Research Article

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Solitary wave solutions of two KdV-type equations

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Abstract: The present paper investigates the solitary wave solutions of the nonlinear evolution equations with power nonlinearities. The study has been carried out for two examples of KdV-type equations, namely, the nonlinear dispersive equation and the generalised KdV equation. To achieve our goal, we have applied the projective Riccati equation method. As a result, many exact solutions in the form of solitary wave solutions and combined formal solitary wave solutions are obtained

Keywords: Solitary wave solutions, projective Riccati equation, the nonlinear dispersive equation, the generalised KdV equation

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1 Introduction

Recently, the nonlinear evolution equations (NEEs) with power nonlinearities have been under a considerable concentration since they are widely used to describe complex phenomena in various fields of science, such as fluid mechanics, plasma physics, solid state physics and optical fibres, etc. Thus, the search for explicit exact solutions, in particular, solitary wave solutions, is very important, as it allows to understand these nonlinear phenomena and to provide valuable reference for other related research. Many powerful and efficient methods have been developed, such as Backlund transformation method [1, 2], Hirota's direct method [3], tanh method [4, 5], extended tanh method [6–8], sine-cosine method [9–11], F-expansion method [12–15], (G'/G) -expansion method [16–18], exp-function method [19–21], rational function method [22], first integral method [23, 24], auxiliary equation method [25–28], and so on. Furthermore, the fractional differential equations have been found to govern some physical models in the natural science. Such equations have frac-

tional derivatives with arbitrary order and different kinds in physical applications. Thus, the use of fractional calculus tools may depend on the type of fractional derivatives. For more information on the fractional calculus, see [29–31].

In the last three decades, one of the more effective methods that has been developed to seek many kinds of exact solutions to NEEs is the projective Riccati equation method. For example, in 1992, Conte and Musette [32] introduced this method to find more new solitary wave solutions to NEEs that can be expressed as polynomial in two elementary functions which satisfy a projective Riccati equation. Thereafter, many studies have been carried out to generalise the projective Riccati equation method and to obtain a variety of exact solutions to nonlinear partial differential equations. For more details, see [33–38]. Recently, Kumar and Chand [39] have examined the exact traveling wave solutions of some nonlinear evolution equations. They assumed that the projective Riccati equations are given in the form

$$\begin{aligned} f'(\xi) &= pf(\xi)g(\xi), & g'(\xi) &= R + pg^2(\xi) - rf(\xi), \\ g^2(\xi) &= -p \left[R - 2rf(\xi) + \frac{r^2 + \delta}{R} f^2(\xi) \right], \end{aligned} \quad (1)$$

where p , R and r are constants and $\delta = \pm 1$. It is found that the system of Eq. (1) admits solutions in terms of hyperbolic functions when $p = -1$ while the system possess solutions in terms of trigonometric functions once $p = 1$.

In this article, we utilise the projective Riccati system (1) with $p = -1$ and $R = 1$ to seek various solitary wave solutions for two KdV-type equations with power nonlinearities. In Section (2), we describe the projective Riccati equation method and its technique for solving NEEs. Then, we apply the method to solve the nonlinear dispersive equation in Section (3) and the generalised KdV equation in Section (4). Finally, in Section (5) we present our discussion and conclusion.

2 Analysis of the method

Consider a nonlinear partial differential equation (NLPDE) for $u(x, t)$ in the form

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, u_{xxx} \dots) = 0, \quad (2)$$

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where P is a polynomial in its arguments. The principle of the generalised projective Riccati method will be demonstrated as follows.

Since we seek for travelling wave solutions, we introduce the wave variable

$$u(x, t) = U(\xi), \quad \xi = x - ct, \quad (3)$$

and this transforms the NLPDE (2) to the following ordinary differential equation (ODE)

$$Q(U, U', U'', U''', \dots) = 0, \quad (4)$$

where prime denotes the derivative with respect to ξ . Then, integrate (4), if possible, to reduce the order of differentiation.

