Research Article

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Scale-3 Haar wavelet-based method of fractal-fractional differential equations with power law kernel and exponential decay kernel

https://doi.org/10.1515/nleng-2022-0380
received May 17, 2023; accepted April 8, 2024

Abstract: In this study, wavelet method has been proposed to solve fractal-fractional differential equations (FFDEs) with power law kernel (FFDPL) and exponential decay kernel (FFDED). The proposed method is based on scale 3 Haar wavelets with collocation method, and fractional integral operational matrices for derivatives of Caputo and Caputo–Fabrizio sense are derived to solve FFDPL and FFDED. The applicability of the proposed method is shown by solving some numerical examples, and the obtained results are compared with available solutions in the literature. The solutions are presented in the graphical and tabular forms also.

Keywords: fractal-fractional differential equations, Haar wavelet, singular and nonsingular kernels, collocation method

1 Introduction

Differential equations have always been used as an efficient tool to represent real-world problems in mathematical manner in order to find various possible solutions in analytical and numerical forms. Owing to distinct physical, biological, and chemical nature of problems, a wide range of differential equations can be found in the existing literature such as equations with integer [1,2] and non-integer order derivatives [3–5], ordinary and partial derivatives [6,7], inclusion of delay and impulse [8], variable order derivatives [9], fractal derivative [10], linear and nonlinear differential equations, stochastic differential equations [11–13], and integro-differential equations. Derivatives of non-integral orders have been developed, analysed, and applied for this purpose in past few decades, namely: fractional order derivatives with singular and non-singular kernels [14–18], fractal derivative [19–21], and fractal-fractional derivatives [10,22,23]. Fractional differential operator works as a useful tool for simulating the dynamics of systems with memory or hereditary features. They help to frame mathematical models pertaining numerous problems associated with viscoelastic materials [17,24], bioengineering [25], fluid mechanics [26], solid mechanics [27], finance [28], waves with electromagnetic impacts [29], damping effects [30], study of neurons [31], signal processing [32], control theory [33], etc. These derivatives have been used to describe double pendulum systems, control systems, coloured noise, nonlinear earthquake oscillation, stable heat conduction, electrochemical process, rheology, artificial neural networks, etc. Among various numerical techniques: collocation and interpolation techniques to solve fractional differential equations, a few mentioned are Grünwald–Letnikov method [30], finite difference methods [34], Laplace transform method (LTM) [35], predictor–corrector approach [36], wavelet methods [37], Adomian decomposition method [38], spline collocation methods [39], homotopy perturbation method [27], etc.

 order derivatives [9], fractal derivative [10], linear and nonlinear differential equations, stochastic differential equations [11–13], and integro-differential equations. Derivatives of non-integral orders have been developed, analysed, and applied for this purpose in past few decades, namely: fractional order derivatives with singular and non-singular kernels [14–18], fractal derivative [19–21], and fractal-fractional derivatives [10,22,23]. Fractional differential operator works as a useful tool for simulating the dynamics of systems with memory or hereditary features. They help to frame mathematical models pertaining numerous problems associated with viscoelastic materials [17,24], bioengineering [25], fluid mechanics [26], solid mechanics [27], finance [28], waves with electromagnetic impacts [29], damping effects [30], study of neurons [31], signal processing [32], control theory [33], etc. These derivatives have been used to describe double pendulum systems, control systems, coloured noise, nonlinear earthquake oscillation, stable heat conduction, electrochemical process, rheology, artificial neural networks, etc. Among various numerical techniques: collocation and interpolation techniques to solve fractional differential equations, a few mentioned are Grünwald–Letnikov method [30], finite difference methods [34], Laplace transform method (LTM) [35], predictor–corrector approach [36], wavelet methods [37], Adomian decomposition method [38], spline collocation methods [39], homotopy perturbation method [27], etc. Non-local fractional differential operators often fall into one of two categories: with singular kernels, and with non-singular kernels. The latest literature includes the development of fractional derivatives of the Caputo–Fabrizio (CF) fractional derivative, the Atangana–Baleanu (AB) fractional derivative, and the weighted AB fractional derivative. In this study, a novel method has been developed for solving differential equations with fractional derivative in CF sense. Fractal derivative is a novel differentiation concept as an extension of standard derivative for discontinuous fractal media. The idea of Hausdorff derivative of a function with regard to a fractal measure was first suggested by Chen in 2006 [19]. The Leibniz
derivative for discontinuous fractal media is naturally extended to the fractal derivative in the study by Yang [40], which falls into the category of unique local fractional derivative.

