Research Article

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MHD and Slip Effect on Two-immiscible Third Grade Fluid on Thin Film Flow over a Vertical Moving Belt

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Abstract: The present paper related to thin film flows of two immiscible third grade fluids past a vertical moving belt with slip conditions in the presence of uniform magnetic field. Immiscible fluids we mean superposed fluids of different densities and viscosities. The basic governing equations of continuity, momentum and energy are incorporated. The modeled coupled equations are solved analytically by using Adomian Decomposition Method (ADM) along with Homotopy Analysis Method (HAM). The residual errors show the authentication of the present work. For comparison, numerical method (ND-Solve) is also applied and good agreement is found. The effects of model parameters on velocity, skin friction and temperature variation have been studied. At the end, the present study is also compared with single layer flow and revealed in close agreement with the result available in the literature.

Keywords: MHD flow, Two-immiscible fluid, Third grade fluid, Thin film flow, Analytical and Numerical Approach

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1 Introduction

The flow and heat transfer inside thin films are ubiquitous in civil, environmental sciences, mechanical engineering, biological sciences geophysics and elsewhere. This is due

to their central applications at large scale such as wire and fiber coating, reactor fluidization, paper production, different food stuffs like ketchup, sauce and honey, transpiration coating, gaseous diffusion, drilling muds, oil wells, heat pipes and fluid cells. Khaled and Vafai [1] studied heat transfer inside thin film flow. Unsteady thin film flow with variable viscosity was carried out by Nadeem and Awais [2]. Thin film flow of a Newtonian fluid was investigated by Munson and Young [3]. For lifting and drainage problem Alam *et al.* [4] investigated the thin flow of Johnson-Segalman fluids. Thin film flow of a power law liquid falling from an inclined plate was studied by Maladinora *et al.* [5]. Taza Gul *et al.* [6] investigated thin film flow of third grade fluid with slip boundary conditions. Thin film flow of non-Newtonian fluid over a moving belt was investigated by Siddique *et al.* [7]. Another study, Siddique *et al.* [8] investigated the thin film flow of a fourth-grade fluid using a vertical cylinder.

The interest in the outer magnetic field effects on heat-physical processes appeared seventy years ago. Blum [9] carried out one of the first works in the field of heat and mass transfer in the presence of a magnetic field. The application of magnetohydrodynamic (MHD) flow control in aerospace engineering was already considered in the mid-1950s. This was coincident with the first studies on the problem of an aerospace vehicle reentering the atmosphere from space. The high temperature reached at the surface of the vehicle flying at hypersonic speed causes the ionization of the surrounding air molecules and the consequent formation of plasma. By imposing a suitable magnetic field, it is possible to modify the aerodynamic forces and heat transfer rates in a convenient way. The increasing interest in the study of MHD phenomena is related to the development of fusion reactors where plasma is confined by a strong magnetic field (Hunt *et al.* [10]). Morley *et al.* [11] studied MHD effects in the so-called blanket. The blanket is located between the plasma and the magnetic field coils, and absorbs neutrons transforming their energy into heat, which is then carried away by a suit-

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able coolant, preventing neutrons from reaching the magnets, thus avoiding radiation damages. Many exciting innovations have been put forth in the areas of MHD propulsion [12] and remote energy deposition for drag reduction [13]. Double-diffusive MHD convection is significant for material solidification processes [14] and fluid flow over a flat surface or stretching sheet in the presence of a magnetic field finds applications in manufacturing processes such as the cooling of the metallic plate, rolling, purification of molten metals, extrusion of polymers, wire and fiber coating, hydro-magnetic lubrication [15–18]. Extensive research is presented in MHD control of flow and heat transfer in the boundary layer [19–22], enhanced plasma ignition [23], and combustion modeling [24]. The analysis of the flow on an inclined porous plate [25] has become the basis of several scientific and engineering applications. Ahmad *et al.* [26] discussed MHD fluid flow over a stretching sheet in the presence of magnetic field. Different numerical techniques for the MHD flow by Hussain and Ahmad [27, 28] on a stretching sheet with porosity.

The subject of two-fluid flow and heat transfer has been extensively studied due to its importance in chemical and nuclear industries [29]. The immiscibility of two fluids is a result of strong cohesion forces between their molecules, and depends on the nature of the fluids. The ease with which the fluids can be mixed is expressed with an experimentally determined coefficient known as the surface tension. The larger value of this coefficient is, the stronger the resistance is to be mixed. A negative value indicates no resistance to mixing [30]. The flow of immiscible fluids can be classified into three groups based on the interfacial structures and topographical distributions of the phases, namely, segregated flows, transitional or mixed flows, and dispersed flows. The three classes can be explained by considering a closed container filled with a liquid and a gas. Segregated flows occur when the container is oscillating very gently with a low amplitude and frequency and the liquid and the gas remain separated with a single well defined interface. Mixed or transitional flows occur when the frequency and amplitude are increased to the extent that the waves become unstable and break which results that parts of the interface break up and small bubbles are trapped in the liquid. Dispersed flows occur when the container is shaken violently and the gas is suspended as small bubbles within the liquid. Meyer and Garder [31] were the first authors to publish a paper on the mechanics of two immiscible fluids in porous media. Loharsabi and Sahai [32] analyzed the flow of two immiscible fluids in a parallel plate channel by the assumption that the velocity and thermal equilibrium at the interface were continuous. Akai *et al.* [33] studied the two-phase magne-