Now, assume that the solution of (4) can be expressed in the finite series form

$$U(\xi) = \sum_{i=0}^m a_i f^i(\xi) + \sum_{j=1}^m b_j f^{j-1}(\xi) g(\xi), \quad (5)$$

where $a_0, a_i, b_j, (i, j = 1, 2, \dots, m)$ are constants to be determined. The parameter m , which is a positive integer, can be determined by balancing the highest order derivative terms with the nonlinear terms in (4).

The variables $f(\xi), g(\xi)$ satisfy the projective Riccati equations

$$\begin{aligned} f'(\xi) &= -f(\xi)g(\xi), & g'(\xi) &= 1 - g^2(\xi) - rf(\xi), \\ g^2(\xi) &= 1 - 2rf(\xi) + (r^2 + \epsilon)f^2(\xi), \end{aligned} \quad (6)$$

where $\epsilon = \pm 1$ and r is arbitrary constant. The set of equations (6) is found to accept the following solutions

$$\begin{aligned} f_1(\xi) &= \frac{\alpha}{\beta \cosh(\xi) + \gamma \sinh(\xi) + \alpha r}, \\ g_1(\xi) &= \frac{\beta \sinh(\xi) + \gamma \cosh(\xi)}{\beta \cosh(\xi) + \gamma \sinh(\xi) + \alpha r}, \end{aligned} \quad (7)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$f_2(\xi) = \frac{1}{\sinh(\xi) + r}, \quad g_2(\xi) = \frac{\cosh(\xi)}{\sinh(\xi) + r}, \quad (8)$$

demands $\epsilon = 1$.

$$f_3(\xi) = \frac{1}{\cosh(\xi) + r}, \quad g_3(\xi) = \frac{\sinh(\xi)}{\cosh(\xi) + r}, \quad (9)$$

implies $\epsilon = -1$.

The substitution of (5) along with (6) into (4) generates a polynomial in $f^i(\xi)g^j(\xi)$. Equating each coefficient of $f^i(\xi)g^j(\xi)$ in this polynomial to zero, yields a set of algebraic equations for a_i, b_j . Solving this system of equations,

we can obtain many exact solutions of (2) according to (7)–(9).

In the following two sections, we will apply this method for the nonlinear dispersive equation and the generalised KdV equation.

3 The nonlinear dispersive equation

Let us consider the nonlinear dispersive equation of the form

$$u^n(u^n)_t + a(u^{3n})_x + bu^n(u^n)_{xxx} = 0, \quad n > 0, \quad (10)$$

where a, b are constants and the dependent variable $u(x, t)$ represents the wave profile. The parameter n is a positive integer with $n > 0$. The effect of nonlinearity of the middle term of (10) on the physical structure of the resulting solutions has been previously studied [40–42]. It is found that the exponent of the middle term manipulates the existence of compactons, solitons and periodic solutions.

In order to deal with the complex form of (10), we use the transformation (3) and then suppose that

$$v = U^n. \quad (11)$$

Hence, we arrive at

$$-cv(v)' + 3av^2(v)' + bv(v)''' = 0. \quad (12)$$

Dividing both sides by v and integrating once with respect to ξ , we obtain

$$k - cv + \frac{3}{2}av^2 + bv'' = 0, \quad (13)$$

where k is the integration constant. Now, we assume that the solutions of (13) can be expressed by

$$v(\xi) = \sum_{i=0}^m a_i f^i + \sum_{j=1}^m b_j f^{j-1} g, \quad (14)$$

where $f = f(\xi), g = g(\xi)$ satisfy (6). The homogeneous balance between the highest order derivative term and nonlinear term in (13) leads to $m = 2$. So we have

$$v(\xi) = a_0 + a_1 f + a_2 f^2 + b_1 g + b_2 fg. \quad (15)$$

Substituting (15) along with (6) into (13) and collecting all the terms with the same power of $f^i g^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations, leads to