A theory of fractal-fractional operator that integrates the theory of fractal operator and the theory of the fractional operator had been developed in the past few years. Researchers have examined the convolution of power law, Mittag–Leffler law, and exponential law with fractal-fractional differential operator [41,42]. These operators are considered as extremely advanced mathematical instruments for applying more realistic solutions to a variety of issues such as fractal-fractional model of the stem cell population dynamics with time-dependent delay [43], reaction–diffusion model, and self-similarities in the chaotic attractors in previous studies [44–47]. Developing analytical and numerical methods to solve these novel types of problems is a wider area of research.

In this study, Haar wavelet fractal-fractional method (HWFFM) has been developed to solve fractal-fractional differential equations (FFDEs) for numerical solutions under fractal-fractional derivatives with the exponential decay kernel (FFDEDs) and fractional differential equation under power law kernel (FFDPLs). Wavelet theory is developed through dilation and translation of a mother wavelet, transformed into a family of functions known as wavelets. Many wavelets such as Morlet wavelet, Legendere’s wavelets, Bernoulli wavelets, and Haar wavelets were broadly discussed and applied in previous studies [6,7,48–52] to find solutions of differential equations. A piecewise continuous wavelet function, the Haar wavelet, has been applied to derive results in the proposed study. Haar wavelet has been found effective in analysing local behaviour of signal function. In order to overcome the discontinuous nature of Haar wavelets, Chen and Hsiao [53] expanded the higher order of the equation as a Haar series and obtained the lower derivatives by integrating. The orthogonal functions basis used by Haar wavelets transforms the differential problem into an algebraic equation. Fast convergence of the Haar wavelet method has successfully derived solutions of different differential models discussed in previous studies [8,9,52,54]. Haar wavelets of scale 3 have been analysed and implemented in previous studies [55–60] with faster convergence rate. This study aims at developing a method to solve FFDEs with Haar wavelet scale 3 involving fractional derivative of Caputo, and CF derivatives. In the existing literature, not any research can be found for operational matrix for CF derivative using the family of Haar wavelets. This article includes the construction of novel Haar operational matrix for fractional order in CF sense and applied to solve FFDEs, which have successfully presented the results in line with the existing literature and reliability for future predictions. The following sections are arranged as preliminaries, Haar wavelet approach on FFDE, convergence of Haar wavelet, numerical experiments, and conclusion.

2 Preliminaries

2.1 Fractional derivative with different kernels

**Definition 2.1.** The Riemann–Liouville fractional derivative is defined as in [41] for $n - 1 < a < n$ by

$$\frac{R_l^a}{D_x^a}f(x) = \frac{d^n}{dx^n} \left[ \frac{1}{\Gamma(a)} \int_a^x (x-t)^{a-1} f(t) dt \right]. \quad (2.1)$$

**Definition 2.2.** The derivative of fractional order in sense of Caputo type [41] is defined as

$$\frac{C_D^a}{D_x^a}f(x) = \begin{cases} \frac{1}{\Gamma(n-a)} \int_a^x (x-t)^{n-a-1} f(t) dt, & \text{for } n-1 < a < n, n \in \mathbb{N}, \\ \frac{d^n}{dx^n} [f(x)], & \text{for } n = a. \end{cases} \quad (2.2)$$

Particularly, in case of $0 < a < 1$,

$$\frac{C_D^a}{D_x^a}f(x) = \frac{1}{\Gamma(1-a)} \int_a^x (x-t)^{-a} f'(t) dt. \quad (2.3)$$

**Definition 2.3.** For $f$ to be continuous and differentiable in $(a, b)$, the FFDPL is defined as in the study by Akgul [23]:

$$\frac{F_{FP}^\beta}{D_x^a}f(x) = \frac{1}{\Gamma(1-a)} \frac{d}{dx^\beta} \left[ \int_a^x (x-t)^{-a} f(t) dt \right]. \quad (2.4)$$

for $0 < a, \beta \leq 1$, where

$$\frac{df(x)}{dx^\beta} = \lim_{\alpha \to 1} \frac{f(x) - f(t)}{x^\beta - t^\beta}.$$
where we will consider \( N(\alpha) = 1 \), which is a constant of normalization.

