Research Article

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Constructing analytic solutions on the Tricomi equation

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Abstract: In this paper, homotopy analysis method (HAM) and variational iteration method (VIM) are utilized to derive the approximate solutions of the Tricomi equation. Afterwards, the HAM is optimized to accelerate the convergence of the series solution by minimizing its square residual error at any order of the approximation. It is found that effect of the optimal values of auxiliary parameter on the convergence of the series solution is not negligible. Furthermore, the present results are found to agree well with those obtained through a closed-form equation available in the literature. To conclude, it is seen that the two are effective to achieve the solution of the partial differential equations.

Keywords: Homotopy analysis method (HAM), variational iteration method (VIM), square residual error, auxiliary parameter, Lagrange multiplier

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1 Introduction

The elliptic-hyperbolic classic Tricomi equation on a region Ω in R^2 may be written in the form [1],

$$w_{xx} - xw_{yy} = 0, (x, y) \in \Omega, \tag{1}$$

which is usually used to study characteristics of the solution between subsonic and supersonic flows of the compressible gas, namely transonic flow. Compared to the Chaplygin equation [2], neither the velocity of the sound nor the nonlinear boundary conditions affects this equation. If $x \to x^m$, the principal part of Eq. (1), *i.e.* $w_{xx} - x^m w_{yy} = 0$, is called the Keldysh equation. Herein, the most pioneering researches undertaken on the Tricomi equation are reviewed.

Singh *et al.* [3] employed the homotopy perturbation Sumudu transform method (HPSTM) to solve the local fractional Tricomi equation in the fractal transonic flow. They reported that their proposed model provides the results without any transformation of the equations into the discrete counterparts and is completely free of the round-off errors. Yagdjian [4] analyzed Eq. (1) by using the integral transform approach which was introduced in Ref. [5] for the hyperbolic domain. Nazipov [6] presented solution of the spatial Tricomi problem for a single mixed-type equation in which the study was carried out in a bound spatial domain. In this regard, Sabitov [7] presented a similar work for a mixed parabolic-hyperbolic equation. Quintanilla [8] analyzed the spatial behavior of the Tricomi equation by reaching the decay for an equation which may be elliptic, hyperbolic and parabolic depending on the different points of the region. Zhang [9] converted the linear Tricomi equation into a confluent hypergeometric equation using partial Fourier-transformation. He could also capture an explicit solution of the initial value problem in terms of two integral operators.

Recently, the analytic approximation of the partial differential equations has been extensively studied through some nonperturbative techniques: Adomian decomposition method (ADM) [10, 11], VIM [12, 13], inverse scattering method [14], δ-expansion method [15] etc. The HAM [16–20] is one of the most effective methods for finding the analytic solution of the partial differential equations. It is required to set $\hbar \neq 0$ as an auxiliary parameter for the case in which the variation of this parameter may lead to a uniformly continuous family of functions. In this regard, Sajid and Hayat [21] presented an analytic solution to the heat conduction/convection equation when the effects of nonlinearity had been taken into account. They also compared the HAM with the homotopy perturbation method (HPM) and drew a conclusion that if $\hbar = -1$, the HPM is a special case of the HAM. Odibat [22] captured the Legendre polynomials to solve the nonlinear fractional differential equations by approximating the non-homogeneous

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and nonlinear terms within the HAM and VIM. Liao [23] introduced a relationship between the HAM and Euler's transformation via a nonlinear ordinary differential equation. He concluded that the Euler's transformation is a special case of the HAM for some special choices of the initial approximation and auxiliary parameter. Shukla *et al.* [24] investigated the HAM by adding a non-homogenous term to the auxiliary linear operator L and concluded that this term plays a significant role in reducing the square residual error. Mastroberardino [25] studied a nonlinear boundary value problem for the electrohydrodynamic flow of a fluid in a circular cylindrical conduit using the HAM. He also showed that the HPM solution yields divergent in all cases. Motsa *et al.* [26] studied a fully-developed parallel flow through a horizontal channel using spectral homotopy analysis method (SHAM). They showed that the SHAM can utilize any form of the initial approximation and concluded that the 4th-order approximation of SHAM is in good agreement with the numerical findings.

