Research Article

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Computational methods and traveling wave solutions for the fourth-order nonlinear Ablowitz-Kaup-Newell-Segur water wave dynamical equation via two methods and its applications

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Abstract: The aim of this article is to construct some new traveling wave solutions and investigate localized structures for fourth-order nonlinear Ablowitz-Kaup-Newell-Segur (AKNS) water wave dynamical equation. The simple equation method (SEM) and the modified simple equation method (MSEM) are applied in this paper to construct the analytical traveling wave solutions of AKNS equation. The different waves solutions are derived by assigning special values to the parameters. The obtained results have their importance in the field of physics and other areas of applied sciences. All the solutions are also graphically represented. The constructed results are often helpful for studying several new localized structures and the waves interaction in the high-dimensional models.

Keywords: Simple equation method, modified simple equation method, fourth-order nonlinear AKNS equation, exact solutions, solitary wave

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1 Introduction

Several dynamical nonlinear problems in the field of applied physics and other areas of natural science are generally characterized by nonlinear evolution of partial differential equations (PDEs) well-known as governing equations [1–6]. Such nonlinear PDEs play a vital role in physical science to understand the nonlinear complex physical phenomena. The analytical solutions of nonlinear PDEs have their own importance in the various branches of mathematical physical sciences, applied sciences and other areas of engineering to understand their physical interpretation. Therefore, the search for an exact solution of nonlinear systems has been an interesting and important topic for mathematicians and physicists in nonlinear science. Many powerful and systemic methods have been developed to construct the solutions of nonlinear PDEs such as, Hirota bilinear method, tanh-coth method, Exp-function method, multi linear variable separation approach method, modified extended direct algebraic method, simple equation method, modified simple equation method, variable coefficients method, extended auxiliary equation method [7–19].

In PDEs, the AKNS equations are very important and have many applications in the field of physics and other nonlinear sciences. These equations are reduced the some nonlinear evolution equations such as sine-Gordon equations, the nonlinear Schrödinger equation, KdV equation etc. Different methods have been used to acquire explicit solutions of the AKNS equations, inverse scattering transformation, the Bäcklund transformation, the Darboux transformation [19–28].

In the current work, we consider the well known fourth-Order nonlinear AKNS water wave equation [19] with a perturbation parameter β in the form of

$$4v_{xt} + v_{xxx} + 8v_x v_{xy} + 4v_{xx} v_y - \beta v_{xx} = 0. \quad (1)$$

we have employed proposed simple equation method and modified simple equation method on equation (1) to obtain new exact solitary wave solutions with different parameters. To the best of our knowledge, no work has been done in previous study by employing the current proposed methods. The obtained solutions are useful in physical sciences and help to understand the physical phenomena.

The article is structured as follows. The main steps of the proposed methods are given in Section 2. In Section 3 we apply the proposed methods to Eq. (1) for constructing solitary wave solution. Finally, the summary of this work is given in Section 4.

2 Description of proposed methods

2.1 Simple equation method

In this section, simple equation method (SEM) will be applied to obtain the solitary wave solutions for Ablowitz-Kaup-Newell-Segur water wave equation. Consider the nonlinear PDE in form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \dots) = 0, \quad (2)$$

where F is called a polynomial function of $u(x, y, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The basic key steps of SEM are as follows:

Step 1. Consider traveling wave transformation

$$u(x, y, t) = V(\xi), \quad \xi = x + y + \omega t, \quad (3)$$

by utilizing the above transformation, the Eq. (2) is reduced into ODE as:

$$G(V, V', V'', V''', \dots) = 0, \quad (4)$$

where G is a polynomial in $V(\xi)$ its derivatives with respect to ξ .

Step 2. Let us assume that the solution of Eq. (4) has the form:

$$V(\xi) = \sum_{i=-M}^M A_i \psi^i(\xi), \quad (5)$$

where A_i ($i = -M, -M+1, \dots, -1, 0, 1, \dots, M$) is arbitrary constants which can be determined latter and M is a positive integer, which can be calculated by homogeneous balance principle on Eq. (4).