tohydrodynamic (MHD) flow and heat transfer in a parallel plate channel with one of the fluids being electrically conducting. Several other researchers have assumed that the separated two-phase flow can be well represented by the superimposition of two single-phase flows separated by a flat interface [34, 35]. In general, multi-phase flows can be driven by gravitational and viscous flows. The laminar flow of two immiscible fluids in a horizontal pipe has been widely studied theoretically and experimentally [36, 37]. Loharsbi and Sahai [32] studied the two-phase magnetohydrodynamic (MHD) flow and heat transfer in a parallel plate channel with one of the fluids being electrically conducting. Later on, Malashetty *et al.* [38, 39] and Umavathi *et al.* [40, 41] studied the two-fluid flow and heat transfer with various geometries. Recently, Prathap-Kumar *et al.* [42] studied the mixed convective flow of immiscible viscous fluids in a vertical channel. Kim *et al.* [43] examined the theoretical prediction on the double-layer coating in wet-on-wet optical fiber coating process. Double-layer coating liquid flows were used by Kim *et al.* [44] in optical fiber manufacturing. For this purpose, power-law fluid model was used. Recently Zeeshan *et al.* [45] used Phan-Thien-Tanner fluid in double-layer optical fiber coating. The same author [46] investigated double-layer resin coating of optical fiber glass using wet-on-wet coating process with constant pressure gradient. Two-phase flow of an elastic-viscous fluid investigated by Zeeshan *et al.* [47] for double-layer optical fiber coating. Flow and heat transfer of two immiscible fluids in double-layer optical fiber coating is investigated by Zeeshan *et al.* [48]. Recently, Zeeshan *et al.* [49] studied Steady flow and heat transfer analysis of Phan-Thien-Tanner fluid in double-layer optical fiber coating analysis with slip conditions.

In view of the above motivation, here in the present study we discuss the magnetohydrodynamic flow of two immiscible and incompressible thin flow of a third-grade fluid in the presence of slip conditions. No one has investigated thin film of two immiscible third grade fluid in the presence of electrically conducting fluid with slip boundary condition. The analytical solution is obtained by utilizing Adomian Decomposition Method. For validation of the results Homotopy Analysis Method (HAM) and numerical method (ND-Solve) are also applied. At the end for clarity the present work is also compared with published results for limiting cases and good agreement is found [50].

2 Modeling of the left problem

Consider incompressible and immiscible double layer flows on a vertical moving belt through a large bath of third grade fluid. The fluids are electrically conducting in the presence of uniform magnetic field and the belt moves vertically upward at constant speed V . The belt carries with itself a layer of liquid of constant thickness h_1 taking as first liquid layer. The uniform total thickness is δ . The thickness of the secondary layer is $\delta - h_1$ on the free surface of the belt [6]. The Cartesian coordinate system is chosen in which x -axis is perpendicular to the belt and y -axis is taking parallel to the belt. We assume that the both layers are steady, laminar and axisymmetric and satisfy the constitutive equation of the third-grade fluid. Further, we assume that the thickness h_1 and $\delta - h_1$ of the both layers are uniform.

Figure 1: Geometry of the lifting problem of two-phase flow when the belt is moving up

The liquids are parameterized by temperature $\theta^{(i)}$, the fluid density $\rho^{(i)}$, the viscosity $\mu^{(i)}$, thermal conductivity $k^{(i)}$ and the specific heat $c_p^{(i)}$. The governing equations for the two fluids are the continuity, momentum, and energy equations given as:

$$\nabla \cdot \mathbf{u}^{(i)} = 0, \quad (1)$$

$$\rho^{(i)} \frac{D\mathbf{u}^{(i)}}{Dt} = \nabla \cdot \mathbf{T}^{(i)} \quad (2)$$

$$\rho^{(i)} c_p^{(i)} \frac{D\theta^{(i)}}{Dt} = k^{(i)} \nabla^2 \theta^{(i)} + \text{tr}(\mathbf{T}^{(i)} \cdot \mathbf{L}^{(i)}) \quad (3)$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}^{(i)} \cdot \nabla$ is the substantive acceleration.

Slip boundary conditions are taken at moving belt while $\theta^{(1)}$ and $\theta^{(2)}$ are temperatures of the first and second layer respectively. At the fluid interface, we utilize the assumptions that the velocity, the shear stress, the pressure gradient, the temperature and the heat flux are continuous.