The main purpose of the present paper is to provide an analytic solution of the Tricomi equation. To this end, the HAM is optimized to accelerate convergence of the series solution. Due to the fact that the VIM is a special case of the HAM [27], the HAM results are compared with those results obtained through this method as well as those obtained through a closed-form equation. To the best of the authors' knowledge, no previous work has been conducted in open literature.

2 Series solution to the Tricomi equation using HAM

Let's consider the following nonlinear algebraic equation,

$$\mathrm{N}\left[w(x,y)\right]=0, \tag{2}$$

where N and $w(x,y)$ are a nonlinear differential operator and an unknown function, respectively. Using $q \epsilon [0,1]$ as an embedding parameter in topology, the following zero-order deformation equation is constructed,

$$(1-q)\,\mathrm{L}\left[\psi(x,y;q)-w_0(x,y)\right]=q\hbar\mathrm{N}\left[\psi(x,y;q)\right], \tag{3}$$

where $\psi(x,y;q)$ and $w_0(x,y)$ are an unknown function and an initial approximation of $w(x,y)$, respectively. It is to be noted that by setting $q=0$ and $q=1$, it holds,

$$\psi(x,y;0)=w_0(x,y),\ \psi(x,y;1)=w(x,y), \tag{4}$$

respectively. As q varies from 0 to 1, $\psi(x,y;q)$ varies from the initial approximation $w_0(x,y)$ to the solution $w(x,y)$. By expanding $\psi(x,y;q)$ in a Taylor's series with respect to q, the following homotopy-series is constructed,

$$\psi(x,y;q)=w_0(x,y)+\sum_{i=1}^{+\infty}w_i(x,y)\,q^i, \tag{5}$$

where,

$$w_i(x,y)=\frac{1}{i!}\frac{\partial^i\psi(x,y;q)}{\partial q^i}\Big|_{q=0}. \tag{6}$$

If Eq. (5) converges at $q=1$, one has,

$$\psi(x,y;q)=w_0(x,y)+\sum_{i=1}^{+\infty}w_i(x,y). \tag{7}$$

Differentiating the zero-order deformation Eq. (3) i-times with respect to q, dividing by $i!$ and setting $q=1$, the ith-order deformation equation will be constructed,

$$\mathrm{L}\left[w_i(x,y)-\chi_i w_{i-1}(x,y)\right]=\mathrm{R}_i\left(w_{i-1}(x,y)\right), \tag{8}$$

where,

$$\mathrm{R}_i\left(w_{i-1}(x,y)\right)=\frac{1}{(i-1)!}\frac{\partial^{i-1}\mathrm{N}\left[\psi(x,y;q)\right]}{\partial q^{i-1}}\Big|_{q=0}, \tag{9}$$

and,

$$\chi_i=\begin{cases}0, & i\le 1,\\ 1, & i>1.\end{cases} \tag{10}$$

To apply the HAM on the present problem, consider Eq. (1) and its corresponding boundary conditions given,

$$\psi(x,0;q)=0,\ \frac{\partial\psi(x,y;q)}{\partial y}\Big|_{y=0}=0. \tag{11}$$

It should be noted that the HAM enables us to choose an auxiliary linear operator [17]. To this end, the auxiliary linear operator L will be defined,

$$\mathrm{L}\left[\psi(x,y;q)\right]=\frac{\partial^2\psi(x,y;q)}{\partial y^2}, \tag{12}$$

which has the property of,

$$\mathrm{L}\left[b_0+b_1 y\right]=0, \tag{13}$$

where b_0 and b_1 are integration constants to be determined by the corresponding boundary conditions. Furthermore, the nonlinear differential operator N can be chosen in terms of Eq. (1),

$$\mathrm{N}\left[\psi(x,y;q)\right]=\frac{\partial^2\psi(x,y;q)}{\partial x^2}-x\frac{\partial^2\psi(x,y;q)}{\partial y^2}. \tag{14}$$