**Definition 2.5.** The CF derivative in the Riemann–Liouville sense of order \( \alpha \) is defined as in the study by Attia et al. [61]:

\[
\frac{\text{CF} a D_0^\alpha f(x)}{a} = \frac{N(\alpha)}{(1 - \alpha)} \frac{d}{dx} \left[ \int_0^x f(t)e^{\frac{\alpha}{\alpha}(x-t)}dt \right],
\]

(2.6)

where we will consider \( N(\alpha) = 1 \), which is a constant of normalization.

**Remarks.** The relationship between CFR and CF is

\[
\frac{\text{CF} a D_0^\alpha f(x)}{a} = \frac{\text{CF} D_0^\alpha f(x)}{a} + \frac{N(\alpha)}{(1 - \alpha)} f(0)e^{-\alpha(x-t)}.
\]

(2.7)

### 2.2 Haar wavelet family

The structure of the Haar wavelet family has been formed by the multiresolution analysis, which is the foundation of decomposing the function in smaller waves. A sequence of nested spaces on \( V_j \subseteq V_{j+1} \) at various levels \( j \) whose union is dense in \( L_2(\mathbb{R}) \), forms a multiresolution analysis (MRA) of \( X = \{V_j \subseteq L_2| j \in \mathbb{Z}\} \) of \( X \). Let \( \langle , \rangle \) be an inner product on \( X \) and \( L_2(X) \) be the space of functions with finite energy defined over the domain \( X \subseteq \mathbb{R}^n \). The sets of scaling basis functions \( \{\psi_{k,j}| k \in \kappa(j)\} \) in the full orthonormal system, as stated by Christov, form the bases of the spaces \( V_j \), \( \kappa(j) \) is an index set defined over all basis functions on level \( j \). Since

\[
V_j \oplus W_j = V_{j+1},
\]

the strictly nested structure of the \( V_j \) presupposes the presence of difference spaces.

The Haar scaling function, which is represented by the symbol \( h_0(x) \) and is typically stated as the following, appears as a square wave over the interval \( x \in [0, 1] \):

\[
h_0(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere}, \end{cases}
\]

(2.8)

The wavelet at level zero, known as the Haar father wavelet, has no displacement and unit magnitude dilation.

Let \( \phi(x) \) be a scaling function satisfying dilation and translation as follows:

\[
\phi(x) = \sum_{k=1}^{m} a_k \phi(3x - k),
\]

for \( a_k \)'s nonzero works as bases of spaces \( V_j \)'s at different level \( j \). MRA aims to express any function at several levels of \( j \). The symmetric wavelet \( \psi_1 \), anti-symmetric wavelet \( \psi_2 \), and scale 3 Haar Wavelet scaling function \( h_i \) are defined as follows:

\[
h_i(x) = \begin{cases} 1, & a(i) \leq x \leq b(i), \\ 0, & \text{elsewhere}, \end{cases}
\]

(2.9)

\[
\psi_1(x) = \frac{1}{\sqrt{2}} \begin{cases} 1, & 0 \leq x < 1/3, \\ 2, & 1/3 \leq x < 2/3, \\ -1, & 2/3 \leq x < 1, \end{cases}
\]

(2.10)

\[
\psi_2(x) = \frac{1}{\sqrt{3}/2} \begin{cases} 1, & 0 \leq x < 1/3, \\ 0, & 1/3 \leq x < 2/3, \\ -1, & 2/3 \leq x < 1/1. \end{cases}
\]

(2.11)

For the more generalized form for interval \( [a, b) \), the wavelets are defined as follows:

\[
h_i(x) = \begin{cases} 1, & a(i) \leq x \leq b(i), \\ 0, & \text{elsewhere}, \end{cases}
\]

(2.12)

\[
\psi_1(x) = \frac{1}{\sqrt{2}} \begin{cases} 1, & c(i) \leq x < d(i), \\ 2, & c(i) \leq x < d(i), \\ -1, & d(i) \leq x < b(i), \end{cases}
\]