The pertinent boundary and interface conditions on the velocity are [29, 30, 43–49]

$$u^{(1)} = V - \gamma S_{xy}^{(1)} \text{at } x = 0, \quad \frac{du^{(2)}}{dx} = 0 \text{ at } x = \delta, \quad (4)$$

$$u^{(1)} = u^{(2)} \text{ and } S_{xy}^{(1)} = S_{xy}^{(2)} \text{ at } x = h_1. \quad (5)$$

The pertinent boundary and interface conditions on the temperature are [29, 30, 43–49]

$$\theta^{(1)}(0) = \theta_0^{(1)} \text{ and } \theta^{(2)}(\delta) = \theta_0^{(2)}, \quad (6)$$

$$\theta^{(1)} = \theta^{(2)} \text{ and } k^{(1)} \frac{d\theta^{(1)}}{dx} = k^{(2)} \frac{d\theta^{(2)}}{dx} \text{ at } x = h_1 \quad (7)$$

Here h shows interface between thin films. $S_{xy}^{(1)}$ and $S_{xy}^{(2)}$ are the shear rates of both thin fluid films and $k^{(1)}$, $k^{(2)}$ are known as flow consistency indexes.

The Cauchy stress $\mathbf{T}^{(i)}$ given in Eq. (2) is:

$$\mathbf{T}^{(i)} = -p\mathbf{I} + \mathbf{S}^{(i)} \quad (8)$$

In the above equation p is the pressure and $\mathbf{I}$ and $\mathbf{S}^{(i)}$ are the identity and extra stress tensor respectively.

For third grade fluid $\mathbf{S}^{(i)}$ defines as [6–10].

$$\begin{aligned} \mathbf{S}^{(i)} = & \mu^{(i)} \mathbf{A}_1^{(i)} + \alpha_1^{(i)} \mathbf{A}_2^{(i)} + \alpha_2^{(i)} \mathbf{A}_1^{(i)2} + \tau_1^{(i)} \mathbf{A}_3^{(i)} \\ & + \tau_2^{(i)} (\mathbf{A}_2^{(i)} \mathbf{A}_1^{(i)} + \mathbf{A}_1^{(i)} \mathbf{A}_2^{(i)}) + \tau_3^{(i)} \left(\text{tr} \mathbf{A}_1^{(i)2} \right) \mathbf{A}_1^{(i)}, \end{aligned} \quad (9)$$

in which $\alpha_1^{(i)}$, $\alpha_2^{(i)}$, $\tau_1^{(i)}$, $\tau_2^{(i)}$, $\tau_3^{(i)}$ are constants and $\mathbf{A}_1^{(i)}$, $\mathbf{A}_2^{(i)}$, $\mathbf{A}_3^{(i)}$ are kinematic tensors defined by

$$\mathbf{A}_1^{(i)} = \mathbf{L}^{(i)T} + \mathbf{L}^{(i)}, \quad (10)$$

$$\mathbf{A}_n^{(i)} = \mathbf{A}_{n-1}^{(i)} \mathbf{L}^{(i)T} + \mathbf{L}^{(i)} \mathbf{A}_{n-1}^{(i)} + \frac{D\mathbf{A}_{n-1}^{(i)}}{Dt}, \quad n = 2, 3. \quad (11)$$

where T stands for transpose of a matrix.

For steady and unidirectional flow velocity, temperature and stress fields are defined as:

$$\mathbf{u}^{(i)} = [0, u^{(i)}(x), 0], \quad \theta^{(i)} = \theta^{(i)}(x), \quad S^{(i)} = S^{(i)}(x). \quad (12)$$

In view of Eq. (11), Eq. (1) fulfilled identically and from Eqs. (2), (3) and (8-11) we have

$$S_{xx}^{(i)} = -p + (2\alpha_1^{(i)} + \alpha_2^{(i)}) \left(\frac{du^{(i)}}{dx} \right)^3, \quad (13)$$

$$S_{yy}^{(i)} = -p + \alpha_2^{(i)} \left(\frac{du^{(i)}}{dx} \right)^2, \quad (14)$$

$$S_{xy}^{(i)} = \mu^{(i)} \frac{du^{(i)}}{dx} + 2(\tau_2^{(i)} + \tau_3^{(i)}) \left(\frac{du^{(i)}}{dx} \right)^3. \quad (15)$$

In view of Eqs. (13) to (15), the momentum and energy equations reduce to:

$$\mu^{(i)} \frac{d^2 u^{(i)}}{dx^2} + 6(\tau_2^{(i)} + \tau_3^{(i)}) \left(\frac{du^{(i)}}{dx} \right)^2 \left(\frac{d^2 u^{(i)}}{dx^2} \right) - \rho^{(i)} g - \sigma^{(i)} B_0^2 u^{(i)}(x) = 0, \quad (16)$$

$$k^{(i)} \frac{d^2 \theta^{(i)}}{dx^2} + \mu^{(i)} \left(\frac{du^{(i)}}{dx} \right)^2 + 2(\tau_2^{(i)} + \tau_3^{(i)}) \left(\frac{du^{(i)}}{dx} \right)^4 = 0. \quad (17)$$