Let ψ satisfies the following equation:

$$\psi'(\xi) = b_0 + b_1 \psi + b_2 \psi^2 + b_3 \psi^3, \quad (6)$$

where b_0, b_1, b_2, b_3 , are arbitrary constants.

Step 3. Substituting Eq. (5) along with Eq. (6) into Eq. (4), and collecting the coefficients of $(\psi)^j$, then setting coefficients equal to zero, we obtained a system of algebraic equations in parameters $b_0, b_1, b_2, b_3, \omega$ and A_i . The system of algebraic equations is solved with the help of Mathematica and we get the values of these parameters.

Step 4. By substituting all these values of parameters and ψ into Eq. (5), we obtained the required solutions of Eq. (2).

2.2 Modified simple equation method

In this section, we describe the algorithm of modified simple equation method (MSEM) to obtain the solitary wave solutions of nonlinear evolution equations. Consider the nonlinear evolution equation in the form

$$G(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \dots) = 0, \quad (7)$$

where G is a polynomial function of $u(x, t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The basics keysteps are::

Step 1. Consider travelling wave transformation

$$u(x, y, t) = U(\xi), \quad \xi = x + y + \omega t, \quad (8)$$

By utilizing the above transformation into Eq. (7), the Eq. (7) is reduced into ODE as

$$H(U, U', U'', U''', \dots) = 0, \quad (9)$$

where H is a polynomial in $U(\xi)$ its derivatives with respect to ξ .

Step 2. Let us assume that the solution of Eq. (9) has the form:

$$U(\xi) = \sum_{M=0}^N B_M \left[\frac{\Psi'(\xi)}{\Psi(\xi)} \right]^M, \quad (10)$$

where B_M are arbitrary constants to be determined, such that $B_N \neq 0$ and $\Psi(\xi)$ is to be determined.

Step 3. The postive integer N can be determined by applying the homogeneous balance technique between the highest order derivatives and nonlinear terms as in Eq. (7).

Step 4. We calculate all the required derivatives of $U', U'', U''' \dots$ and substitute into Eq. (10) and (9). We obtain a polynomial of $\Psi^{-j}(\xi)$ with the derivatives of $\Psi(\xi)$.

We equate all the coefficients of $\Psi^{-j}(\xi)$ to zero, where $j \geq 0$. This procedure yields a system of equations which can be solved to find B_M , $\Psi(\xi)$ and $\Psi'(\xi)$.

Step 5. We substitute the values of B_M , $\Psi(\xi)$ and $\Psi'(\xi)$ into Eq. (10) and (8) to complete the determination of exact solution of Eq. (1).

3 Applications of descriptions method on AKNS

3.1 Applications of SEM

In this section we apply the method which is described in Section 2.1 on Eq. (1). Consider the traveling waves transformation

$$v(x, y, t) = V(\xi), \quad \xi = x + y + kt, \quad (11)$$

where k is arbitrary constant, which can be determined later. By using the above transformation to the Eq. (1) into the following ordinary differential equation and integrated

$$(4k - \beta)V' + 6(V')^2 + kV''' = 0. \quad (12)$$

Now applying the homogeneous balance principle between $(V')^2$ and V''' in Eq. (12), we get $M = 2$. We suppose the solution of Eq. (12) has the form:

$$V(\xi) = A_{-2}\psi^{-2} + A_{-1}\psi^{-1} + A_0 + A_1\psi + A_2\psi^2. \quad (13)$$

Substituting Eq. (13) along Eq. (6) into Eq. (12), we get system of algebraic equation in parameters $b_0, b_1, b_2, b_3, \beta, k, A_0, A_{-1}, A_{-2}, A_1, A_2$. The system of algebraic equations can be solved for these parameters, we have following solutions cases.

