Research Article

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An implication of magnetic dipole in Carreau Yasuda liquid influenced by engine oil using ternary hybrid nanomaterial

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Abstract: The aim of this work was to study the enhancement of thermal transportation in Carreau Yasuda liquid passed over a vertical surface in the presence of magnetic dipole. A mixture of tri-hybrid nanoparticles (AlO₂, MoS₂, TiO₂) is inserted into the Carreau Yasuda liquid. The transport phenomenon of heat is derived in the presence of heat source/sink contribution. The concept boundary layer theory is engaged to derive the mathematical expression for momentum and energy in the form of coupled partial differential equations. The derivations are transformed into a set of coupled nonlinear ordinary differential equations (ODEs) with the help of suitable similarity transformation. These converted ODEs have been handled numerically via finite element method. The grid-independent analysis is established for 300 elements. The impact of numerous involved parameters on temperature and velocity solution is plotted and their contribution is recorded. Temperature profile is inclined versus the higher values of heat generation and viscous dissipation numbers while thermal layers are also increasing the behavior. A vital role of magnetic dipole is examined to raise the production of thermal layers but declination is noticed in flow profile.

Keywords: magnetic dipole, Carreau Yasuda liquid, ternary hybrid nanofluid, thermal performance, engine oil, heat source sink, vertical surface

Nomenclature

v, u velocity components
ρ fluid density
H₀ magnetic permeability
H magnetic field
Λ Carreau Yasuda number
C_p specific heat
Q₀ heat source number
T fluid temperature
u_e free stream velocity
β ferrohydrodynamic interaction number
π Pι
We Weissenberg number
λ viscous dissipation number
ϕ₁, ϕ₂, ϕ₃ volume fractions
σ electrical conductivity
Re Reynolds number
a, b space coordinates
P pressure
M magnetization
γ strength of the magnetic dipole
d fluid number
k thermal conductivity
S₁ stretching ratio number
T_w wall temperature
ψ trial shape function

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1 Introduction

Numerous research works have been carried out on nanomaterials due to their wider applications in medicine, energy system and different industrial mechanisms. The involvement of nanoparticles has also been tested for the treatment of cancer. Researchers used these nanoparticle mixtures in different liquids to study the thermal performance. For instance, Shah et al. [1] worked on ecosystem by studying the inclusion of titanium dioxide particles. Khan et al. [2] examined the contribution of slip effects on Eyring–Powell liquid with heat transport in which graphene particles are mixed. The phenomenon of thin film is further discussed under time-dependent magnetic field. The flow-governing equations have been solved analytically via homotopic solution scheme. They noted the decline in temperature field against Prandtl number. Rehman et al. [3] numerically solved the magnetized non-Newtonian Casson model passed over a radial rotating stretching sheet. They handled the derived boundary layer transformed nonlinear set of differential equations via shooting scheme. They observed the decline in fluid velocity by mounting the values of slip parameter, Casson fluid parameter and magnetic parameter. Albaid et al. [4] presented the experimental study of carbon-based nanoparticles to examine the degraded soil properties. Ali et al. [5] presented the involvement of thermal radiation and heat generation for the stagnation point flow of viscous liquid past over a stretching cylinder by including the involvement of Brownian motion and thermophoresis. They used shooting scheme to solve the coupled nonlinear ordinary differential equations (ODEs) in unbounded domain. They recorded the decline in velocity against the mounting values of curvature parameter and also the decrease in temperature field is monitored against Prandtl number and radiation parameter. Boaescu et al. [6] scrutinized the involvement of nanomaterials in making the different medicines and drugs. Pop et al. [7] presented the experimental survey by considering the involvement of nanoparticles in nerve injury treatment. Iacoviță et al. [8] examined the involvement of silica nanoparticles to study the cancer cells in human body. Swain et al. [9] studied MWCNT/Fe₃O₄ hybrid nanofluid past over an exponentially stretched porous sheet under slip boundary conditions and chemical reaction. They considered the influence of radiation and heat generations in energy equation. They solved the system of nonlinear transformed ODEs via BVP4C procedure in MATLAB computational package. They noticed the decrease in velocity, temperature and concentration fields against slip parameter. Furthermore, they monitored the enhancement in heat transfer rate against Prandtl number. Gul and Firdous [10] presented the experimental, theoretical and analytical investigation to discuss the comportment of graphene nanoparticles in water and the flow is produced due to the rotation of disks. They presented the convergence and error analysis. They have shown the rise in fluid velocity against Reynolds number. Tassaddiq et al. [11] studied the hybrid nanofluid flow between rotating disks and established the analytical solution. They have considered the involvement of viscous dissipation in energy equation. They noticed the decline in velocity for magnetic parameter. Also, they observed that against volume fraction, temperature field increases but concentration profile decreases. Shafiq et al. [12] studied bio-convective and buoyancy-driven flow phenomena for second-grade liquid with chemical reaction. They solved the resulting equations numerically. They noticed the enhancement in motile density profile against thermophoresis parameter, whereas reverse behavior is recorded for Pecklet number. Nisar et al. [13] studied the inclusion of gold nanoparticles in micro-polar liquid past over a stretched sheet. They computed the numerical solution for coupled nonlinear transformed ODEs. Khan et al. [14] examined the involvement of nanoparticles in bio-convective flow of modified second-grade liquid. They used shooting approach to solve the transformed modeled ODEs. They found the decrease in motile density profile against Lewis number. Waqas et al. [15] studied numerically the flow of nanoparticles past over a slandering surface under radiation effect. Some important contributions are reported in refs [16–19] and references therein. Sadat et al. [20] performed the model based on 3D Euler equations using invariant solution and lie symmetry study in view of incompressible flow via cylindrical coordinates. Mousa et al. [21] investigated MHD convection in square cavities using localized heating approach. Ali et al. [22] used a new wavelet approach to solve boundary value problems based on adiabatic tubular chemical reactor theory. Sabir et al. [23] simulated fourth-order nonlinear system using Levenberg–Marquardt backpropagation approach. Baleanu et al. [24] studied the role of