(2.13)

\[
\psi_2(x) = \frac{1}{\sqrt{3}/2} \begin{cases} 1, & a(i) \leq x < c(i), \\ 0, & c(i) \leq x < d(i), \\ -1, & d(i) \leq x < b(i), \end{cases}
\]

(2.14)

where

\[
a(i) = a + (b - a) \frac{k}{m},
\]

\[
c(i) = a + (b - a) \frac{k + 1/3}{m},
\]

\[
d(i) = a + (b - a) \frac{k + 2/3}{m},
\]

\[
b(i) = a + (b - a) \frac{k + 1}{m},
\]

where \( j = 0, 1, 2, ..., J \), and \( m = 3^j \). The index \( i > 1 \) is calculated from the relation \( m + 2k = i \) for even index and \( i = m + 2k + 1 \) for index having odd number. Mother wavelets \( \psi_1 \) and \( \psi_2 \) are, respectively, representations of even and odd wavelets.

### 3 Haar wavelet fractal fractional method (HWFFM)

Consider the FFDPL as follows:

\[
\frac{\text{FFDPL} D_0^{\alpha, \beta} f(x)}{a} = F(x, f(x)).
\]

(3.1)
Using Definition 2.3, it has been converted into
\[ \frac{R}{a} D_x^\alpha f(x) = \beta x^{\beta - 1} F(x, f(x)), \] (3.2)
as explained in the study by Akgul [23], which can further be reduced using (2.7) into
\[ \frac{D_x^\alpha f(x)}{a} = \beta x^{\beta - 1} F(x, f(x)) - \frac{f(0)}{\Gamma(1 - a)} x^{-a}. \] (3.3)

Over the interval \([a, b]\), consider a square integrable function \(f(x)\) whose derivative with highest order can be expressed as a linear combination of Haar wavelet family as
\[ f^n(x) = \sum_{i=0}^{3p} a_i h_i(x). \]

Integrating the aforementioned equation from 0 to \(x\), the lower-order derivatives can be evaluated as
\[ f^n(x) = \sum_{i=0}^{3p} a_i P_{k,i}(x) + \sum_{k=1}^n \frac{\Gamma(1 - x)}{\Gamma(k)} f^{n-k}(0) + \ldots + f^{n-k}(0), \]
i.e., by integrating the assumed higher-order derivative. Here, \(P_{k,i}\) is the matrix of integration of \(h_i(x)\). For fractional order derivatives, the definitions discussed in Section 2 are used to obtain matrix of integration. As we are considering \(0 < a, \beta \leq 1\), we will consider
\[ f'(x) = \sum_{i=0}^{3p} a_i h_i(x). \]

Therefore,
\[ f(x) = \sum_{i=0}^{3p} a_i P_{1,i}(x) + f(0) \]
and
\[ f^\alpha(x) = \sum_{i=0}^{3p} a_i P_{1-\alpha,i}(x), \]
whose operational matrix will be calculated in the following for \(i = 1:\)
\[ P_{1,i}(x) = \frac{1}{\Gamma(a + 1)}(x - a)^{\alpha}. \] (3.4)

For even and odd values of \(i\), the operational matrices have been obtained as (3.9) and (3.10). These matrices have been calculated using definition (2.2), the derivative of fractional order in Caputo sense, and \(\psi(x)\) and \(\psi_\alpha(x)\) are the Haar wavelet functions defined in (2.10) and (2.12).

Now, consider FFDED as follows:
\[ \frac{FFT}{a} D_x^{\alpha, \beta} f(x) = F(x, f(x)), \] (3.5)
which can be converted to
\[ \frac{FFT}{a} D_x^{\alpha, \beta} f(x) = \beta x^{\beta - 1} F(x, f(x)), \] (3.6)
as explained in previous studies [23, 61], which can further be reduced using (2.7):
\[ \frac{CF}{a} D_x^{\alpha, \beta} f(x) = \beta x^{\beta - 1} F(x, f(x)) - \frac{N(a)}{(1 - a)} f(0) e^{-\frac{\Gamma(\alpha)}{\Gamma(\alpha - 1)}}. \] (3.7)

