results, we obtain

\[
\begin{align*}
(\Delta y)(\tau - 1) &\geq -\zeta(\Delta y)(a) - \lambda(\tau - 1 - a)(1 + \lambda)^{r-2-a}(\Delta y)(a) \\
&\quad \geq \zeta(\Delta y)(a) \text{ by (iii)} \\
&\quad - \sum_{s=a+1}^{\tau-2} (\Delta y)(s)(1 + \lambda)^{r-s-1} \\
&\quad - \sum_{r=a+1}^{\tau-2} (\Delta y)(r)(1 + \lambda)^{r-2-s}(\tau - r - 1) \tag{3.10} \\
&\quad \geq - \sum_{s=a+1}^{\tau-2} (\Delta y)(s)(1 + \lambda)^{r-2-s}(\lambda(\tau - s - 1) + 1 + \lambda). \\
\end{align*}
\]

According to (2.6), we know that

\[
-\lambda(1 + \lambda)^{r-2-a}(\lambda(\tau - s - 1) + 1 + \lambda) \geq 0.
\]

Also, we know for each \( \tau \in \mathbb{N}_{a+1} \) that

\[
-\lambda(\tau - a)(1 + \lambda)^{r-2-a} = \frac{v}{1-v} (\tau - a)(1 + \lambda)^{r-2-a} \geq 0.
\]

So, these confirm that (3.10) is well defined. Thus, we can continue: by using \( \tau = a + 2 \) into (3.10), we obtain

\[
(\Delta y)(a + 1) \geq -\lambda \sum_{s=a+1}^{a+1} \sum_{r=s}^{a+1} (\Delta y)(s)(1 + \lambda)^{r-s} \text{ by assumption (i)} \\
= -(\Delta y)(a + 1)(2\lambda + 1) \geq 0.
\]

We can continue by the same techniques for the next steps with reusing their previous steps and (2.6), and we can deduce that \( (\Delta y)(\tau) \geq 0 \) for each \( \tau \in \mathbb{N}^T_{a+2} \). Also, we have \( (\Delta y)(a) \geq 0 \) by assumption (i). Thus, \( (\Delta y)(\tau) \geq 0 \) for each \( \tau \in \mathbb{N}^T_{a+1} \) as required. \( \square \)
Remark 3.2. The reason why we have two cases for $v \neq \mu$ and $v = \mu$ is due to the condition (iii). If we look at the condition (iii) in Theorem 3.1, we see that there are no $v$ and $\mu$ in the inequality. So, we could not conclude a special case when $v = \mu$. Furthermore, there is no guarantee to have the same condition of Theorems 3.1 in 3.2.

4 Main theorem test

In this section, the results of the proposed Theorems 3.1 and 3.2 are discussed by means of an example. The figures are drawn by using MATLAB R2018b.

Example 4.1. Suppose that $y$ is a function $y : \mathbb{N}_0 \to \mathbb{R}$ defined by

$$y(\tau) = \tau^{v+\mu} = \frac{\Gamma(\tau + 1)}{\Gamma(\tau + 1 - v - \mu)}.$$

First, for $v = 0.40$ and $\mu = 0.45$, we see that

$$1 < \frac{v}{1 - v} + \frac{\mu}{1 - \mu} < \frac{3}{2} \Rightarrow 1 < 1.4848 < \frac{3}{2},$$

which verifies that $(v, \mu) \in \mathcal{D}_1$.

Now, by choosing $\zeta = 0.001$ and $\tau = a + 2$ with $a = 0$, we obtain

$$(\mathcal{C}^{\tau}_{\mathcal{A}^{v+\mu}_{c}}) \mathcal{D}(y)(2) = \frac{B(v)B(\mu)}{(1 - 2v)(1 - \mu)} \left[ \lambda \sum_{s=1}^{1} (\lambda y)(s)(1 + \lambda)^{2-s} + \frac{\lambda^2}{1 + \lambda} \sum_{s=1}^{1} (\lambda y)(s)(1 + \lambda)^{2-s} \right]$$

$$= 0.1418B(v)B(\mu) \geq -0.0028B(v)B(\mu).$$

Similarly, we can deduce that

$$(\mathcal{C}^{\tau}_{\mathcal{A}^{v+\mu}_{c}}) \mathcal{D}(y)(\tau) \geq 0,$$

for each $\tau \in \mathbb{N}_0$, and hence, the second condition of Theorem 3.1 is satisfied. Furthermore, the first condition of Theorem 3.1,

$$(\lambda y)(0) = 0.9110 \geq 0,$$

holds. The last condition,

$$-\lambda \left( \frac{(1 + \lambda)^{\tau-1-a} - (1 + \lambda)^{\tau-1-a}}{\lambda_2 - \lambda_1} \right) = 0.8182 \geq 0.001,$$

is satisfied. Hence, $\tau^{v+\mu}$ is increasing on $\mathbb{N}_0$ according to Theorem 3.1. Moreover, its plot is shown in Figure 3.

Example 4.2. Consider the same function in Example 4.1. Here, if we choose $v = \mu = 0.4$, $\zeta = 0.001$ and $\tau = a + 2$ with $a = 0$, then we have

$$1 < \frac{v}{1 - v} + \frac{\mu}{1 - \mu} < \frac{3}{2} \Rightarrow 1 < 1.3333 < \frac{3}{2},$$

which verifies that $(v, \mu) \in \mathcal{D}_2$. Moreover,

$$(\lambda y)(0) = 0.8713 \geq 0$$

and

$$(\mathcal{C}^{\tau}_{\mathcal{A}^{v+\mu}_{c}}) \mathcal{D}(y)(2) = \frac{B(v)}{(1 - 2v)(1 - \mu)} \left[ \lambda \sum_{s=1}^{1} (\lambda y)(s)(1 + \lambda)^{2-s} + \frac{\lambda^2}{1 + \lambda} \sum_{s=1}^{1} (\lambda y)(s)(1 + \lambda)^{2-s} \right]$$

$$= 0.4034B(v) \geq -0.0073B(v),$$

and similarly, we can obtain

$$(\mathcal{C}^{\tau}_{\mathcal{A}^{v+\mu}_{c}}) \mathcal{D}(y)(\tau) \geq 0,$$

for each $\tau \in \mathbb{N}_0$. In addition,

$$-\lambda(\tau - 1 - a)(1 + \lambda)^{2-a} = 0.6667 \geq 0.001.$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, the increase of $\tau^{v+\mu}$ is proved on $\mathbb{N}_0$. For more clarification, see Figure 4.
5 Conclusion and future extensions

In this study, we successfully analysed a sequential fractional difference operator \((\text{CF}^\mu \Delta_a t^\nu \text{CF}^\mu \Delta_a t^\nu)\) in the sense of Liouville–Caputo type operators for \(v \neq \mu\) and \(v = \mu\) in Theorems 3.1 and 3.2, respectively. In the first theorem, we used the orders to be \(0 < v, \mu < 1\), however, in the next theorem, we had to use a restriction that \(\frac{1}{2} < v = \mu < 1\).

Furthermore, the study investigated the positivity analysis between the sign of the fractional difference operator \((\text{CF}^\mu \Delta_a t^\nu \text{CF}^\mu \Delta_a t^\nu)\) and the negative lower boundedness of the function \(y\) itself. Based on the example in Section 4, it can be noted that the main theorems are applicable to obtain a monotonicity function.

Our obtained results in this article can be extended to the fractional differences or generalised fractional differences including Mittag-Leffler in their kernels; see studies by Abdeljawad et al. [1,32], for further information about these fractional differences and their main properties.

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