An extensive research has been conducted on nanoparticles so far due to their wider applications and utilization. This research is conducted in the presence of magnetic dipole for the inclusion of ternary hybrid nanoparticle mixture in Carreau Yasuda liquid, and numerical computation is performed by using finite element method (FEM) tool in MAPLE18.0 package. The influence of different emerging parameters has been displayed and discussed.

## 2 Mathematical analysis

A 2D heat transfer model is carried out in Carreau Yasuda liquid past a stretching surface. A Carreau Yasuda liquid is immersed along with base fluid based on engine oil. Three kinds of nanoparticles (aluminum oxide, MoS2 and TiO2) are inserted into base liquid as shown in Figure 1. A wall is considered as stretchable to bring motion into fluid particles. A magnetic dipole is considered while the center related to magnetic dipole is placed at the horizontal direction. The flow development is assumed by Figure 2. Transfer of heat energy is considered as absorption and generation into fluid particles. A set of partial differential equations [46,47] is modeled using considerations.

![Figure 1: Mixture of nanoparticles resulting in tri-hybrid nanoparticles.](image-url)
\[
\frac{\partial u}{\partial a} + \frac{\partial v}{\partial b} = 0, \quad (1)
\]

\[
\rho_{\text{Thnf}} \left( \frac{\partial u}{\partial a} + \frac{\dot{v}}{\partial b} \right) = -\frac{\partial P}{\partial a} + \mu_{\nu M} \frac{\partial H}{\partial a} + \nu_{\text{Thnf}} \left[ \frac{\partial^2 u}{\partial b^2} + \left( \lambda \right) \left( \frac{m-1}{d} \right) (d+1) \frac{\partial^2 u}{\partial b^2} \left( \frac{\partial u}{\partial b} \right) \right], \quad (2)
\]

\[
J_{\text{Thnf}} = \frac{1}{\partial T} \frac{\partial T}{\partial b} + \left( \frac{\mu_{\nu M}}{\partial a} + \nu_{\text{Thnf}} \frac{\partial H}{\partial b} \right) \mu_{\nu T} \frac{\partial M}{\partial T} + K_{\text{Thnf}} \frac{\partial^2 T}{\partial b^2} + Q_0 (T - T_\infty). \quad (3)
\]

Figure 2 captures the behavior of developed model. It is mentioned that horizontal surface is assumed where \( a \)-axis is taken along the horizontal direction and \( b \)-axis is considered along the vertical direction. The motion into fluidic particles is produced using movement of wall. Moreover, direction of magnetic dipole is visualized along \( a \)-axis due to an implication of magnetic dipole.