By solving the ith-order deformation Eq. (8), the corresponding ith-order approximated solution can be obtained,

$$w_i(x,y)=\chi_i w_{i-1}(x,y)+\hbar\int_0^y\int_0^y \mathrm{R}_i\left(w_{i-1}(x,y)\right)dy\,dy, \tag{15}$$

and,

$$R_i\left(w_{i-1}(x,y)\right)=\frac{\partial^2 w_{i-1}(x,y)}{\partial x^2}-x\frac{\partial^2 w_{i-1}(x,y)}{\partial y^2}. \quad (16)$$

Therefore, the ith-order approximated solution of $w(x,y)$ will be obtained,

$$w_k(x,y)\approx\sum_{i=0}^{k}w_i(x,y). \quad (17)$$

In theory, at the pth-order of approximation, the square residual error can be defined [28],

$$\Delta_p=\int_0^\infty\left(\mathrm{N}\left[\sum_{i=0}^{p}w_i(\xi)\right]\right)^2 d\xi, \quad (18)$$

where $\xi=\xi(x,y)$. It should be noted that by decreasing the values of Δ_p, the convergence for corresponding series solution would be faster [28].

3 VIM based Lagrange multiplier method

Generating the correction functional for the VIM to solve the partial differential equations is one of the procedures which can ensure a rapid convergence of the series solution. It is to be noted that neither the small parameter nor the very large one affects the solution [12].

By introducing the differential equation as a combination of linear and nonlinear operators in the form of $\mathrm{L}[w(x,y)]+\mathrm{N}[w(x,y)]=f(x,y)$, where $f(x,y)$ is a known analytic function, the correction functional can be written,

$$w_{i+1}(x,y)=w_i(x,y)+\int_0^y\lambda(\xi)\left(\mathrm{L}[w_i(\xi)]+\mathrm{N}[w_i(\xi)]-f(\xi)\right)d\xi,\quad i\geq 0, \quad (19)$$

where λ is a general Lagrange multiplier which can be calculated by variational theory, $w_i(\xi)$ is considered as a restricted variation which means $\delta w_i(\xi)=0$. It is noted that the exact solution can be determined by $w(x,y)=\lim_{i\to\infty}w_i(x,y)$. The application of the restricted variation in Eq. (19) is to simplify the determination of the multiplier [12].

By substituting Eq. (1) into Eq. (19), the correction functional of the present problem can be obtained,

$$w_{i+1}(x,y)=w_i(x,y)+\int_0^y\lambda(\xi)\left(\frac{\partial^2 w_i(\xi)}{\partial x^2}-x\frac{\partial^2 w_i(\xi)}{\partial y^2}\right)d\xi. \quad (20)$$

Calculating the variation with respect to $w_i(x,y)$ and $\delta w_{i+1}(x,y)=0$ yields,

$$\delta w_{i+1}(x,y)=\delta w_i(x,y)+\delta\int_0^y\lambda(\xi)\left(\frac{\partial^2 w_i(\xi)}{\partial x^2}-x\frac{\partial^2 w_i(\xi)}{\partial y^2}\right)d\xi=0. \quad (21)$$

To find the explicit form of Eq. (21), it is sufficient to express the general Lagrange multiplier λ explicitly. To this end, using He and Wu's paper [29], the Lagrange multiplier can be identified,

$$\lambda(y,\xi)=\frac{(-1)^k(\xi-y)^{k-1}}{(k-1)!}, \quad (22)$$

where k is the highest-order derivative. By setting $k=2$, the Lagrange multiplier for the present problem can be determined,

$$\lambda(y,\xi)=(\xi-y). \quad (23)$$

After finding Eq. (23), the iteration formula can be given,

$$w_{i+1}(x,y)=w_i(x,y)+\int_0^y(\xi-y)\left(\frac{\partial^2 w_i(\xi)}{\partial x^2}-x\frac{\partial^2 w_i(\xi)}{\partial y^2}\right)d\xi. \quad (24)$$