Case 1.

$$a_0 = \frac{c-4b}{3a}, a_2 = -\frac{4b\epsilon}{a}, k = \frac{c^2-16b^2}{6a}, r=0, \quad (16)$$

$$a_1 = b_1 = b_2 = 0.$$

Case 2.

$$a_0 = \frac{c-b}{3a}, a_2 = -\frac{2b\epsilon}{a}, b_2 = \pm \frac{2b\sqrt{\epsilon}}{a}, k = \frac{c^2-b^2}{6a}, \quad (17)$$

$$r=0, a_1 = b_1 = 0.$$

Case 3.

$$a_0 = \frac{c-b}{3a}, a_1 = \pm \frac{2b\sqrt{-\epsilon}}{a}, k = \frac{c^2-b^2}{6a}, \quad (18)$$

$$r = \pm\sqrt{-\epsilon}, a_2 = b_1 = b_2 = 0.$$

Case 4.

$$a_0 = \frac{c-b}{3a}, a_1 = \frac{2br}{a}, a_2 = -\frac{2b(r^2+\epsilon)}{a}, \quad (19)$$

$$b_2 = \pm \frac{2b\sqrt{r^2+\epsilon}}{a}, k = \frac{c^2-b^2}{6a}, b_1 = 0.$$

According to the above results combined with (7)-(9), (11) and (15) we can obtain the following families of solitary wave solutions to the nonlinear dispersive equation.

From Case 1 we reach the solutions

$$u_1 = \left\{ \frac{c-4b}{3a} - \frac{4b\alpha^2\epsilon}{a[\beta \cosh(\xi) + \gamma \sinh(\xi)]^2} \right\}^{1/n}, \quad (20)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_2 = \left\{ \frac{c-4b}{3a} - \frac{4b}{a[\sinh(\xi)]^2} \right\}^{1/n}, \quad \epsilon = 1. \quad (21)$$

$$u_3 = \left\{ \frac{c-4b}{3a} + \frac{4b}{a[\cosh(\xi)]^2} \right\}^{1/n}, \quad \epsilon = -1. \quad (22)$$

From Case 2 we obtain

$$u_4 = \left\{ \frac{c-b}{3a} - \frac{2b\alpha^2\epsilon}{a[\beta \cosh(\xi) + \gamma \sinh(\xi)]^2} \right. \\ \left. \pm \frac{2b\sqrt{\epsilon}}{a} \frac{\alpha[\beta \sinh(\xi) + \gamma \cosh(\xi)]}{[\beta \cosh(\xi) + \gamma \sinh(\xi)]^2} \right\}^{1/n}, \quad (23)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_5 = \left\{ \frac{c-b}{3a} - \frac{2b}{a} \frac{[1 \pm \cosh(\xi)]}{[\sinh(\xi)]^2} \right\}^{1/n}, \quad \epsilon = 1. \quad (24)$$

$$u_6 = \left\{ \frac{c-b}{3a} + \frac{2b}{a} \frac{[1 \pm i \sinh(\xi)]}{[\cosh(\xi)]^2} \right\}^{1/n}, \quad \epsilon = -1. \quad (25)$$

From Case 3 we arrive at

$$u_7 = \left\{ \frac{c-b}{3a} \pm \frac{2b\alpha\sqrt{-\epsilon}}{a[\beta \cosh(\xi) + \gamma \sinh(\xi) \pm \alpha\sqrt{-\epsilon}]} \right\}^{1/n}, \quad (26)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_8 = \left\{ \frac{c-b}{3a} \pm \frac{2ib}{a[\sinh(\xi) \pm i]} \right\}^{1/n}, \quad \epsilon = 1. \quad (27)$$

$$u_9 = \left\{ \frac{c-b}{3a} \pm \frac{2b}{a[\cosh(\xi) \pm 1]} \right\}^{1/n}, \quad \epsilon = -1. \quad (28)$$