Case 1. $b_3 = 0$,

Family-I

$$k = \frac{\beta}{b_1^2 - 4b_0b_2 + 4}, \quad A_{-1} = \frac{\beta b_0}{b_1^2 - 4b_0b_2 + 4}, \quad (14)$$

$$A_2 = 0, \quad A_1 = 0, \quad A_{-2} = 0.$$

Substituting Eq. (14) into Eq. (6), then the solution of Eq. (1) becomes:

$$v_1(x, y, t) = \frac{-2b_2b_0\beta}{(b_1^2 - 4b_0b_2 + 4)(b_1 - \sqrt{4b_0b_2 - b_1^2} \tan(\frac{\sqrt{4b_0b_2 - b_1^2}}{2}(\xi + \xi_0)))} + A_0, \quad 4b_0b_2 > b_1^2, \quad (15)$$

where $\xi = x + y + \frac{\beta}{b_1^2 - 4b_0b_2 + 4}t$.

Family-II

$$k = \frac{\beta}{b_1^2 - 4b_0b_2 + 4}, \quad A_1 = \frac{\beta b_2}{-b_1^2 + 4b_0b_2 - 4}, \quad (16)$$

$$A_{-1} = 0, \quad A_{-2} = 0, \quad A_2 = 0.$$

Substituting Eq. (16) into Eq. (6), then the solution of Eq. (1) becomes:

$$v_2(x, y, t) = A_0 \quad (17)$$

$$+ \frac{\beta(b_1 - \sqrt{4b_0b_2 - b_1^2} \tan(\frac{\sqrt{4b_0b_2 - b_1^2}}{2}(\xi + \xi_0)))}{(2b_1^2 - 8b_0b_2 + 8)},$$

$$4b_0b_2 > b_1^2,$$

where $\xi = x + y + \frac{\beta}{b_1^2 - 4b_0b_2 + 4}t$.

(a)

(b)

Figure 1: Exact solitary wave solutions of Eq. (15) and Eq. (17) are plotted by choosing the values of parameters $A_0 = 1.5$, $b_1 = 0.5$, $\beta = 1$, $\xi_0 = 1$: (a) periodic solitary wave of v_1 at $b_2 = 1$, $b_0 = 1$ and (b) periodic solitary of v_2 at $b_2 = -2$, $b_0 = -1$

Case 2. $b_0 = b_3 = 0$,

$$k = \frac{\beta}{b_1^2 + 4}, \quad A_1 = -\frac{\beta b_2}{b_1^2 + 4}, \quad (18)$$

$$A_{-1} = 0, \quad A_{-2} = 0, \quad A_2 = 0.$$

Substituting Eq. (18) into Eq. (6) the solution of Eq. (1) becomes:

$$v_{31}(x, y, t) = A_0 + \frac{-\beta b_2 b_1 e^{b_1(\xi + \xi_0)}}{(b_1^2 + 4)(1 - b_2 e^{b_1(\xi + \xi_0)}), \quad (19)$$

$$b_1 > 0;$$

$$v_{32}(x, y, t) = A_0 + \frac{\beta b_2 b_1 e^{b_1(\xi + \xi_0)}}{(b_1^2 + 4)(1 + b_2 e^{b_1(\xi + \xi_0)}), \quad (20)$$

$$b_1 < 0;$$

where $\xi = x + y + \frac{\beta}{b_1^2 + 4}t$.

(a)

(b)

Figure 2: Exact solitary wave solutions of Eq. (19) and Eq. (20) are plotted by choosing the values of parameters as: $A_0 = 0.5$, $b_2 = 0.5$, $\beta = -0.5$, $\xi_0 = -0.5$: (a) solitary wave of v_{31} at $b_1 = 0.5$ and (b) solitary wave of v_{32} at $b_1 = -0.5$

