Research Article

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Hybrid censoring samples in assessment the lifetime performance index of Chen distributed products

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Abstract: The performance and potential of a process of industrial products is assessed under lower specification limit L by lifetime performance index (C_L). In this paper, under consideration the independent lifetimes Chen products with known one shape parameters the C_L of the performance of a process is evaluated. For the hybrid censoring scheme the maximum likelihood (ML) estimate of C_L is constructed as well as confidence interval for C_L is developing. Also, Bayesian approach is adopted to estimates C_L and credibly interval is constructed. Some theoretical results of hypothesis tests of C_L is adopted. Finally, our obtaining results will be assessed and compared through Monte Carlo simulation study and numerical example.

Keywords: Chen distribution; Process capability indices; Hybrid progressive censoring; Bayesian estimation; Hypothesis tests

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1 Introduction

The problem of determining the quality of the industrial products before they are put on the market requires some tests and calculation of some statistical measures. One of statistical measures of process capability analysis is called process capability index C_L which can be used to describe the potential capability and process performance. The C_L in industry service is utilized to assess whether the prod-

uct quality meets the required level. For the applications of C_L on electronic components see [1]. The minimum variance unbiased estimator of C_L and confidence interval of exponential lifetime products are derived by [2]. Also, in normal lifetime case or exponential, one can see [3]. The applications of C_L of Burr XII, Rayleigh and Weibull models discussed respectively by [4, 5] and [6].

In lifetime experiments, censoring is defined when the exact lifetimes of experimental units are known for some units but not all units. The common ones in lifetime testing are called type-I and typ-II censoring schemes. Firstly, in type-I censoring scheme the life testing experiment stop at prescribed time τ and the failure numbers occur randomly. In type-II censoring scheme, life testing experiment stop at prescribed number m of failures and the experiment time is random. The last two types of censoring schemes do not allow the experimental units to be removed at time other than the final point of the experiment. The general scheme which allows experimental units to be removed through the experiment is called progressive type-I or type-II censoring see [7]. The hybrid censoring scheme is defined as a joint of type-I and typ-II censoring schemes, for more details of hybrid censoring scheme see [8] and [9]. In type-II progressive hybrid censoring scheme, prior to the experiment, the censoring scheme $\mathbf{R} = (R_1, R_2, \dots, R_m)$ and the total life testing experiment time τ are given. Type-II progressive hybrid censoring scheme is described as follows.

Suppose $T_1, T_2, \dots, T_n$ are n continuous identically distributed failure times of n independent units put in a life testing experiment. Let m, τ and $\mathbf{R} = \{R_1, R_2, \dots, R_m\}$ be previously fixed. At the first failure $T_{1;m,n}$ is recorded, the surviving units with size R_1 are randomly removed from the experiment. When the second failure $T_{2;m,n}$ is recorded, then randomly R_2 removed from the experiment. The experiment continues until one of the two time $(T_{m;m,n}, \tau)$ is observed and the reaming units are randomly removed from the experiment. The conventional of type-II progressive hybrid censoring scheme is more similar to type-I censoring. The ordered observed failure times are denoted by $(T_{1;m,n}, T_{2;m,n}, \dots, T_{r;m,n})$ and called type-

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Figure 1: Schematic description of data under type-II progressive hybrid censoring scheme ($r < m$)

Figure 2: Schematic description of data under type-II progressive hybrid censoring scheme ($r = m$)

II progressive hybrid censored order statistics with size r where $r = m$ at $T_{m;m,n} < \tau$ and $r < m$ at $T_{m;m,n} > \tau$. Figures 1 and 2 show the schematic description of type-II progressive hybrid censoring. It is clearly that at $\tau \rightarrow \infty$ this scheme reduces to the ordinary type-II progressive censoring scheme