4 Results and discussion

To validate the present analytic solutions (*i.e.* HAM and VIM), present findings are compared with the separable solution of the generalized Tricomi equation, $w_{xx}-f(x)w_{yy}=0$, which can be determined [30],

$$w=\left[c_1e^{\lambda y}+c_2e^{-\lambda y}\right]H(x), \quad (25)$$

where c_1, c_2, and λ are arbitrary constants, and the function $H=H(x)$ is calculated by the ordinary differential equation $H''_{xx}+\lambda^2 f(x)H=0$ [30]. In Table 1, results of the present analytic solutions are compared with those obtained through Eq. (25) for solving the Tricomi Eq. (1). The initial approximation in this case is taken as $w_0(x,y)=y^2$. Based on the results of Table 1, by increasing in the values of x, $w(x,y)$ will be decreased in all cases. It is seen that although the convergence is occurred at the 11th-order approximation of the HAM, the results of the 9th- and 7th-order approximated solution only suffer from a relative error of at most 1.127% and 1.737%, respectively, compared with the results of Eq. (25).

To determine the valid values of the auxiliary parameter $\hbar$, one needs to examine the properties of the series

Table 1: The solutions for the Tricomi Eq. (1) obtained by the proposed techniques and those obtained through Eq. (25), when they are subjected to $y = 1$

x	HAM ($\hbar = 1$)			VIM	Eq. (25)
	$i = 7$	$i = 9$	$i = 11$		[30]
0	0.495	0.497	0.499	0.504	0.502
0.2	0.493	0.495	0.497	0.502	0.500
0.4	0.484	0.487	0.489	0.495	0.492
0.6	0.458	0.461	0.464	0.471	0.466
0.8	0.411	0.414	0.417	0.425	0.419
1	0.328	0.331	0.334	0.343	0.336

solution. It should be noted that since the auxiliary parameter $\hbar$ significantly influences on the convergence of the series solution [31], it is straightforward to use $\hbar = 1$ which is seen in Table 1. Furthermore, if the auxiliary parameter $\hbar$ is properly chosen, the series solution may converge fast. However, for cases subjected to obtain higher convergence, the HAM may be optimized.

To investigate the accuracy of the VIM findings from Table 1, the relative error between VIM and Eq. (25) does not exceed 0.985%. It should be noted that although the present VIM findings suffer from a large error compared with the 11th-order approximation of the HAM (=0.551%), using this technique for analyzing the partial differential equations is still recommended. It is due to the fact that this scheme provides high convergence with more iteration by introducing a Lagrange multiplier λ [32] which is given in Eq. (23) for this problem. Moreover, solving Eq. (24) in terms of x and y and using the fact that $\int y^k d\xi = \frac{c_1 \xi^{(k-1)}}{(k-1)!} + \frac{c_2 \xi^{(k-2)}}{(k-2)!} + \cdots + \frac{c_m \xi^{(k-i)}}{(k-i)!}$ [33], may lead to the threshold of λ. In this way, one would replace above series with Eq. (23), and vary the integral part of Eq. (24) to find its more convergent values in which the Lagrange multiplier takes a value in the range $0 \le \lambda \le \frac{c_1 \xi^{(k-1)}}{(k-1)!}$. This fact can be considered as verification of the VIM given in Eq. (24) for solving the Tricomi Eq. (1).

The optimal values of $\hbar$ can be found by minimizing the square residual error Δ_p, which is given in Eq. (18). In this case, the variation of Δ_p at each order of the HAM solution should be plotted versus the variation of $\hbar$ numerically. Hence, the optimal values of $\hbar$ can be obtained at the minimum of the resulting graph which is illustrated in Figure 1 for 11th-order approximation of the HAM. As Figure 1 depicts, Δ_p indicates descending linear behaviour for $\hbar \le 1.017$ while $\hbar > 1.017$ suggests ascending linear behaviour of this curve.