From Case 4 we obtain

$$u_{10} = \left\{ \frac{c-b}{3a} + \frac{2bra}{a[\beta \cosh(\xi) + \gamma \sinh(\xi) + ar]} \right. \\ \left. - \frac{2b(r^2+\epsilon)\alpha^2}{a[\beta \cosh(\xi) + \gamma \sinh(\xi) + ar]^2} \right. \\ \left. \pm \frac{2b\sqrt{r^2+\epsilon}}{a} \frac{\alpha[\beta \sinh(\xi) + \gamma \cosh(\xi)]}{[\beta \cosh(\xi) + \gamma \sinh(\xi) + ar]^2} \right\}^{1/n}, \quad (29)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_{11} = \left\{ \frac{c-b}{3a} + \frac{2br}{a[\sinh(\xi) + r]} \right. \\ \left. - \frac{2b\sqrt{r^2+1}}{a} \frac{[\sqrt{r^2+1} \pm \cosh(\xi)]}{[\sinh(\xi) + r]^2} \right\}^{1/n}, \quad \epsilon = 1. \quad (30)$$

$$u_{12} = \left\{ \frac{c-b}{3a} + \frac{2br}{a[\cosh(\xi) + r]} \right. \\ \left. - \frac{2b\sqrt{r^2-1}}{a} \frac{[\sqrt{r^2-1} \pm \sinh(\xi)]}{[\cosh(\xi) + r]^2} \right\}^{1/n}, \quad \epsilon = -1. \quad (31)$$

Remarks:

- I. Case 2 can be recovered from Case 4 when $r = 0$ and therefore the solutions (29) - (31) are converted to (23) - (25).
- II. Case 3 can be recovered from Case 4 if $r = \pm\sqrt{-\epsilon}$, so the solutions (29) - (31) reduce to (26) - (28).
- III. Once $\beta = 0$ in $\gamma^2 = \alpha^2 + \beta^2$ then $\gamma = \alpha$. Therefore, solutions (20), (23), (26) and (29) change into solutions (21), (24), (27) and (30), respectively.
- IV. If $\gamma = 0$ in $\beta^2 = \alpha^2 + \gamma^2$ so $\beta = \alpha$ and hence solutions (20), (23), (26) and (29) are transformed into solutions (22), (25), (28) and (31), respectively.

To shed light on the behaviour of solitary wave solutions, we shall present the profile of some obtained solutions of the nonlinear dispersive equation in Figures 1 - 4.

Figure 1: Graphical representation of solution (21)

Figure 2: Graphical representation of solution (22)

Figure 3: Graphical representation of solution (24)

For example, the solutions u_2 and u_5 display the singular waves while the solutions u_3 and u_{12} show the bell-shaped solitary waves. The graphs are depicted with $a = 1$, $b = 1$, $n = 2$ and $c = 6$ for the solutions u_2 and u_3 whereas

Figure 4: Graphical representation of solution (31)

$c = 4$ for the solutions u_5 and u_{12} . The value of constant r in the solution u_{12} is taken to be 2.

4 The generalised KdV equation

In this section we study the generalised KdV equation with two power nonlinearities [43] given by

$$u_t + (au^n - bu^{2n})u_x + u_{xxx} = 0, \quad n > 0, \quad (32)$$

where a, b are constants. This equation describes the propagation of nonlinear long acoustic-type waves. Once the amplitude is not supposed to be small, Eq. (32) serves as an approximate model for the description of weak dispersive effects on the propagation of nonlinear waves along a characteristic direction [43]. It can be noted that for $n = 1$, (32) represents the standard Gardner equation, or the combined KdV-mKdV equation.