Using the process discussed earlier with operational matrix for CF kernel for \(0 < a \leq 1\) is for \(i = 1:\)
\[ P_{i,a}(x) = \frac{1}{\Gamma(\alpha + 1)} \left[ \frac{1}{\sqrt{2}}(x - a)^{\alpha} \right], \]
\[ \begin{align*}
- (x - a)^{\alpha} & \quad \text{if } x \in [a, b] \\
- (x - a)^{\alpha} + 3(x - c(i))^\alpha & \quad \text{if } x \in [c(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- (x - b(i))^\alpha & \quad \text{if } x \in [b(i), 1] \\
0 & \quad \text{elsewhere.}
\end{align*} \] (3.8)

Similarly, for even and odd values of \(i\), the operational matrices have been obtained as (3.9) and (3.10). These operational matrices have been calculated using definitions (2.4) and \(\psi(i), \psi_\alpha(i)\), and \(\psi_\beta(i)\), Haar wavelet functions defined in (2.10) and (2.12) discussed in Section 2.

Integral in Caputo derivative for \(i = 2, 4, 6, \ldots\)
\[ P_{i,a}(x) = \frac{1}{\sqrt{2}} \Gamma(\alpha + 1) \left[ \frac{1}{\Gamma(\alpha + 1)} \right], \]
\[ \begin{align*}
(x - a)^{\alpha} & \quad \text{if } x \in [a, b] \\
- (x - a)^{\alpha} + 3(x - c(i))^\alpha & \quad \text{if } x \in [c(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- (x - b(i))^\alpha & \quad \text{if } x \in [b(i), 1] \\
0 & \quad \text{elsewhere.}
\end{align*} \] (3.9)

Integral in Caputo derivative for \(i = 3, 5, 7, \ldots\)
\[ P_{i,a}(x) = \frac{1}{\sqrt{2}} \Gamma(\alpha + 1) \left[ \frac{1}{\Gamma(\alpha + 1)} \right], \]
\[ \begin{align*}
(x - a)^{\alpha} & \quad \text{if } x \in [a, b] \\
- (x - a)^{\alpha} + 3(x - c(i))^\alpha & \quad \text{if } x \in [c(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- 3(x - d(i))^\alpha & \quad \text{if } x \in [d(i), b(i)] \\
- (x - b(i))^\alpha & \quad \text{if } x \in [b(i), 1] \\
0 & \quad \text{elsewhere.}
\end{align*} \] (3.10)
Integral in CF derivative for \( i = 2, 4, 6, \ldots \)

\[
P_{i,a}(x) = \frac{1}{\sqrt{2}a} \left[ \begin{array}{c} \left[ -1 + e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e(a(i), c(i))], \\
\left[ 2 + e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e(c(i), d(i))], \\
\left[ -3e^{-\frac{x}{\pi a}(x-c(i))} \right] x[e[d(i), b(i))], \\
\left[ -1 + e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e(b(i), 1)], \\
\left[ -3e^{-\frac{x}{\pi a}(x-c(i))} \right] x[e[1]], \\
\left[ + 3e^{-\frac{x}{\pi a}(x-d(i))} \right] \end{array} \right]
\]

(3.11)

Integral in CF derivative for \( i = 3, 5, 7, \ldots \)

\[
P_{i,a}(x) = \frac{1}{\sqrt{2}a} \left[ \begin{array}{c} \left[ 1 - e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e(a(i), c(i))], \\
\left[ -e^{-\frac{x}{\pi a}(x-a(i))} + e^{-\frac{x}{\pi a}(x-c(i))} \right] x[e(c(i), d(i))], \\
\left[ -1 - e^{-\frac{x}{\pi a}(x-d(i))} \right] x[e[e(d(i), b(i))], \\
\left[ -e^{-\frac{x}{\pi a}(x-b(i))} + e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e[b(i), 1)], \\
\left[ -e^{-\frac{x}{\pi a}(x-c(i))} + e^{-\frac{x}{\pi a}(x-a(i))} \right] x[e[1]], \\
\left[ + e^{-\frac{x}{\pi a}(x-d(i))} \right] \end{array} \right]
\]

(3.12)

The following equation

\[
||f(x) - f_{3m}(x)|| \leq \frac{M}{\sqrt{2A 3^l}}
\]

gives the error bound determined for the Haar wavelet approximation of the function \( f(x) \) by \( L_2 \)-norm. This indicates that if we know the precise value of \( M \), we can obtain the correct error bound for the approximation. Additionally, as the level of resolution increased, the amount of error reduced, demonstrating the convergence of approximations to exact solutions.