For given type-II progressive hybrid censoring data $\underline{T} = (T_{1;m,n}, T_{2;m,n}, \dots, T_{r;m,n})$ the likelihood function of PDF $f(t)$ and CDF $F(t)$ is defined by

$$L(\theta|\underline{t}) = \frac{n!(S(\tau))^{\delta_j \left(\sum_{i=r+1}^m (R_i+1) \right)}}{(n-r)!} \prod_{i=1}^r f(t_i) (S(t_i))^{R_i}, \quad (1)$$

where θ is the vector of parameters, r is the number of failures occurred before τ ,

$$\delta_j = \begin{cases} 0, & \text{if } r = m \\ 1, & \text{if } r < m, \end{cases}, \quad (2)$$

and

$$t_{1;m,n} < t_{2;m,n} < \dots < t_{r;m,n}.$$

Chen Distribution has a bathtub shaped or increasing failure rate function which is more useful in analysis of reliability. This type of distributions is discussed in different papers, for example, [10–13] and [14]. Recently, estimating the parameters of Chen distribution under the assumption

that data are type-I progressive hybrid censored [15]. In this paper, we discuss the Chen distribution with two parameters as the bathtub shape or increasing failure rate function presented by [16].

The random variable T is said to be Chen random variable with two shape and scale parameters a and b , respectively if it has probability density function (PDF) given by

$$f(t) = abt^{a-1} \exp [t^a + b(1 - \exp(t^a))], \quad (3)$$

$$t > 0, a, b > 0,$$

and cumulative distribution function (CDF)

$$F(t) = 1 - \exp [b(1 - \exp(t^a))]. \quad (4)$$

Also, the reliability and hazard rate functions respectively are given by

$$S(t) = \exp [b(1 - \exp(t^a))], \quad (5)$$

$$h(t) = abx^{a-1} \exp(x^a). \quad (6)$$

The Chen distribution with different values of shape parameter a has the following properties

1. For $a < 1$, Chen distribution has bathtub-shaped failure rate function.
2. For $a \geq 1$, Chen distribution has increasing failure rate function.
3. For $a = 1$, Chen distribution becomes exponential power distribution.

The optimal parameters estimation of this distributions are discussed by [17] under doubly type-II censored scheme.

Statistical inference of C_L for the Chen distribution with known shape parameter has been considered under type-II progressive hybrid censoring scheme. The concept of max p -value method discussed in [6] is adopted to select the optimum value of the shape parameter. Then, we constructed the ML estimator for C_L and developed a testing procedure for the lifetime performance index of the products with Chen distribution. Also, Bayesian approach is developed of estimation procedure of C_L on the basis of the type-II progressive hybrid censored sample with max p -value method.

This article is organized in successive sections as follows, the C_L and its properties introduced in Section 2. The ML and Bayes estimators of C_L are discussed in Section 3. Testing hypothesis of C_L in Section 4 are developed over a lower bound for the C_L . In Section 5 numerical example is used to illustrate our purpose. In Section 6 Monte Carlo method is conducted to assess our results. Finally, simple comments are used to describe the numerical results in concluding remarks in Section 7.

Table 1: The C_L values and the corresponding conforming rate P_r for $a = 0.38$

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.0000	0.10	0.40657	0.45	0.5770	0.80	0.8187
-10	0.0000	0.15	0.4274	0.50	0.6065	0.85	0.8607
-5	0.0025	0.20	0.4493	0.55	0.6376	0.90	0.9048
-2	0.0498	0.25	0.4724	0.60	0.6703	0.95	0.9512
-1	0.1353	0.30	0.4966	0.65	0.7047	0.97	0.9705
-0.05	0.3499	0.35	0.5221	0.70	0.7408	0.99	0.9901
0	0.36788	0.40	0.5488	0.75	0.7788	1.00	1.0000

2 The lifetime performance index

The quality of products show that the longer lifetime product is a larger-the-better type quality characteristic. The lifetime performance of products can be described with lifetime performance index C_L . Then random lifetime T is used to denote the lifetime of Chen product with PDF given by (3). Generally, for products to satisfy customers and be economically profitable the lifetime is required to exceed L unit times. The larger-the-better quality characteristic measured by capability index C_L [1]. Under the process mean μ , standard deviation σ and lower specification limit L the C_L is defined by

$$C_L = \frac{\mu - L}{\sigma}. \quad (7)$$

The assessment problem of the C_L can be defined as the lifetime performance index.