Case 3. $b_1 = b_3 = 0$,

Family-I

$$k = -\frac{\beta}{4(b_0 b_2 - 1)}, \quad A_1 = 0, \quad (21)$$

$$A_{-1} = -\frac{\beta b_0}{4(b_0 b_2 - 1)}, \quad A_{-2} = 0, \quad A_2 = 0.$$

Substituting Eq. (21) into Eq. (6) the solution of Eq. (1) becomes:

$$v_{41}(x, y, t) = \frac{b_2 b_0 \beta}{(-4b_0 b_2 + 4)\sqrt{b_0 b_2} \tan(\sqrt{b_0 b_2}(\xi + \xi_0))} + A_0, \quad b_0 b_2 > 0; \quad (22)$$

$$v_{42}(x, y, t) = \frac{b_2 b_0 \beta}{(4b_0 b_2 - 4)\sqrt{-b_0 b_2} \tanh(\sqrt{-b_0 b_2}(\xi + \xi_0))} + A_0, \quad b_0 b_2 < 0; \quad (23)$$

where $\xi = x + y - \frac{\beta}{4(b_0 b_2 - 1)}t$.

(a)

(b)

Figure 3: Exact solitary wave solutions of Eq. (22) and Eq. (23) are plotted by choosing these values of parameters at: $A_0 = -1.5$, $\beta = 1$, $\xi_0 = 1$: (a) periodic solitary wave of v_{41} at $b_0 = -0.5$, $b_2 = -1$ and (b) solitary wave of v_{42} at $b_0 = -0.5$, $b_2 = 1$

Family-II

$$k = -\frac{\beta}{4(b_0 b_2 - 1)}, \quad A_1 = \frac{\beta b_2}{4(b_0 b_2 - 1)}, \quad (24)$$

$$A_{-1} = 0, \quad A_{-2} = 0, \quad A_2 = 0.$$

Substituting Eq. (24) into Eq. (6) the solution of Eq. (1) becomes:

$$v_{51}(x, y, t) = A_0 + \frac{\beta \sqrt{b_0 b_2} \tan(\sqrt{b_0 b_2}(\xi + \xi_0))}{4(b_0 b_2 - 1)}, \quad (25)$$

$$b_0 b_2 > 0;$$

$$V_{52}(x, y, t) = A_0 + \frac{\beta \sqrt{-b_0 b_2} \tanh(\sqrt{-b_0 b_2}(\xi + \xi_0))}{(-4b_0 b_2 + 4)}, \quad (26)$$

$$b_0 b_2 < 0;$$

where $\xi = x + y - \frac{\beta}{4(b_0 b_2 - 1)}t$.

(a)

(b)

Figure 4: Exact solutions of Eq. (25) and Eq. (26) are plotted at $A_0 = 1.5$, $\beta = -0.75$, $\xi_0 = 1$: (a) periodic solitary wave of v_{51} at $b_0 = 0.75$, $b_2 = 0.5$ and (b) solitary wave of v_{52} at $b_0 = 0.75$, $b_2 = -0.5$

Family-III

$$k = -\frac{\beta}{4(4b_0 b_2 - 1)}, \quad A_1 = \frac{\beta b_2}{4(4b_0 b_2 - 1)}, \quad (27)$$

$$A_{-1} = -\frac{\beta b_0}{4(4b_0 b_2 - 1)}, \quad A_{-2} = 0, \quad A_2 = 0.$$

Substituting Eq. (27) into Eq. (6) the solution of Eq. (1) becomes:

$$v_{61}(x, y, t) = \frac{b_2 b_0 \beta}{(-16b_0 b_2 + 4)\sqrt{b_0 b_2} \tan(\sqrt{b_0 b_2}(\xi + \xi_0))} + A_0 + \frac{\beta \sqrt{b_0 b_2} \tan(\sqrt{b_0 b_2}(\xi + \xi_0))}{4(4b_0 b_2 - 1)}, \quad b_0 b_2 > 0; \quad (28)$$

$$v_{62}(x, y, t) = \frac{b_2 b_0 \beta}{(16b_0 b_2 - 4)\sqrt{-b_0 b_2} \tanh(\sqrt{-b_0 b_2}(\xi + \xi_0))} + A_0 + \frac{\beta \sqrt{-b_0 b_2} \tanh(\sqrt{-b_0 b_2}(\xi + \xi_0))}{(-16b_0 b_2 + 4)}, \quad b_0 b_2 < 0; \quad (29)$$

where $\xi = x + y - \frac{\beta}{4(b_0 b_2 - 1)}t$.