5 Numerical experiments and error analysis

To test the effectiveness and applicability of proposed method, some initial value problems have been solved. To check the efficiency of the proposed method, the following errors are calculated:

Absolute error (AE) = \( |f_{\text{exact}}(x_l) - f_{\text{num}}(x_l)| \),

\[
L_m = \max \{ |f_{\text{exact}}(x_l) - f_{\text{num}}(x_l)| \},
\]

\[
L_2 = \sqrt{\sum_{l=1}^{m} |f_{\text{exact}}(x_l) - f_{\text{num}}(x_l)|^2}.
\]

In case of nonavailability of exact solution, for absolute and relative errors, the numerical results have been compared with the existing solutions in the literature and used in place of exact solution in error calculations.

Example 1. In Eq. (3.1) for \( f(0) = 0 \) and \( F(x, f(x)) = x^2 \), the fractional differential equation with derivative of Caputo sense becomes

\[
\frac{D^x_{a}f}{x} = \beta x^{\beta+1} - \frac{f(0)}{\Gamma(1 - \alpha)}.
\]

The available Laplace transformation solution is

\[
f(x) = \frac{\beta\Gamma(2 + \beta)x^{\alpha+\beta+1}}{\Gamma(2 + \alpha + \beta)}.
\]

Using the HWFFM, approximate solution has been obtained and comparison between the proposed method and LTM [23] is presented in Figures 1 and 2. Table 1 depicts \( L_m \) and \( L_2 \) errors for collocation points at different level of Haar wavelet.

We can see with an increase in collocation points, errors decreases rapidly and solution converges to exact solution.
Example 2. In Eq. (3.5) for $f(0) = 0, F(x, f(x)) = x$ with exact solution
\[ f(x) = \frac{\beta \Gamma(1 + \beta)}{N(\alpha)} \left[ \frac{1 - \alpha}{\Gamma(1 + \beta)} + \frac{\alpha x}{\Gamma(1 + \beta)} \right] x^{\beta}, \]
the fractional differential equation with derivative of CF sense becomes
\[ \frac{cF}{d\alpha} f(x) = \beta x^\beta - \frac{f(0)}{\Gamma(1 - \alpha)}, \quad 0 < \alpha < 1, \ 0 < x \leq 1. \]

Using the HWFFM approximate solution has been obtained, and comparison between the proposed method and exact solution [61] is presented in Figure 3 and absolute error in Figure 4. $L_\infty$ and $L_2$ errors for collocation points at different levels of Haar wavelet are presented in Table 2.

**Table 1: Error analysis of solution of Example 1 for order $\alpha = \beta = 0.1$**

<table>
<thead>
<tr>
<th>Coll. points</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.00338077</td>
<td>0.000396290</td>
</tr>
<tr>
<td>9</td>
<td>0.00090463</td>
<td>0.000001110</td>
</tr>
<tr>
<td>27</td>
<td>0.000024206</td>
<td>0.000005129</td>
</tr>
<tr>
<td>81</td>
<td>0.000006477</td>
<td>0.000000726</td>
</tr>
<tr>
<td>243</td>
<td>0.000001733</td>
<td>0.000000099</td>
</tr>
</tbody>
</table>

**Figure 1:** Comparison of LTM and HWFFM solutions for $\alpha = \beta = 0.1$ for $m = 27$.

**Figure 2:** AE for $\alpha = \beta = 0.1$ for $m = 27$.

**Figure 3:** Comparison of exact and HWFFM solutions for $\alpha = \beta = 0.7$ for $m = 27$.

**Figure 4:** AE for $\alpha = \beta = 0.7$ for $m = 27$.

Figure 2: AE for $\alpha = \beta = 0.1$ for $m = 27$.  