Introducing the dimensionless parameters

$$\begin{aligned} u^{(i)} &= \frac{u^{(i)}}{V}, \quad \bar{x} = \frac{x}{\delta}, \quad \theta^{(1)}(0) = \frac{\theta^{(1)} - \theta_0}{\theta_1 - \theta_0}, \quad (18) \\ \theta^{(2)}(1) &= \frac{\theta^{(2)} - \theta_0}{\theta_1 - \theta_0}, \quad \Lambda = \frac{\mu^{(1)} \gamma}{h}, \\ Br^{(i)} &= \frac{\mu^{(i)} V^2}{k^{(i)} (\theta_1 - \theta_0) \delta^2}, \quad m^{(i)} = \frac{\rho^{(i)} \delta^2 g}{\mu^{(i)} V}, \\ M^{(i)} &= \frac{\sigma^{(i)} B_0^2 \delta^2}{\mu^{(i)}}, \quad \beta^{(i)} = \frac{(\tau_2^{(i)} + \tau_3^{(i)}) V^2}{\mu^{(i)} \delta^2}, \\ \alpha^{(i)} &= \frac{\delta V}{\nu^{(i)}}, \quad \nu^{(i)} = \frac{\mu^{(i)}}{\rho^{(i)}}, \quad k = \frac{k^{(2)}}{k^{(1)}}. \end{aligned}$$

In view of Eq. (18), Eqs. (16)-(17) and (4)-(7) become

$$\frac{d^2 u^{(i)}}{dx^2} + 6\beta^{(i)} \left(\frac{du^{(i)}}{dx} \right)^2 \left(\frac{d^2 u^{(i)}}{dx^2} \right) - m^{(i)} - M^{(i)} u^{(i)}(x) = 0, \quad (19)$$

$$\frac{d^2 \theta^{(i)}}{dx^2} + \lambda^{(i)} \left[\left(\frac{du^{(i)}}{dx} \right)^2 + 2\beta^{(i)} \left(\frac{du^{(i)}}{dx} \right)^4 \right] = 0 \quad (20)$$

$$u^{(1)}(0) = \alpha - \Lambda \left(\frac{du^{(1)}}{dx} + 2\beta^{(1)} \left(\frac{du^{(1)}}{dx} \right)^3 \right), \quad (21)$$

$$\frac{du^{(1)}}{dx}(1) = 0,$$

$$u^{(1)} = u^{(2)} \text{ and } S_{xy}^{(1)} = S_{xy}^{(2)} \text{ at } x = h. \quad (22)$$

$$\theta^{(1)}(0) = 0, \text{ and } \theta^{(2)}(1) = 1, \quad (23)$$

$$\theta^{(1)} = \theta^{(2)} \text{ and } \frac{d\theta^{(1)}}{dx} = k \frac{d\theta^{(2)}}{dx} \text{ at } x = h. \quad (24)$$

3 ADM solution of the Left problem

By applying the inverse operator ζ^{-1} = of the Adomian decomposition method on (19), we have

$$\begin{aligned} u^{(1)} &= C_1 x + C_2 + m^{(1)} \frac{x^2}{2} + \zeta^{-1} \left[M^{(1)} u^{(1)} \right] \\ &\quad - 6\beta^{(1)} \zeta^{-1} \left[\left(\frac{du^{(1)}}{dx} \right)^2 \frac{d^2 u^{(1)}}{dx^2} \right], \end{aligned} \quad (25)$$

$$\begin{aligned} u^{(2)} &= C_3 x + C_4 + m^{(2)} \frac{x^2}{2} + \zeta^{-1} \left[M^{(2)} u^{(2)} \right] \\ &\quad - 6\beta^{(2)} \zeta^{-1} \left[\left(\frac{du^{(2)}}{dx} \right)^2 \frac{d^2 u^{(2)}}{dx^2} \right], \end{aligned} \quad (26)$$

Applying summation to the Eqs. (25) and (26) we have

$$\begin{aligned} \sum_{n=0}^{\infty} u^{(1)}_n &= C_1 x + C_2 + m^{(1)} \frac{x^2}{2} \\ &\quad + M^{(1)} \zeta^{-1} \left(\sum_{n=0}^{\infty} u^{(1)}_n \right) - 6\beta^{(1)} \zeta^{-1} \\ &\quad \times \left[\frac{d}{dx} \left(\sum_{n=0}^{\infty} u^{(1)}_n \right)^2 \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u^{(1)}_n \right) \right], \end{aligned} \quad (27)$$

$$\begin{aligned} \sum_{n=0}^{\infty} u^{(2)}_n &= C_3 x + C_4 + m^{(2)} \frac{x^2}{2} \\ &\quad + M^{(2)} \zeta^{-1} \left(\sum_{n=0}^{\infty} u^{(2)}_n \right) - 6\beta^{(2)} \zeta^{-1} \\ &\quad \times \left[\frac{d}{dx} \left(\sum_{n=0}^{\infty} u^{(2)}_n \right)^2 \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u^{(2)}_n \right) \right], \end{aligned} \quad (28)$$