(a)

(b)

Figure 5: Exact solutions of Eq. (28) and Eq. (29) are plotted at $A_0 = 1$, $\beta = 2$, $\xi_0 = -0.5$: (a) periodic solitary wave of v_{61} at $b_0 = 1$, $b_2 = 0.5$ and (b) solitary wave of v_{62} at $b_0 = -1$, $b_2 = 0.5$

Case 4. $b_0 = b_2 = 0$,

$$k = \frac{\beta}{(4 + 4b_1^2)}, \quad A_2 = \frac{-\beta b_3}{(2 + 2b_1^2)}, \quad (30)$$

$$A_{-1} = 0, \quad A_{-2} = 0, \quad A_1 = 0.$$

where $\xi = x + y \frac{\beta}{(4 + 4b_1^2)} t$. Substituting Eq. (30) into Eq. (6) the solution of Eq. (1) becomes:

$$v_{71}(x, y, t) = \frac{-b_1 b_3 \beta}{(2 + 2b_1^2)(e^{-2(\xi + \xi_0)b_1} + b_3)} + A_0, \quad (31)$$

$$b_1 < 0,$$

$$v_{72}(x, y, t) = \frac{-\beta b_3 b_1 e^{2(\xi + \xi_0)b_1}}{(2 + 2b_1^2)1 - e^{2(\xi + \xi_0)b_1} b_3} + A_0, \quad (32)$$

$$b_1 > 0.$$

where $\xi = x + y + \frac{\beta}{(4 + 4b_1^2)} t$.

(a)

(b)

Figure 6: Exact solutions of Eq. (31) and Eq. (32) are plotted at $A_0 = 0.5$, $\beta = 3$, $\xi_0 = -0.5$, $b_1 = -0.5$, $b_3 = -0.5$ at (a) and $A_0 = -0.5$, $\beta = 3$, $\xi_0 = -0.5$, $b_1 = 1$, $b_3 = -1$ at (b), respectively are solitary waves of v_{71} and v_{72}

3.2 Application of MSEM

In this section we apply method which is described in Section 2.2, on Eq. (1). Consider the traveling waves transformation

$$v(x, y, t) = U(\xi), \quad \xi = x + y + \omega t, \quad (33)$$

where ω is arbitrary constant which can be determined later. By using the above transformation to the Eq. (1) into the following ordinary differential equation and integrated we have

$$(4\omega - \beta)U' + 6(U')^2 + \omega U''' = 0. \quad (34)$$

Now applying the homogeneous balance principle between $(U')^2$ and U''' in Eq. (34), we get $N = 1$. We suppose the solution of Eq. (34) has the form as:

$$U = B_0 + B_1 \frac{\Psi'(\xi)}{\Psi(\xi)}. \quad (35)$$

Where B_0, B_1 are constant such that $B_1 \neq 0$, substituting Eq. (35) into Eq. (34) and equating the coefficients of Ψ^{-4} , Ψ^{-3} , Ψ^{-2} , Ψ^{-1} we get the following equations:

$$(-\omega + B_1)B_1(\Psi')^4 = 0, \quad (36)$$

$$(\omega - B_1)B_1(\Psi')^2 \Psi'' = 0, \quad (37)$$

$$(\beta B_1 - 4\omega B_1)(\Psi')^2 + (6B_1^2 - 3\omega B_1)(\Psi'')^2 - 4\omega B_1 \Psi' \Psi''' = 0, \quad (38)$$