Example 1 (Absolute error)
Example 3. In Eq. (3.5) for \( f(0) = 0, F(x, f(x)) = x^2 \), the given Laplace transformation solution is

\[
f(x) = \frac{\beta x^{\alpha+1} [2 + \alpha(-2 - \beta + x)] + \beta}{N(\alpha) (2 + \beta)}. \]

The fractional differential equation with derivative of CF sense becomes

\[
_0^C D_x^\alpha f(x) = \beta x^\alpha - \frac{f(0)}{\Gamma(1 - \alpha)}, \quad 0 < \alpha < 1 \quad 0 < x \leq 1. \tag{5.3}
\]

Using the HWFFM approximate solution has been obtained, and comparison between the proposed method and LTM is presented in Figure 5. The values of \( L_\infty \) and \( L_2 \) errors in Table 3 explain good convergence and reliability.

Example 4. In Eq. (3.1) for \( f(0) = 0, F(x, f(x)) = x \), the fractional differential equation with derivative of Caputo sense becomes

\[
_0^C D_x^\alpha f(x) = \beta x^\alpha - \frac{f(0)}{\Gamma(1 - \alpha)}. \tag{5.4}
\]

Using HWFFM, several results have been obtained and comparison between the proposed method and reproducing kernel Hilbert space method (RKHM) [23] is presented in Figures 6 and 7 and Tables 4 and 5 with AE for different values of \( \alpha \) and \( \beta \).

Here, exact solution for the problem is not available. The AE has been calculated by comparing with available values from RKHM solution with HWFFM values for \( m = 243 \).

For \( \alpha = \beta = 0.1 \), the errors according to available data are \( L_\infty = 0.00010054 \) and \( L_2 = 0.00037683 \). For \( \alpha = \beta = 0.5 \), \( L_\infty = 0.000019880 \) and \( L_2 = 0.00002220 \).

Example 5. In Eq. (3.5) for \( f(0) = 0, F(x, f(x)) = \sin x \), using HWFFM, approximate solutions have been obtained and are presented in Figure 8 and Table 6 for different values of \( \alpha \) and \( \beta \).

Example 6. In Eq. (3.5) for \( f(x, f(x)) = x^2 + x f^2(x) \) and \( f(0) = 0 \), applying quasilinearization technique as explained in [52] to convert the nonlinear equation into linear equation, the given equation becomes

\[
_0^C D_x^{\alpha+1} f(x) + 2 \beta x^\beta f(x) f_x(x) = \beta x^\beta (x + f_x(x)), \tag{5.5}
\]

Using HWFFM, several results have been obtained and comparison between the proposed method and reproducing kernel Hilbert space method (RKHM) [23] is presented in Figures 6 and 7 and Tables 4 and 5 with AE for different values of \( \alpha \) and \( \beta \).
subject to boundary condition \( f_{r+1}(x) = 0 \) for \( r = 0, 1, 2, \ldots \). The quasilinearization process estimates the solution to aforementioned differential equation as \((r + 1)^{th}\) iterative approximation \( f_{r+1}(x) \). Using HWFFM, approximate solutions have been obtained and are presented in Figure 9 for different values of \( \alpha \) and \( \beta \).

**Table 4:** Comparison of solution of Example 4 for order \( \alpha = 0.1 \)

<table>
<thead>
<tr>
<th>x</th>
<th>HWFFC</th>
<th>RKHSM</th>
<th>AE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0386</td>
<td>0.0385</td>
<td>0.000051117</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0534</td>
<td>0.0533</td>
<td>0.000068131</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0626</td>
<td>0.0627</td>
<td>0.000100548</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0695</td>
<td>0.0696</td>
<td>0.000087689</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0751</td>
<td>0.0750</td>
<td>0.000051553</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0796</td>
<td>0.0796</td>
<td>0.000017264</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0834</td>
<td>0.0835</td>
<td>0.000066789</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0869</td>
<td>0.0869</td>
<td>0.000002014</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0926</td>
<td>0.0926</td>
<td>0.000036199</td>
</tr>
</tbody>
</table>