From Eqs. (27) and (28), the Adomian polynomials are expressed as:

$$A_n^{(1)} = \left(\frac{d}{dx} \left(\sum_{n=0}^{\infty} u^{(1)}_n \right)^2 \right) \left[\frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u^{(1)}_n \right) \right], \quad (29)$$

$$A_n^{(2)} = \left(\frac{d}{dx} \left(\sum_{n=0}^{\infty} u^{(2)}_n \right)^2 \right) \left[\frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} u^{(2)}_n \right) \right], \quad (30)$$

$$n \geq 0,$$

In components form from Eqs. (29), and (30) the Adomian polynomial are when $n \geq 0$,

$$A_0^{(1)} = \left(\frac{du_0^{(1)}}{dx} \right)^2 \frac{d^2 u_0^{(1)}}{dx^2}, \quad (31)$$

$$A_1^{(1)} = \left(\frac{du_0^{(1)}}{dx} \right)^2 \frac{d^2 u_1^{(1)}}{dx^2} + 2 \frac{du_0^{(1)}}{dx} \frac{du_1^{(1)}}{dx},$$

$$A_2^{(1)} = \left(\frac{du_0^{(1)}}{dx} \right)^2 \frac{d^2 u_2^{(1)}}{dx^2} + \left(\frac{du_1^{(1)}}{dx} \right)^2 \frac{d^2 u_0^{(1)}}{dx^2} + 2 \frac{du_0^{(1)}}{dx} \frac{du_2^{(1)}}{dx} \frac{d^2 u_0^{(1)}}{dx^2}, \quad (32)$$

$$A_0^{(2)} = \left(\frac{du_0^{(2)}}{dx} \right)^2 \frac{d^2 u_0^{(2)}}{dx^2}, \quad A_1^{(2)} = \left(\frac{du_0^{(2)}}{dx} \right)^2 \frac{d^2 u_1^{(2)}}{dx^2} + 2 \frac{du_0^{(2)}}{dx} \frac{du_1^{(2)}}{dx}, \quad (33)$$

$$A_2^{(2)} = \left(\frac{du_0^{(2)}}{dx} \right)^2 \frac{d^2 u_2^{(2)}}{dx^2} + \left(\frac{du_1^{(2)}}{dx} \right)^2 \frac{d^2 u_0^{(2)}}{dx^2} + 2 \frac{du_0^{(2)}}{dx} \frac{du_2^{(2)}}{dx} \frac{d^2 u_0^{(2)}}{dx^2}. \quad (34)$$

The series solution becomes:

$$u_0^{(1)} + u_1^{(1)} + u_2^{(1)} + \dots = C_1 x + C_2 + m^{(1)} \frac{x^2}{2} + M^{(1)} \zeta^{-1} [u_0^{(1)} + u_1^{(1)} + u_2^{(1)} + \dots] - 6\beta^{(1)} \zeta^{-1} [A_0 + A_1 + A_2 + \dots], \quad (35)$$

$$u_0^{(2)} + u_1^{(2)} + u_2^{(2)} + \dots = C_3 x + C_4 + m^{(2)} \frac{x^2}{2} + M^{(2)} \zeta^{-1} [u_0^{(2)} + u_1^{(2)} + u_2^{(2)} + \dots] - 6\beta^{(2)} \zeta^{-1} [A_0 + A_1 + A_2 + \dots] \quad (36)$$

By comparing both sides of Eqs. (35) and (36), the velocity components are:

Zero Components Problem.

$$u_0^{(1)}(x) = C_1 x + C_2 + m^{(1)} \frac{x^2}{2}, \quad (37)$$

$$u_0^{(2)}(x) = C_3 x + C_4 + m^{(2)} \frac{x^2}{2}. \quad (38)$$

From Eqs. (21) and (22), the boundary and interface conditions are

$$u_0^{(1)}(0) = \alpha - \Lambda^{(1)} \left(\frac{du_0^{(1)}}{dx} + 2\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^3 \right), \quad (39)$$

$$\frac{du_0^{(2)}(1)}{dx} = 0.$$

$$u_0^{(1)}(h) = u_0^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (40)$$

First Component Problem.

$$u_1^{(1)}(x) = M^{(1)} \zeta^{-1} [u_0^{(1)}] - 6\beta^{(1)} \zeta^{-1} [A_0^{(1)}], \quad (41)$$

$$u_1^{(2)}(x) = M^{(2)} \zeta^{-1} [u_0^{(2)}] - 6\beta^{(2)} \zeta^{-1} [A_0^{(2)}]. \quad (42)$$

The boundary and interface conditions are:

$$u_1^{(1)}(0) = -\Lambda^{(1)} \left(\frac{du_1^{(1)}}{dx} + 6\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^2 \frac{du_1^{(1)}}{dx} \right), \quad (43)$$

$$\frac{du_1^{(2)}(1)}{dx} = 0,$$

$$u_1^{(1)}(h) = u_1^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (44)$$

Second Component Problem.