$$(-\beta B_1 + 4\omega B_1)\Psi'' + \omega B_1 \Psi^{(4)} = 0. \quad (39)$$

From Eq. (36) and Eq. (37) we obtain, $B_1 = \omega$, Now integrating Eq. (39) and substituting into Eq. (38), we get:

$$\frac{\Psi'''}{\Psi''} = \mu, \quad \text{where } \mu = \pm \sqrt{\frac{3\beta - 16\omega}{3\omega}}, \quad (40)$$

consequently, we obtain

$$\Psi'(\xi) = c_1 e^{\mu \xi}, \quad (41)$$

$$\Psi = c_2 + \frac{c_1}{\mu} c_1 e^{\mu \xi}, \quad (42)$$

where c_1 and c_2 are constant of integration. Now substituting the values of B_1, Ψ and Ψ' into Eq. (35), we get the exact solution of Eq. (1), as follows

$$v(x, y, t) = B_0 + \omega \left(\frac{\mu c_1 e^{\mu \xi}}{\mu c_2 + c_1 e^{\mu \xi}} \right), \quad (43)$$

where $\xi = x + y + \omega t$.

(a)

(b)

Figure 7: Exact solitary wave solution of Eq. (43) is plotted in different shapes at : $B_0 = 1$, $\beta = 2$, $c_1 = 0.5$, $c_2 = -0.5$, $\omega = 0.25$:(a) solitary wave and (b) one dimensional solitary wave

Simplifying Eq. (43), we obtain

$$v(x, y, t) = B_0 + \omega \left(\frac{\mu c_1 \left(\cosh\left(\frac{\mu\xi}{2}\right) + \sinh\left(\frac{\mu\xi}{2}\right) \right)}{(c_1 + \mu c_2) \cosh\left(\frac{\mu\xi}{2}\right) + (c_1 - c_2 \mu) \sinh\left(\frac{\mu\xi}{2}\right)} \right), \quad (44)$$

where $\xi = x + y + \omega t$.

Similarly we can randomly choose parameter c_1 and c_2 , by setting $c_1 = \mu c_2$, we obtain the following solitary solution

$$v(x, y, t) = B_0 + \sqrt{\frac{3\beta\omega - 16\omega^2}{12}} \left(\tanh\left(\sqrt{\frac{3\beta - 16\omega}{12\omega}} \xi\right) \pm 1 \right), \quad (45)$$

where $\xi = x + y + \omega t$.

Now again we choose $c_1 = -c_2\mu$, we obtain following solitary waves solution.

$$v(x, y, t) = B_0 + \sqrt{\frac{3\beta\omega - 16\omega^2}{12}} \left(\coth\left(\sqrt{\frac{3\beta - 16\omega}{12\omega}} \xi\right) \pm 1 \right), \quad (46)$$

$$B_0 + \sqrt{\frac{3\beta\omega - 16\omega^2}{12}} \left(\coth\left(\sqrt{\frac{3\beta - 16\omega}{12\omega}} \xi\right) \pm 1 \right),$$

where $\xi = x + y + \omega t$ and B_0 is left as a free parameter.

(a)

(b)

Figure 8: Exact solitary wave solution of Eq. (46) is plotted in different shapes at : $B_0 = 1$, $\beta = 1$, $\omega = 0.5$:(a) periodic solitary wave and (b) one-dimensional solitary wave

4 Conclusion

In the current paper, proposed methods such as simple equation method (SEM) and modified simple equation method (MSEM) have been successfully employed to obtain the exact solitary wave solutions for the fourth-order nonlinear ablowitz-Kaup-Newell-Segur water equation. The AKNS equations have a wide application in field of physical sciences and the obtained solitary wave solutions in different form help to understand the physical phenomenon in various aspects. All the solutions are also prescribed graphically by assigning special values to the

parameters. Mathematica facilitate us to handle all the calculations. As the extensive applications of the solitary wave theory, it is valuable to study further the localized excitation and its applications in the future.

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