Let T be the lifetime random variable with Chen distribution (3) the random variable Z defined by $Z = \exp(T^a) - 1$, $a > 0$ is distributed with exponential form with PDF given by

$$f(z) = b \exp(-bz). \quad (8)$$

This transformation present some important properties, as follows

1. The random variable Z has CDF and $h(z)$ described respectively by

$$F(z) = 1 - \exp(-bz), \quad z > 0, \quad (9)$$

and

$$h(z) = b, \quad b > 0. \quad (10)$$

2. The transform lifetime performance index C_L of Chen distribution with process mean $\mu = E(z) = \frac{1}{b}$ and process standard deviation $\sigma = \frac{1}{b}$ is given by

$$C_L = \frac{\mu - L}{\sigma} = [1 - bL], \quad -\infty < C_L < 1, \quad (11)$$

where L is the lower specification limit.

Transformation $z = \exp(t^a) - 1$, $a > 0$ present one-to-one and strictly increasing of t to z , so data set of t transformed to data set z and have the same effect in assessing the C_L . Also, this transformation enables easy calculation, the $C_L > 0$ for the value of mean $\frac{1}{b} > L$. The equations (10) and (11) show that, the smaller the value of b the smaller the failure rate and the larger the C_L , and the larger the value of b the larger the failure rate and the smaller the C_L . Hence, the C_L is used as accurately description to the lifetime performance of a new product. If the lifetime of a product exceeds the lower specification limit (i.e. $Z \geq L$), the product is labeled as a conforming product. Otherwise, the product is labeled as a nonconforming product. The conforming rate (P_r) is calculated as the ratio of conforming products that defined by

$$P_r = P_r(z \geq L) = \exp(-bL) = \exp(C_L - 1), \quad (12)$$

$$-\infty < C_L < 1.$$

Hence, we obtained the strictly increasing relationship between P_r and C_L , for known $b > 0$. For example, if $a = 0.38$, some different values C_L and the corresponding value of P_r are summarized in Table 1. Other values of P_r can be calculated by Eq. (12). Since P_r and C_L have an one-to-one relationship, therefore, the lifetime performance index can be a flexible and effective tool, in evaluating product quality and estimating the conforming rate P_r .

3 Bayesian and Non-Bayesian estimations of C_L

3.1 Maximum likelihood approach in estimation of C_L

Let $\underline{T} = (T_{1;m,n}, T_{2;m,n}, \dots, T_{r;m,n})$ be the type-II progressive hybrid censored sample from Chen lifetime distribution with the PDF and CDF given by (3) and (4). For the prior censoring scheme $\mathbf{R} = \{R_1, R_2, \dots, R_m\}$ and pre-

scribed time of the experiment τ the joint likelihood function without normalized constant is given by

$$L(b|a, \underline{t}) = (ab)^r \left(\prod_{i=1}^r t_i^{a-1} \right) \exp \left\{ \sum_{i=1}^r \{t_i^a + b(R_i + 1)(1 - \exp(t_i^a))\} + b\delta_j \right. \\ \left. \times (1 - \exp(\tau^a)) \sum_{i=r+1}^m (R_i + 1) \right\}, \quad (13)$$

where δ_j are given by (2) and

$$0 < t_1 < t_2 < \dots < t_r < \infty.$$

So, the log-likelihood function of (13) is given by

$$\ell(b|a, \underline{t}) = r \log(ab) + \sum_{i=1}^r \left\{ (a-1) \log t_i + t_i^a \right. \\ \left. + b(R_i + 1)(1 - \exp(t_i^a)) \right\} + b\delta_j (1 - \exp(\tau^a)) \sum_{i=r+1}^m (R_i + 1). \quad (14)$$