Boundary conditions \([46,47]\) are

\[
u = (s \nu) \frac{1}{2} f, \quad \theta = \frac{T_{\infty} - T}{T_{\infty} - T_w}, \quad \eta = \frac{t}{b} \frac{1}{2}. \quad (10)
\]

A set of dimensionless ODEs with boundary conditions \([46,47]\) is derived as (Table 1)

\[
\begin{align*}
&f'''' + \frac{(We)^{d}(m-1)(d+1)}{d} f''''(f'')^d \\
&- \frac{\nu_{\text{Thnf}}}{\nu_{f}} \left\{ f' f'' f'''' + \frac{\theta_{\nu} \frac{2}{\theta}}{(\lambda + \gamma)^d} \right\},
\end{align*} \quad (11)
\]

\[
\begin{align*}
&f'(0) = 1, \quad f'(\infty) = 0, \quad f(0) = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0.
\end{align*}
\]

\[
H_b = \frac{\partial \delta}{\partial a} = \frac{\delta}{2\pi} \left( \frac{a - (b + d)^2}{(a^2 + (b + d)^2)^2} \right), \quad (6)
\]

\[
\begin{align*}
&H_a = \frac{\partial \delta}{\partial b} = \frac{\delta}{2\pi} \left( \frac{a - (b + d)^2}{(a^2 + (b + d)^2)^2} \right), \quad (7)
\end{align*}
\]

Magnitude of magnetic induction is

\[
H = \left( \frac{\partial \delta}{\partial a} \right)^2 + \left( \frac{\partial \delta}{\partial b} \right)^2 \gamma. \quad (8)
\]

Using Binomial series and expanding it up to \( a^2 \),

\[
H_b = \frac{\partial \delta}{\partial b} - \frac{2a}{(b + d)^2} \gamma. \quad (9)
\]

Transformations \([46,47]\) are defined as

\[
u = (s \nu) \frac{1}{2} f, \quad \theta = \frac{T_{\infty} - T}{T_{\infty} - T_w}, \quad \eta = \frac{t}{b} \frac{1}{2}. \quad (10)
\]
It is noticed that Eqs. (11)–(12) are known as non-Newtonian model in the presence of occurrence of Carreau Yasuda liquid. The present non-Newtonian study is reduced into a case of Newtonian model considering $\text{We} = 0$ and $\beta = 0$.

The correlations for ternary hybrid nanoparticles [48 and 49] are

$$\rho_{\text{Thnf}} = (1 - \varphi_f)\{(1 - \varphi_f)(1 - \varphi_f)\rho_f + \varphi_f\rho_s\} + \varphi_f\rho_1,$$

and

$$\frac{\mu_f}{(1 - \varphi_f)^2(1 - \varphi_f)^2(1 - \varphi_f)^2} = \frac{K_{\text{Thnf}}}{K_{nf}} = \frac{K_1 + 2K_{nf} - 2\varphi_f(K_{nf} - K)}{K_{nf} + 2K_{nf} + \varphi_f(K - K}),$$

$$\frac{K_{nf}}{K_f} = \frac{K_{3} + 2K_f - 2\varphi_f(K_f - K)}{K_f + 2K_f + \varphi_f(K_f - K)}.$$  \hspace{1cm} (13)

$$\sigma_{\text{Thnf}} = \frac{\sigma_f(1 + 2\varphi_f) - \varphi_{\text{Thnf}}(1 - 2\varphi_f)}{\sigma_f(1 - \varphi_f) + \varphi_{\text{Thnf}}(1 + \varphi_f)},$$

$$\frac{\sigma_{\text{Thnf}}}{\sigma_f} = \frac{\sigma_f(1 + 2\varphi_f) + \varphi_{\text{Thnf}}(1 - 2\varphi_f)}{\sigma_f(1 - \varphi_f) + \varphi_{\text{Thnf}}(1 + \varphi_f)},$$

$$\frac{\sigma_{\text{Thnf}}}{\sigma_f} = \frac{\sigma_f(1 + 2\varphi_f) + \varphi_{\text{Thnf}}(1 - 2\varphi_f)}{\sigma_f(1 - \varphi_f) + \varphi_{\text{Thnf}}(1 + \varphi_f)}.$$  \hspace{1cm} (14)