**Table 5:** Comparison of solution of Example 1 for order \( \alpha = 0.5 \)

<table>
<thead>
<tr>
<th>x</th>
<th>HWFFC</th>
<th>RKHSM</th>
<th>AE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.030746341</td>
<td>0.030704377</td>
<td>0.000041964</td>
</tr>
<tr>
<td>0.2</td>
<td>0.061417073</td>
<td>0.061401056</td>
<td>0.000006917</td>
</tr>
<tr>
<td>0.3</td>
<td>0.092107317</td>
<td>0.092115688</td>
<td>0.000008371</td>
</tr>
<tr>
<td>0.4</td>
<td>0.122819512</td>
<td>0.122821151</td>
<td>0.000016393</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1535</td>
<td>0.153526584</td>
<td>0.000026584</td>
</tr>
<tr>
<td>0.6</td>
<td>0.18425122</td>
<td>0.184239294</td>
<td>0.000019218</td>
</tr>
<tr>
<td>0.7</td>
<td>0.214931707</td>
<td>0.21493741</td>
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<tr>
<td>0.8</td>
<td>0.245673171</td>
<td>0.245642812</td>
<td>0.000030359</td>
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<tr>
<td>0.9</td>
<td>0.276353659</td>
<td>0.276348209</td>
<td>0.000005450</td>
</tr>
<tr>
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<td>0.3064</td>
<td>0.307053603</td>
<td>0.000065360</td>
</tr>
</tbody>
</table>

**Table 6:** Approximate solution of Example 5 for different orders of \( \alpha \) and \( \beta \) for \( m = 27 \)

<table>
<thead>
<tr>
<th>x</th>
<th>( \alpha = 0.8 )</th>
<th>( \alpha = 0.7 )</th>
<th>( \alpha = 0.9 )</th>
<th>( \alpha = 0.75 )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.016678934</td>
<td>0.039006236</td>
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<tr>
<td>0.2</td>
<td>0.063415531</td>
<td>0.086239294</td>
<td>0.040964798</td>
<td>0.074865466</td>
</tr>
<tr>
<td>0.3</td>
<td>0.100565947</td>
<td>0.126013649</td>
<td>0.072947847</td>
<td>0.113670901</td>
</tr>
<tr>
<td>0.4</td>
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<td>0.167634813</td>
<td>0.11223048</td>
<td>0.15569214</td>
</tr>
<tr>
<td>0.5</td>
<td>0.18822287</td>
<td>0.210983953</td>
<td>0.158172052</td>
<td>0.200620584</td>
</tr>
<tr>
<td>0.6</td>
<td>0.23783262</td>
<td>0.255855164</td>
<td>0.21034956</td>
<td>0.248114747</td>
</tr>
<tr>
<td>0.7</td>
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<td>0.301965745</td>
<td>0.268112708</td>
<td>0.297808764</td>
</tr>
<tr>
<td>0.8</td>
<td>0.346189129</td>
<td>0.348854547</td>
<td>0.330826379</td>
<td>0.349139135</td>
</tr>
<tr>
<td>0.9</td>
<td>0.403805847</td>
<td>0.396181718</td>
<td>0.397856757</td>
<td>0.401688303</td>
</tr>
<tr>
<td>1</td>
<td>0.461793667</td>
<td>0.442604106</td>
<td>0.467066963</td>
<td>0.453904749</td>
</tr>
</tbody>
</table>

**Figure 7:** AE for \( \alpha = \beta = 0.1 \) and \( \alpha = \beta = 0.5 \).

**Figure 8:** Approximate solutions by HWFFM for \( \alpha = \beta = 0.7, \alpha = \beta = 0.8, \alpha = \beta = 0.9, \) and \( \alpha = \beta = 0.75 \).

**Figure 9:** Approximate solutions by HWFFM for different values of \( \alpha \) and \( \beta \).

**6 Conclusion**

Fractal-fractional derivatives in Riemann-Liouville sense with singular and nonsingual kernels have been tackled efficiently by proposed method. A novel operational matrix for integral of Haar wavelet has been developed to handle CF derivative for fractional order \( \alpha \). The scheme has been successfully applied to solve six numerical problems and
presented comparison with exact and available results. The benefits of the proposed technique can be seen as easy applicability, non-complex calculations, conversion of problem into algebraic equation, reliable solutions, good convergence, and less computation time. Therefore, the technique could be extended and suggested to solve partial differential equations with fractal-fractional derivatives.

**Funding information:** This work has not received any external funding.

**Author contributions:** All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

**Conflict of interest:** The authors declare that there is no conflict of interest.

**Data availability statement:** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

**References**


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