The use of wave transformation (3) reduces (32), after integrating once, to an ODE in the form

$$-cU + \frac{a}{n+1}U^{n+1} - \frac{b}{2n+1}U^{2n+1} + U'' = 0, \quad (33)$$

where integration constant is considered zero. It is more convenient to replace the exponents of nonlinear terms by integers in order to seek for a closed form solution. Accordingly assuming that $U = v^{1/n}$, (33) reduces to

$$-Av^2 + Bv^3 - Cv^4 - Dv'^2 + vv'' = 0, \quad (34)$$

where

$$A = cn, \quad B = \frac{an}{n+1}, \quad C = \frac{bn}{2n+1}, \quad D = \frac{n-1}{n}. \quad (35)$$

Similarly, we suppose that (34) has solutions in the form of (14). Then, by the homogeneous balance principle we have

thus assumed that (34) has the following solution

$$v(\xi) = a_0 + a_1 f + b_1 g, \quad (36)$$

where $f = f(\xi)$, $g = g(\xi)$ satisfy (6). Substituting (36) with (6) into (34) and setting the coefficients of $f^i g^j$ ($i, j = 1, 2, \dots$) to zero generates a system of algebraic equations. Solving this system of equations, gives rise to

Case 1.

$$\begin{aligned} a_1 &= \frac{(n+1)(n+2)r}{an^2}, \quad c = \frac{1}{n^2}, \\ \frac{a^2}{b} &= \frac{(n+1)(n+2)^2 r^2}{(2n+1)(r^2 + \epsilon)n^2}, \quad a_0 = b_1 = 0. \end{aligned} \quad (37)$$

Case 2.

$$\begin{aligned} a_0 &= \frac{2(n+1)(n+2)}{an^2}, \quad b_1 = \pm a_0, \quad c = \frac{4}{n^2}, \\ \frac{a^2}{b} &= \frac{4(n+1)(n+2)^2}{(2n+1)n^2}, \quad r = 0, \quad a_1 = 0. \end{aligned} \quad (38)$$

Case 3.

$$\begin{aligned} a_0 &= \frac{(n+1)(n+2)}{2an^2}, \quad b_1 = \pm a_0, \quad c = \frac{1}{n^2}, \\ \frac{a^2}{b} &= \frac{(n+1)(n+2)^2}{(2n+1)n^2}, \quad r = \sqrt{-\epsilon}, \quad a_1 = 0. \end{aligned} \quad (39)$$

Case 4.

$$\begin{aligned} a_0 &= \frac{(n+1)(n+2)}{2an^2}, \quad b_1 = \pm a_0, \quad a_1 = \pm a_0 \sqrt{\epsilon}, \\ c &= \frac{1}{n^2}, \quad \frac{a^2}{b} = \frac{(n+1)(n+2)^2}{(2n+1)n^2}, \quad r = 0. \end{aligned} \quad (40)$$

Case 5.

$$\begin{aligned} a_0 &= \frac{(n+1)(n+2)}{2an^2}, \quad b_1 = \pm a_0, \quad a_1 = \pm a_0 \sqrt{r^2 + \epsilon}, \\ c &= \frac{1}{n^2}, \quad \frac{a^2}{b} = \frac{(n+1)(n+2)^2}{(2n+1)n^2}. \end{aligned} \quad (41)$$

Case 6.

$$\begin{aligned} a_0 &= \frac{(n+1)(n+2)}{2an^2}, \quad b_1 = \pm a_0, \\ a_1 &= \pm a_0 \frac{n+1}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}}, \quad c = \frac{1}{n^2}, \\ \frac{a^2}{b} &= \frac{(n+1)(n+2)^2}{(2n+1)n^2}, \quad r = \pm \frac{(2n-1)}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}}. \end{aligned} \quad (42)$$

Inserting these results along with (7)-(9) into (36), various solitary wave solutions of the generalised KdV equation can be created as follows.