**Table 1:** Thermal properties of hybrid nanoparticles with base fluid [48,49]

<table>
<thead>
<tr>
<th></th>
<th>$k$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engine oil</td>
<td>0.144</td>
<td>0.125 x $10^{-11}$</td>
<td>884</td>
</tr>
<tr>
<td>Aluminum oxide</td>
<td>32.9</td>
<td>5.96 x $10^7$</td>
<td>6,310</td>
</tr>
<tr>
<td>Titanium dioxide</td>
<td>8.953</td>
<td>2.4 x $10^8$</td>
<td>4,250</td>
</tr>
<tr>
<td>Silicon dioxide</td>
<td>1.4013</td>
<td>3.5 x $10^6$</td>
<td>2,270</td>
</tr>
</tbody>
</table>

Drag force coefficient of Carreau Yasuda liquid is formulated as

$$(\text{Re})^{\frac{1}{2}}C_f = -\left[\frac{m - \text{We}f''(0)}{(1 - \varphi_f)^2(1 - \varphi_f)^2(1 - \varphi_f)^2}ight]f''(0).$$

Rate of heat transfer in the presence of tri-hybrid nanoparticles is

$$(\text{Re})^{-0.5}\text{Nu} = -\frac{k_f}{K_{\text{Thnf}}}\theta(0).$$

**Figure 3:** Steps related to FEM.
3 Finite element approach

A strong numerical approach based on FEM is implemented to simulate numerical results of ODEs along with boundary conditions. An FEM is used to conduct the solutions of various CDD problems. It has the capacity to handle complex geometries as well as various types of boundary conditions. Six steps of FEMs are discussed below, while six steps are mentioned in Figure 3. An FEM is observed as a good method in view of accuracy analysis, convergence analysis and stability analysis rather than other numerical methods. The following advantages of implementing FEM are as follows.

- Numerous applications of FEM are investigated in computational fluid mechanics problems;
- Complex types of geometries are tackled by FEM;
- Physical problems based on applied science are developed by FEM;
- It has the ability to discretize the derivatives with very ease;
- An important role of FEM is to solve various types of boundary conditions;
- FEM requires low investment and time rather than other numerical techniques.

Step I: Domain discretization

The first step is about domain discretization of problem domain. Domain is broken into small elements of up to 300 elements. Three hundred elements are enough to simulate the solution of current analysis. It is noticed that a system of ODEs is called strong form, whereas weak form is achieved via the residual method.

Step II: Selection of shape function

A significant role of shape functions is used to obtain approximation solution of current analysis. Various types of shape functions are used in finite element procedure. In this procedure, linear kind of shape functions is used. The desired form of shape functions is defined as

\[ \psi_i = (-1)^{i+j} \left( \frac{-\eta + \eta_{j-1}}{-\eta + \eta_{j+1}} \right), \quad i = 1, 2. \]  

Step III: Weak formulation

Eqs. (12)–(14) are known as strong form along with boundary conditions. In this procedure, weak forms are needed to achieve approximation solution. Collection of all terms is placed on one side and integrated over 300 elements. The desired residuals of present problems are derived as

\[ \int_{n_e}^{n_e+1} w_i (F - S) d\eta = 0, \]  

Step IV: Finite element formulation

In this step, stiffness elements are obtained from current problem. Finally, global stiffness matrices are achieved over each element. The stiffness elements are derived as

\[ K_{ij}^{11} = 0, \quad K_{ij}^{12} = \int_{n_e}^{n_e+1} \left( \frac{d\psi_j}{d\eta} \right) d\eta, \]  

\[ K_{ij}^{22} = \int_{n_e}^{n_e+1} \left( \frac{d\psi_i}{d\eta} \right) \left( \frac{d\psi_j}{d\eta} \right) d\eta. \]  

Step V: Assembly process

Assembly process is an integral part of FEM. Stiffness matrices are formulated using concept of assembly approach.
Step VI: Solution of algebraic equations

Finally, a system of linear algebraic equations is numerically solved within computational tolerance (10^{-5}). The stopping condition is listed below. Flow chart of finite element procedure is given in Figure 4. Furthermore, validation of numerical results in terms of Nusselt number is shown in Table 3. Moreover, programming of FEM is designed on MAPLE 18. Homemade code regarding FEM is developed using MAPLE 18, whereas this code is tested with already published studies.

\[ \left| \frac{\delta_{i+1} - \delta_i}{\delta_i} \right| < 10^{-5}. \]  

(29)

3.1 Mesh-free study

The convergence of problem is investigated through investigation of mesh-free. It is noticed that the present problem becomes grid independent by observing 300 elements. The outcomes of velocity and temperature profiles against 30–300 elements are recorded in Table 2. The convergence of problem is achieved by observing 300 elements. It is included that numerical as well as graphical study is simulated via 300 elements.