Table 2 investigates the optimal values of $\hbar$ as well as the minimum value of the corresponded Δ_p for the Tricomi

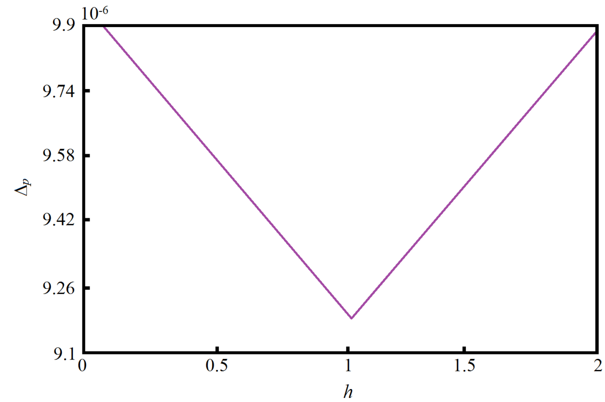

Figure 1: Selection of the optimal values of $\hbar$ for 11th-order approximation of the HAM

Eq. (1) at 7th-, 9th-, and 11th-order of the HAM solution. From this table, Δ_p decreases with an increase of the order of the approximation in series solution which indicates the convergence of the present HAM solution. In other words, using the optimal values of $\hbar$ accelerates the rate of convergence of the series solution. It is seen that by increasing order of the approximation, the auxiliary parameter $\hbar$ becomes more and more close to unity.

A comparison between the values of $w(x, y)$ obtained by different orders of the HAM with and without utilizing the optimal values of $\hbar$ is presented in Table 3. This table also compares the optimized HAM findings with the results obtained through Eq. (25). According to the results of this table, the relative error between the optimized 11th-order approximation of the HAM and Eq. (25) becomes only 0.052%. This is due to the fact that the optimized 11th-order approximation of the HAM can provide more accurate results than the non-optimized one. Therefore, one can conclude that this exhibits the significant effect of the optimal values of $\hbar$ on the series solution.

5 Conclusions

Utilizing the HAM and VIM to solve the Tricomi equation, which is an abstraction of the Euler equation on a 2D fluid motion near the sonic condition, was the main objective of the present study. Due to the fact that the HAM contains the auxiliary parameter $\hbar \neq 0$, the series solution was optimized by minimizing its square residual error at any order of the analytic approximation. It was found that the optimized HAM can accelerate convergence of the series solution. Furthermore, it was shown that the HAM can provide more accurate results than the VIM for solving the

Table 2: The optimal values of $\hbar$ for solving the Tricomi Eq. (1) by applying $y = 1$

x	$i = 7$		$i = 9$		$i = 11$	
	$\hbar_{opt}$	Δ_p	$\hbar_{opt}$	Δ_p	$\hbar_{opt}$	Δ_p
0	1.038	0.00003623	1.028	0.00001631	1.017	0.000009164
0.2	1.038	0.00003669	1.028	0.00001643	1.017	0.000009273
0.4	1.038	0.00003702	1.028	0.00001658	1.017	0.000009302
0.6	1.038	0.00003747	1.028	0.00001710	1.017	0.000009464
0.8	1.038	0.00003798	1.028	0.00001801	1.017	0.000009601
1	1.038	0.00003811	1.028	0.00001869	1.017	0.000009719

Table 3: Effect of the optimal values of $\hbar$ on accelerating the HAM solution for the Tricomi Eq. (1), when they are subjected to $y = 2$. The values in parentheses are those obtained using $\hbar = 1$

x	HAM			Eq. (24) [30]
	$i = 7$	$i = 9$	$i = 11$	
0	1.908 (1.993)	1.931 (1.996)	1.942 (1.998)	1.943
0.2	1.905 (1.990)	1.929 (1.993)	1.939 (1.995)	1.940
0.4	1.899 (1.984)	1.924 (1.986)	1.933 (1.989)	1.934
0.6	1.876 (1.958)	1.897 (1.961)	1.910 (1.964)	1.911
0.8	1.828 (1.910)	1.850 (1.912)	1.862 (1.915)	1.863
1	1.752 (1.828)	1.773 (1.831)	1.780 (1.834)	1.781

Tricomi equation. Comparison between the results found by the present analytic solutions and those prepared by a closed-form equation showed that such techniques can be considered as a promising tool for analyzing partial differential equations. This fact is due to the existing small differences between the results.

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