From Case 1 we obtain the solutions

$$u_1 = \left\{ \frac{(n+1)(n+2)ra}{an^2[\beta \cosh(\xi) + \gamma \sinh(\xi) + ar]} \right\}^{1/n}, \quad (43)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_2 = \left\{ \frac{(n+1)(n+2)r}{an^2[\sinh(\xi) + r]} \right\}^{1/n}, \quad \epsilon = 1. \quad (44)$$

$$u_3 = \left\{ \frac{(n+1)(n+2)r}{an^2[\cosh(\xi) + r]} \right\}^{1/n}, \quad \epsilon = -1. \quad (45)$$

From Case 2 we arrive at the solutions

$$u_4 = \left\{ \frac{2(n+1)(n+2)}{an^2} \left[1 \pm \frac{\beta \sinh(\xi) + \gamma \cosh(\xi)}{\beta \cosh(\xi) + \gamma \sinh(\xi)} \right] \right\}^{1/n}, \quad (46)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_5 = \left\{ \frac{2(n+1)(n+2)}{an^2} [1 \pm \coth(\xi)] \right\}^{1/n}, \quad \epsilon = 1. \quad (47)$$

$$u_6 = \left\{ \frac{2(n+1)(n+2)}{an^2} [1 \pm \tanh(\xi)] \right\}^{1/n}, \quad \epsilon = -1. \quad (48)$$

From Case 3 we find the solutions

$$u_7 = \left\{ \frac{(n+1)(n+2)}{2an^2} \times \left[1 \pm \frac{\beta \sinh(\xi) + \gamma \cosh(\xi)}{\beta \cosh(\xi) + \gamma \sinh(\xi) \pm \alpha \sqrt{-\epsilon}} \right] \right\}^{1/n}, \quad (49)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_8 = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\cosh(\xi)}{\sinh(\xi) \pm i} \right] \right\}^{1/n}, \quad \epsilon = 1. \quad (50)$$

$$u_9 = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\sinh(\xi)}{\cosh(\xi) \pm 1} \right] \right\}^{1/n}, \quad \epsilon = -1. \quad (51)$$

From Case 4 we acquire the solutions

$$u_{10} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\alpha \sqrt{\epsilon}}{\beta \cosh(\xi) + \gamma \sinh(\xi)} \right] \right\}^{1/n}, \quad (52)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_{11} = \left\{ \frac{(n+1)(n+2)}{2an^2} [1 \pm \operatorname{csch}(\xi) \pm \coth(\xi)] \right\}^{1/n}, \quad (53)$$

$$\epsilon = 1.$$

$$u_{12} = \left\{ \frac{(n+1)(n+2)}{2an^2} [1 \pm \operatorname{isech}(\xi) \pm \tanh(\xi)] \right\}^{1/n}, \quad (54)$$

$$\epsilon = -1.$$

From Case 5 we obtain the solutions

$$u_{13} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\alpha\sqrt{r^2 + \epsilon}}{\beta \cosh(\xi) + \gamma \sinh(\xi) + \alpha r} \pm \frac{\beta \sinh(\xi) + \gamma \cosh(\xi)}{\beta \cosh(\xi) + \gamma \sinh(\xi) + \alpha r} \right] \right\}^{1/n}, \quad (55)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_{14} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\sqrt{r^2 + 1}}{\sinh(\xi) + r} \pm \frac{\cosh(\xi)}{\sinh(\xi) + r} \right] \right\}^{1/n}, \quad \epsilon = 1. \quad (56)$$

$$u_{15} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{\sqrt{r^2 - 1}}{\cosh(\xi) + r} \pm \frac{\sinh(\xi)}{\cosh(\xi) + r} \right] \right\}^{1/n}, \quad \epsilon = -1. \quad (57)$$