3.2 Validation of numerical results

Table 3 is prepared to investigate the numerical results of present problem with already published work [46] considering \( Ec = H \alpha = \lambda = \gamma = \beta = 0 \). It is noticed that numerical results of current analysis are simulated by FEM while published results are derived by optimal homotopy analysis method (OHAM) analysis method. It is observed that good comparison is investigated. Moreover, [0, 8] is considered as computational domain and computational

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**Table 2: Simulations of temperature and velocity at mid of each 300 elements [47,49]**

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>( \theta(\frac{\eta}{2}) )</th>
<th>( f(\frac{\eta}{2}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.3830450314</td>
<td>0.3240812858</td>
</tr>
<tr>
<td>60</td>
<td>0.383760528</td>
<td>0.292231857</td>
</tr>
<tr>
<td>90</td>
<td>0.3831603611</td>
<td>0.2819022485</td>
</tr>
<tr>
<td>120</td>
<td>0.3829060468</td>
<td>0.2768517260</td>
</tr>
<tr>
<td>150</td>
<td>0.3827834669</td>
<td>0.2728692845</td>
</tr>
<tr>
<td>180</td>
<td>0.3826538369</td>
<td>0.2719045859</td>
</tr>
<tr>
<td>210</td>
<td>0.3825512827</td>
<td>0.2705127850</td>
</tr>
<tr>
<td>240</td>
<td>0.3824686218</td>
<td>0.2694762872</td>
</tr>
<tr>
<td>270</td>
<td>0.3824008552</td>
<td>0.2686761587</td>
</tr>
<tr>
<td>300</td>
<td>0.3823104321</td>
<td>0.2683721231</td>
</tr>
</tbody>
</table>

---

**Figure 4: Flow chart via FEM.**
4 Results and discussion

A developing model is analyzed inserting heat generation and heat absorption phenomena in Carreau Yasuda material past a stretching surface. A role of magnetic dipole is implemented toward stretching surface. A viscous dissipation effect is added into heat energy. Such complex-type model is handled with the help of FEM. Graphical simulations and tables are tabulated, whereas graphical discussion of heat energy and velocity fields versus physical parameters is listed below. Here, base fluid is considered as engine oil in ternary hybrid nanofluid. Numerical value of Prandtl fluid [50] is taken as $Pr = 6,450$.

### 4.1 Graphical simulations of velocity field

A variation in $\beta$, $We$, and $m$ is observed against velocity curves inserting ternary hybrid nanomaterials, whereas these simulations are noticed by Figures 5–7. Figure 5 is prepared to notice variation in velocity curves versus the implication of $\beta$. A role of $\beta$ appeared because of magnetic dipole, while a magnetic dipole is applied at the surface of wall. It is noticed that a magnetic dipole attracts fluid particles at the surface of wall and this attraction of fluid particles toward magnetic dipole creates frictional force among particles and layers. So, this attraction force is the reason for slow down velocity of fluid particles. Therefore, it is included that velocity curves have decreasing function against implication of $\beta$. This graph is studied for a case without dipole and presence of magnetic dipole. It is investigated that ferro-hydrodynamic interaction number is a dimensionless parameter. The viscosity of fluid is enhanced when ferro-hydrodynamic interaction parameter is increased. Physically, viscous force is produced into motion via fluid particles. Moreover, thermal layer thickness is reduced using argument values of $\beta$. An influence of $We$ on velocity curves using ternary hybrid nanoparticles is carried out by Figure 6. Physically, a ratio among viscous force and elastic force makes a Weissenberg number. It is visualized that an increment in $We$ results in increment in viscosity of fluid particles. Hence, fluid becomes significantly vicious when $We$ is increased. Moreover, layers

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>$\text{Nadeem et al. [46]}$</th>
<th>FEM results</th>
<th>Present problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.72</td>
<td>0.8086641</td>
<td>0.80855192</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>1.0000000</td>
<td>1.0000000</td>
<td></td>
</tr>
<tr>
<td>3.0</td>
<td>1.923690</td>
<td>1.92394013</td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td>2.003170</td>
<td>2.00300358</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Validation of Nusselt number when $H_0 = \lambda = \gamma = \beta = 0$. 