$$u_2^{(1)}(x) = M^{(1)} \zeta^{-1} [u_1^{(1)}] - 6\beta^{(1)} \zeta^{-1} [A_1^{(1)}], \quad (45)$$

$$u_2^{(2)}(x) = M^{(2)} \zeta^{-1} [u_1^{(2)}] - 6\beta^{(2)} \zeta^{-1} [A_1^{(2)}]. \quad (46)$$

The boundary and interface conditions are

$$u_2^{(1)}(0) = -\Lambda^{(1)} \left(\frac{du_2^{(1)}}{dx} \left(1 + 6\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^2 \right) + 6\beta^{(1)} \left(\frac{du_1^{(1)}}{dx} \right)^2 \frac{du_0^{(1)}}{dx} \right), \quad \frac{du_2^{(2)}(1)}{dx} = 0, \quad (47)$$

$$u_2^{(1)}(h) = u_2^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (48)$$

Solve the above system of equations corresponding to their boundary and interface conditions we get the series solution for velocity profiles for both layers up to second order components as:

$$u^{(1)}(x) = u_0^{(1)}(x) + u_1^{(1)}(x) + u_2^{(1)}(x), \quad (49)$$

$$u^{(2)}(x) = u_0^{(2)}(x) + u_1^{(2)}(x) + u_2^{(2)}(x). \quad (50)$$

The dimensionless shear stress for both layers is

$$\tau_{xy} = \frac{du^{(i)}}{dx} + 2\beta^{(i)} \left(\frac{du^{(i)}}{dx} \right)^3 \bigg|_{x=1}, \quad (51)$$

4 Modeling of Drainage Problem

The assumptions and geometry of the drainage problem is same as discussed in the previous problem. Here the belt is stationary and the fluid drain down due the gravity over the belt as depicted in Figure 2. The coordinate system for the drainage problem is same as discussed for the left problem. Additionally, the flow is laminar, steady and neglected the external pressure. Consider that the fluid shear forces balance gravity and the film thickness remains constant.

Figure 2: Geometry of drainage problem when the belt is stationary

The pertinent boundary and interface conditions on the velocity are [29, 30, 43–49]

$$u^{(1)} = -\gamma S_{xy} \text{ at } x = 0, \quad \frac{du^{(2)}}{dx} = 0 \text{ at } x = \delta, \quad (52)$$

$$u^{(1)} = u^{(2)} \text{ and } S_{xy}^{(1)} = S_{xy}^{(2)} \text{ at } x = h. \quad (53)$$

In view of Eq. (20), omitting the asterisk Eqs. (52) and (53) in dimensionless form become

$$u^{(1)}(0) = -\Lambda \left(\frac{du^{(1)}}{dx} + 2\beta \left(\frac{du^{(1)}}{dx} \right)^3 \right), \quad (54)$$

$$\frac{du^{(2)}}{dx}(1) = 0,$$

$$u^{(1)} = u^{(2)} \text{ and } S_{xy}^{(1)} = S_{xy}^{(2)} \text{ at } x = h. \quad (55)$$

5 ADM solution of drainage problem

To solve Eqs. (19) and (20) along with boundary and interface conditions given in Eqs. (54) and (55) respectively, we apply Adomian Decomposition Method (ADM). The Adomian polynomials in (31)–(34) is same for both problems. The different velocity components are obtained as follows.

Zero Component Problem

$$u_0^{(1)}(x) = C_1 x + C_2 + m^{(1)} \frac{x^2}{2}, \quad (56)$$

$$u_0^{(2)}(x) = C_3 x + C_4 + m^{(2)} \frac{x^2}{2}. \quad (57)$$

Here C_1, C_2, C_3 and C_4 are constants of integrations to be determined by applying the following boundary and interface conditions given below:

$$u_0^{(1)}(0) = -\Lambda^{(1)} \left(\frac{du_0^{(1)}}{dx} + 2\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^3 \right), \quad (58)$$

$$\frac{du_0^{(2)}}{dx}(1) = 0.$$

$$u_0^{(1)}(h) = u_0^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (59)$$

First Component Problem.

$$u_1^{(1)}(x) = M^{(1)} \zeta^{-1} \left[u_0^{(1)} \right] - 6\beta^{(1)} \zeta^{-1} \left[A_0^{(1)} \right], \quad (60)$$

$$u_1^{(2)}(x) = M^{(2)} \zeta^{-1} \left[u_0^{(2)} \right] - 6\beta^{(2)} \zeta^{-1} \left[A_0^{(2)} \right]. \quad (61)$$

The boundary and interface conditions are:

$$u_1^{(1)}(0) = -\Lambda^{(1)} \left(\frac{du_1^{(1)}}{dx} + 6\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^2 \frac{du_1^{(1)}}{dx} \right), \quad (62)$$

$$\frac{du_1^{(2)}}{dx}(1) = 0,$$

$$u_1^{(1)}(h) = u_1^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (63)$$

Second Component Problem.