From Case 6 we come by the solutions

$$u_{16} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{(n+1)}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}} \pm \frac{\alpha}{\beta \cosh(\xi) + \gamma \sinh(\xi) \pm \alpha \frac{(2n-1)}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}}} \pm \frac{\beta \sinh(\xi) + \gamma \cosh(\xi)}{\beta \cosh(\xi) + \gamma \sinh(\xi) \pm \alpha \frac{(2n-1)}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}}} \right] \right\}^{1/n}, \quad (58)$$

when $\epsilon = 1$: α, β, γ satisfy $\gamma^2 = \alpha^2 + \beta^2$ and when $\epsilon = -1$: α, β, γ satisfy $\beta^2 = \alpha^2 + \gamma^2$.

$$u_{17} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm i \frac{(n+1)}{n} \sqrt{\frac{n}{3(n-2)}} \pm \frac{1}{\sinh(\xi) \pm i \frac{(2n-1)}{n} \sqrt{\frac{n}{3(n-2)}}} \pm \frac{\cosh(\xi)}{\sinh(\xi) \pm i \frac{(2n-1)}{n} \sqrt{\frac{n}{3(n-2)}}} \right] \right\}^{1/n}, \quad \epsilon = 1. \quad (59)$$

$$u_{18} = \left\{ \frac{(n+1)(n+2)}{2an^2} \left[1 \pm \frac{(n+1)}{n} \sqrt{\frac{n}{3(n-2)}} \right] \right\}^{1/n}, \quad (60)$$

$$\times \frac{1}{\cosh(\xi) \pm \frac{(2n-1)}{n} \sqrt{\frac{n}{3(n-2)}}} \pm \frac{\sinh(\xi)}{\cosh(\xi) \pm \frac{(2n-1)}{n} \sqrt{\frac{n}{3(n-2)}}} \right\}^{1/n}, \quad \epsilon = -1.$$

Remarks:

- I. Case 3 can be recaptured from Case 5 when $r = \pm\sqrt{-\epsilon}$ and therefore the solutions (55) - (57) are converted to (49) - (51).
- II. Case 4 can be retrieved from Case 5 if $r = 0$, so the solutions (55) - (57) reduce to (52) - (54).
- III. Case 6 can be brought back from Case 5 once $r = \pm \frac{(2n-1)}{n} \sqrt{\frac{-n\epsilon}{3(n-2)}}$ and as a result the solutions (55) - (57) change into (58) - (60).
- IV. Once $\beta = 0$ in $\gamma^2 = \alpha^2 + \beta^2$ then $\gamma = \alpha$. Thus, solutions (43), (46), (49), (52), (55) and (58) are converted to solutions (44), (47), (50), (53), (56) and (59), respectively.
- V. If $\gamma = 0$ in $\beta^2 = \alpha^2 + \gamma^2$ so $\beta = \alpha$ and consequently solutions (43), (46), (49), (52), (55) and (58) become solutions (45), (48), (51), (54), (57) and (60), respectively.

Now, we exhibit the graphical representations for some solutions of the generalised KdV equation. Figures 5-8 illustrate three types of waves such as the bell-shaped solitary waves u_3 , the kink-shaped wave u_6 , u_{18} and the singular wave u_{11} . The solutions u_3 , u_6 and u_{11} are drawn with $n = 2$ which is not valid for u_{18} so n is selected to be

Figure 5: Graphical representation of solution (45)

Figure 6: Graphical representation of solution (48)

Figure 7: Graphical representation of solution (53)

Figure 8: Graphical representation of solution (60)

4 for this case. The dependence of the constant c on the parameter n leads to $c = 1/4$ for u_3 and u_{11} , $c = 1$ for u_6 , and $c = 1/16$ for u_{18} . It is assumed that $r = 2$ in u_3 and $a = 1$ in all cases.

5 Discussion and conclusion

In this paper, we have studied the solitary wave solutions of two KdV-type equations with power nonlinearities. The projective Riccati equation method is applied to solve the two examples of KdV-type equations. Various types of solitary wave solutions of bell-shaped, kink-shaped and singular wave profiles have been obtained. The results have shown that the projective Riccati equation approach can derive many kinds of exact solutions of NEEs.

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