Figure 5: Variation in velocity curves versus $\beta$ when $Pr = 6450$, $m = 0.3$, $d = 0.1$, $\lambda = 1.2$, $H_0 = -1.3$, $\epsilon = 0.4$, $\eta = 1.4$, $\phi_1 = 0.003$, $\phi_3 = 0.94$, $\phi_2 = 0.057$, $We = 2.0$. 

Figure 6: Variation in velocity curves versus $We$ when $Pr = 6450$, $m = 0.5$, $d = 0.7$, $\lambda = 1.3$, $H_0 = 1.5$, $\epsilon = 0.4$, $\eta = 2.4$, $\phi_1 = 0.003$, $\phi_3 = 0.94$, $\phi_2 = 0.057$, $\beta = 3.0$. 

domain is taken as $\eta_{\text{max}} = 8$ while $\eta_{\text{max}}$ is based on asymptotic boundary conditions where $\eta$ is satisfied.
of momentum boundary are also decreasing against higher values of We. The Weissenberg number is a dimensionless parameter, which appeared due to occurrence of Carreau Yasuda liquid in the present problem. The present analysis is called non-Newtonian model in the presence of Carreau Yasuda liquid. Figure 7 is plotted to know variation in velocity curves versus implication of power law number. A power law parameter is used to characterize the behavior of fluid category among layers. It is noticed that \( m \) is a dimensionless number which appeared due to occurrence of Carreau Yasuda liquid. Momentum boundary layers have less thickness when \( m \) is increased. A frictional force is generated among momentum layers, whereas frictional force makes fluid more thick. Furthermore, flow over surface is decreased versus higher values of We.

4.2 Graphical simulations of temperature field

Figures 8–11 are plotted to notice the behavior of heat energy versus \( \beta, H_t \) and Ec, while Figure 9 shows the comparison of tri-hybrid nanoparticles, fluid, nanofluid and hybrid nanomaterials. Increase is investigated into heat energy when \( \beta \) is increased. Appearance of \( \beta \) is occurred using the strength of magnetic dipole. A magnetic dipole is used to slow down velocity in particles. It is predicted that higher values of \( \beta \) lead to enhanced thermal energy in fluid particles. This effect occurred due to interaction of magnetic field and nanoparticles. So, a frictional heating phenomenon is enhanced in fluid particles because of interaction of magnetic field in fluid particles. Thickness regarding thermal layers is declined versus argument numerical values of \( \beta \). Physically, it is a dimensionless parameter which is based on the strength of magnetic dipole. Hence, fluid particles absorbed more heat energy when \( \beta \) is increased. Figure 8 is developed to

**Figure 7:** Variation in velocity curves versus \( m \) when \( Pr = 6450, d = 0.7, \beta = 1.3, \lambda = 1.2, H_t = -1.3, \epsilon = 1.4, \gamma = 1.4, \varphi_1 = 0.003, \varphi_2 = 0.94, \varphi_3 = 0.057, We = 4.0.\)

**Figure 8:** Variation in temperature curves versus \( \beta \) when \( Pr = 6450, m = 0.3, d = 0.2, \beta = 1.7, \lambda = 1.2, H_t = -1.6, \epsilon = 0.4, \gamma = 1.4, \varphi_1 = 0.003, \varphi_2 = 0.94, \varphi_3 = 0.057, We = 3.0.\)

**Figure 9:** Comparative performance of temperature curves in fluid, hybrid nanofluid, tri-hybrid nanomaterials and hybrid nanofluid.
characterize thermal energy among fluid layers, hybrid nanoparticles layers, nano fluid layers and tri-hybrid nano fluid layers. Figure 8 is most significant visualization among layers using hybrid nanoparticles, nano fluid and tri-hybrid nano fluid. It is concluded that tri-hybrid nanoparticles (mixture of TiO$_2$, Al$_2$O$_3$ and SiO$_2$ in engine oil) are observed to be most significant among fluid layers for the development of more heat energy rather than heat energy is manufactured for nano fluid, fluid and hybrid nano fluid. Hence, maximum amount of heat energy is achieved for the case of tri-hybrid nanomaterials. Figure 10 exhibits an effect of heat generation parameter on temperature field. Maximum production in thermal energy is generated when external heat source is implemented at the surface of wall. Physically, it happened due to the occurrence of external heat source. It is noticed that two kinds of heat phenomena occurred based on heat absorption and heat generation. Heat absorption is based on $H<0$, whereas heat generation is based on $H>0$. For both cases, heat energy is augmented by implanted higher values of $H$ because external heat source is placed at wall. Basically, viscous dissipation number is observed as dimensionless parameter based on viscous dissipation. In energy equation, viscous dissipation parameter appeared in viscous