$$u_2^{(1)}(x) = M^{(1)} \zeta^{-1} \left[u_1^{(1)} \right] - 6\beta^{(1)} \zeta^{-1} \left[A_1^{(1)} \right], \quad (64)$$

$$u_2^{(2)}(x) = M^{(2)} \zeta^{-1} \left[u_1^{(2)} \right] - 6\beta^{(2)} \zeta^{-1} \left[A_1^{(2)} \right]. \quad (65)$$

The boundary and interface conditions are:

$$u_2^{(1)}(0) = -\Lambda^{(1)} \left(\frac{du_2^{(1)}}{dx} \left(1 + 6\beta^{(1)} \left(\frac{du_0^{(1)}}{dx} \right)^2 \right) \right) \quad (66)$$

Figure 3: Comparison of ADM and OHAM for lifting velocity

Figure 5: Comparison of ADM and OHAM for temperature profile for lifting problem

Figure 4: Comparison of ADM and OHAM for drainage velocity

Figure 6: Comparison of ADM and OHAM for temperature profile for drainage problem

$$+6\beta^{(1)}\left(\frac{du_1^{(1)}}{dx}\right)^2\frac{du_0^{(1)}}{dx}, \frac{du_2^{(2)}(1)}{dx} = 0$$

$$u_2^{(1)}(h) = u_2^{(2)}(h) \text{ and } S_{xy}^{(1)}(h) = S_{xy}^{(2)}(h). \quad (67)$$

By solving the above system of equations corresponding to their boundary and interface conditions we get the velocity profiles for both layers up to second components as:

$$u^{(1)}(x) = u_0^{(1)}(x) + u_1^{(1)}(x) + u_2^{(1)}(x), \quad (68)$$

$$u^{(2)}(x) = u_0^{(2)}(x) + u_1^{(2)}(x) + u_2^{(2)}(x). \quad (69)$$

6 Results and Discussion

In the present study, we modeled two-immiscible thin film flow of third grade fluid over vertical moving belt in the presence of magnetic field and porous medium. The modeled ordinary differential equations of velocity and temperature profiles are solved analytically by adopting Adomian Decomposition method (ADM) along with Homotopy

Figure 7: Effect of non-Newtonian parameters $\beta^{(1)}$ on lift velocity profiles when $m^{(1)} = 0.1$, $m^{(2)} = 0.2$, $M^{(1)} = 0.1$, $M^{(2)} = 0.2$, $\Lambda^{(2)} = 0.01$, $\Lambda^{(2)} = 0.02$, $\alpha^{(1)} = 0.1$, $\alpha^{(2)} = 0.2$, $\beta^{(2)} = 2$

Analysis Method (HAM). Both lift and drainage velocities and temperature profiles of both layers (primary and secondary layers) are discussed by assigning numerical val-

Figure 8: Effect of non-Newtonian parameters $\beta^{(1)}$ on drainage velocity profiles when $m^{(1)} = 0.1, m^{(2)} = 0.2, M^{(1)} = 0.1, M^{(2)} = 0.2, \Lambda^{(2)} = 0.01, \Lambda^{(2)} = 0.02, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2, \beta^{(2)} = 2$

Figure 10: Effect of magnetic parameters $M^{(1)}$ on drainage velocity profiles when $m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \Lambda^{(2)} = 0.01, \Lambda^{(2)} = 0.02, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2, M^{(2)} = 0.5$

Figure 9: Effect of magnetic parameters $M^{(1)}$ on lift velocity profiles when $m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \Lambda^{(2)} = 0.01, \Lambda^{(2)} = 0.02, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2, M^{(2)} = 0.5$

Figure 11: Effect of magnetic parameters $m^{(1)}$ on lift velocity profiles when $M^{(1)} = 0.1, M^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \Lambda^{(2)} = 0.01, \Lambda^{(2)} = 0.02, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2, m^{(2)} = 1$

ues to the physical parameters of interest involved in the solution such as non-Newtonian parameters $\beta^{(1)}$ and $\beta^{(2)}$, magnetic parameters $M^{(1)}$ and $M^{(2)}$, gravitational parameters $m^{(1)}$ and $m^{(2)}$, slip parameter Λ and Brinkman numbers $\lambda^{(1)}$ and $\lambda^{(2)}$ in Figures 7-16. Before proceeding to the results and their discussion, we first validate our results by comparing with OHAM and numerical results obtained by ND-solve method. To end this, Figures 3-6 are prepared which show the velocity and temperature profiles obtained through both analytical and numerical solutions. These figures clearly demonstrate an excellent correlation between both the solutions. This established the confidence on both analytical and numerical solutions and also on the results prediction by these solutions. Figures 7

and 8 illustrate the effect of non-Newtonian parameters $\beta^{(1)}$ and $\beta^{(2)}$, on lift and drainage velocity profiles respectively. It is investigated that the velocity profiles for both lift and drainage problems, follow an increasing trend with increasing non-Newtonian parameters $\beta^{(1)}$ and $\beta^{(2)}$. From the Figure 6 it is cleared that for small value of non-Newtonian parameter, the velocity profile differs little from the Newtonian one, however, when the non-Newtonian parameter increases these profiles become more flattened showing the effect of shear-thinning effect. The effect of magnetic parameter $M^{(1)}$ and $M^{(2)}$, on lifting and drainage velocity profiles is shown in Figures 9 and 10 respectively, by taking other parameters fixed. It is observed that the bound-