![Figure 10](image1.png) Variation in temperature curves versus $H_t$ when $\varphi_0 = 0.94$, $\varphi_1 = 0.057$, $We = 4.0$, $Pr = 6450$, $m = 0.3$, $d = 0.3$, $\beta = 1.3$, $\lambda = 4.2$, $\epsilon = 0.4$, $\gamma = 1.4$, $\varphi_1 = 0.003$.

![Figure 11](image2.png) Variation in temperature curves versus $Ec$ when $\varphi_0 = 0.94$, $\varphi_1 = 0.057$, $Pr = 6450$, $d = 0.1$, $\beta = 1.3$, $\epsilon = 0.4$, $\gamma = 1.4$, $\varphi_1 = 0.003$, $H_t = -1.3$, $\varphi_1 = 0.003$.

| Table 4: Numerical aspects of Nusselt number and skin friction coefficient against $H_t$, $\beta$, $We$, $m$ and $\lambda$ when $\varphi_0 = 0.94$, $\varphi_1 = 0.057$, $We = 2.0$, $Pr = 6450$, $m = 0.3$, $d = 0.1$ |
|---|---|---|
| Variation in parameters | $-(Re)^{\frac{1}{2}}C_f$ | $-Re^{-\frac{1}{2}}Nu$ |
| $w_0$ | 0.0 | 0.2620890941 | 0.6269023682 |
| 0.5 | 0.2805125031 | 0.6150064882 |
| 1.5 | 0.2983840010 | 0.6038369770 |
| $H_t$ | -1.5 | 0.4793868421 | 0.3524502521 |
| 0.0 | 0.3864742265 | 0.2572577431 |
| 0.7 | 0.1464279369 | 0.2116707874 |
| 0.1 | 0.07467530385 | 0.364197594 |
| $m$ | 0.4 | 0.1547010729 | 0.5585384446 |
| 0.7 | 0.2398573119 | 0.7083760632 |
| 0.0 | 0.2619130138 | 0.7417210656 |
| $\lambda$ | 0.4 | 0.2614998160 | 0.6310409286 |
| 0.8 | 0.2612520196 | 0.5406326980 |
dissipation term. So, a directly proportional relation is investigated among heat energy and viscous dissipation number. Hence, large values of viscous dissipation number bring more enhancements in heat energy. An influence of viscous dissipation number on temperature field is shown in Figure 11. Heat energy is boosted against implication of viscous dissipation.

4.3 Aspects of Nusselt number and skin friction coefficient against various parameters

Table 4 is tabulated to sketch the effects of Nusselt number and skin friction coefficient against variation in Weissenberg number, heat source parameter, viscous dissipation number and power law number. From Table 4, it is estimated that flow rate and heat transfer rate are decreased versus higher numerical values of heat source parameter. But flow rate is enhanced when Weissenberg number is increased. The role of power law number is observed as very significant to develop maximum amount of heat transfer rate and flow rate. Moreover, heat transfer rate is declined versus argument numerical values of viscous dissipation number.

5 Final outcomes

Mathematical model of Carreau Yasuda liquid is developed in the presence of magnetic dipole via stretching surface. Ternary hybrid nano-structures are used to visualize the thermal performance under heat source sink. A finite element scheme is implemented to conduct numerical consequences via flow and temperature profiles. Key remarks are summarized as follows:

• Three hundred elements are ensured for visualization of convergence simulations;
• It is noticed that ternary hybrid nanomaterials are observed as a significant source to conduct maximum inclination into thermal energy rather than hybrid nanostructures and nanofluid;
• Temperature profile is inclined versus the higher values of heat generation and viscous dissipation numbers while thermal layers are also increasing the behavior;
• A vital role of magnetic dipole is examined to raise the production of thermal layers but declination is noticed in flow profile;
• Flow rate and heat transfer rate are declined versus argument numerical values of viscous dissipation parameter but opposite behavior on heat transfer rate and flow is studied versus power law number;
• Heat transfer rate is boosted against higher impact of viscous dissipation.

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