Figure 12: Effect of magnetic parameters $m^{(1)}$ on drainage velocity profiles when $M^{(1)} = 0.1, M^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \Lambda^{(2)} = 0.01, \Lambda^{(2)} = 0.02, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2, m^{(2)} = 0.2$

Figure 14: Effect of magnetic parameters Λ on drainage velocity profiles when $M^{(1)} = 0.1, M^{(2)} = 0.2, m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2$

Figure 13: Effect of magnetic parameters Λ on lift velocity profiles when $M^{(1)} = 0.1, M^{(2)} = 0.2, m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2$

Figure 15: Effect of Brinkman numbers on temperature profiles for lift problem when $M^{(1)} = 0.1, M^{(2)} = 0.2, m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2$

ary layer thickness is reciprocal to the applied transverse magnetic field and as a result the velocity decreases. It is also interested to note that the velocity of the fluid is maximum at the center of the annular gap for different values of magnetic parameter, and then it decreases as to meet the far field boundary conditions for the fixed parameters. Additionally, it is noticed that for large value of M the velocity increases as compared to small values in drainage case. The effect of gravitational parameters $m^{(1)}$ and $m^{(2)}$ on the primary and secondary flows of both lifting and drainage fluid is shown in Figure 11 and 12 respectively. It is observed that the primary and secondary layer velocity of the lifting flow decreases and increases in case of drainage. The effect of slip parameter Λ on the velocity profile is shown in

Figures 13 and 14 respectively. It is observed that the velocity of the fluid increases with increasing the slip parameter. It is also noticed that for large values of slip parameter i.e. $\Lambda^{(1)} = 1.5$ and $\Lambda^{(2)} = 2.0$, the increase can be seen more clearly because the friction goes on decreasing. Figures 15 and 16 depict the temperature profiles for different values of Brinkman numbers $\lambda^{(1)}$ and $\lambda^{(2)}$. It is observed that the temperature distribution increases when Brinkman numbers increase and become more flattened for large value of Brinkman numbers. In Figure 17 we plotted the skin friction versus Reynolds number with selected sets of parameters. It is observed that as the Reynolds number increases the skin friction decreases and for large values of Reynolds number the skin friction vanished. Figure 18 show the vari-

Figure 16: Effect of Brinkman numbers on temperature profiles for drainage problem when $M^{(1)} = 0.1, M^{(2)} = 0.2, m^{(1)} = 0.1, m^{(2)} = 0.2, \beta^{(1)} = 0.1, \beta^{(2)} = 0.2, \alpha^{(1)} = 0.1, \alpha^{(2)} = 0.2$

Figure 17: Skin friction verses Reynolds number

Figure 18: Skin friction verses non-Newtonian parameter when $\beta^{(1)} = \beta^{(2)} = \beta$

ation of skin friction versus the non-Newtonian parameter β . It is observed that the skin friction increases with the increasing values of β . At the end for verification the present

Figure 19: Lift velocity comparison of the present work with published single layer work [50]

Figure 20: Drainage velocity comparison of the present work with published single layer work [50]

Figure 21: Lift temperature comparison of the present work with published single layer work [50]

study is also compared for limiting cases and is revealed in closed agreement with the results available in the literature [50] as cleared from the Figures 19-22.

Figure 22: Drainage temperature comparison of the present work with published single layer work [50]

7 Conclusion

In the present study, we investigate two-immiscible flows of thin film third grade fluid through vertical belt for lifting (when the belt is moving upward) and drainage (when the belt is stationary) problem. The model equations for velocity and temperature profiles in lifting and drainage problems are solved analytically by Adomian Decomposition Method along with homotopy analysis method. For the sake of comparison, the computed results are also compared numerically by adopting ND-Solve method. The effect of physical parameters involved in the solutions is discussed through graphs. It is observed that the velocity profiles for both lifting and drainage problems, increases with increasing the non-Newtonian parameter. It is concluded that the velocity profile decreases when magnetic parameter increases. It is also noticed that gravitational parameter reduces the velocity of the lifting problem and increases in case of drainage problem. It is observed that the velocity of the fluid increases with increasing the slip parameter. The temperature profiles of lifting and drainage problems in both primary and secondary layers increase with the increasing values of Brinkman numbers. At the end, for the accuracy of the present work, a comparison is done with the published single-layer thin film flow and outstanding agreement is found.

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