For simplicity the value of observed value t_i is written instead of $t_{i;m,n}$. After calculating the derivatives of (14) for b and equating to zero, we get the ML estimator of b as

$$\hat{b}(a) = \frac{r}{\eta + A}, \quad (15)$$

where

$$\begin{cases} \eta = \sum_{i=1}^r (R_i + 1) (\exp(t_i^a) - 1) \\ A = \delta_j (\exp(\tau^a) - 1) \sum_{i=r+1}^m (R_i + 1). \end{cases} \quad (16)$$

By using the invariance property of ML estimators see [18], the ML estimator of C_L is given by

$$\hat{C}_L = 1 - \hat{b}L = 1 - \frac{r}{\eta + A}L. \quad (17)$$

Theorem 1: Let $T_1, T_2, \dots, T_r$ be an type-II progressive hybrid censored order statistic from two-parameter Chen distribution (3) and $z = \exp(t^a) - 1$, with censored scheme $\mathbf{R}$, then

$$2b\eta \sim \chi_{1-\alpha(2r)}^2, \quad (18)$$

where η given by (16)

Proof: let $Y_i = bZ_i, i = 1, \dots, r$. It can be seen that $Y_1 < Y_2 < \dots < Y_r$ is an type-II progressive hybrid censored order statistic from standard exponential distribution. Consider the following transformation

$$\begin{cases} \zeta_1 = nY_1, \\ \zeta_2 = (n - R_1 - 1)(Y_2 - Y_1), \\ \zeta_3 = (n - R_1 - R_2 - 2)(Y_3 - Y_2), \\ \vdots \\ \zeta_r = (n - R_1 - R_2 - \dots - R_{r-1} - r + 1)(Y_r - Y_{r-1}). \end{cases} \quad (19)$$

The independent and identically generalized spacings $\zeta_1, \zeta_2, \dots, \zeta_r$ are distributed as standard exponential distribution, see [19]. Hence,

$$2 \sum_{i=1}^r \zeta_i = 2 \sum_{i=1}^r (R_i + 1)Y_i = 2b \sum_{i=1}^r (R_i + 1)Z_i = 2b\eta, \quad (20)$$

has a chi-squared distribution with $2r$ degrees of freedom.

Remark 1: The expectation of Equation (17) of $\hat{C}_L$ presented by

$$E(\hat{C}_L) = 1 - E\left(\frac{r}{\eta + A}\right)L. \quad (21)$$

Hence $\hat{C}_L$ is not unbiased estimator of C_L . But as $r \rightarrow \infty$, $E(\hat{C}_L) \rightarrow C_L$, so $\hat{C}_L$ is asymptotic unbiased estimator, also $\hat{C}_L$ is consistent.

3.2 Bayesian approach in estimation of C_L

Consider the gamma prior distribution of the parameter b given by

$$h^*(b) \propto b^{\varphi-1} \exp(-\psi b), \quad (22)$$

where the parameters φ and ψ are obtained from our previous experience. Then the posterior distribution can be obtained from Eqs (13) and (22) as follows

$$h(b|\underline{t}) \propto b^{r+\varphi-1} \exp \left\{ -b \left\{ \sum_{i=1}^r \{ (R_i + 1) (\exp(t_i^a) - 1) \} \right. \right. \\ \left. \left. + \psi + \delta_j (\exp(\tau^a) - 1) \sum_{i=r+1}^m (R_i + 1) \right\} \right\}. \quad (23)$$

By considering a squared-error loss function, $\phi(b, \hat{b}) = (\hat{b} - b)^2$, hence, the posterior mean is considered as the Bayes estimator of b given by

$$\hat{b}_B = \frac{r + \varphi}{\eta^*}, \quad (24)$$

where

$$\eta^* = \sum_{i=1}^r (R_i + 1) (\exp(t_i^a) - 1) + B, \quad (25)$$

and

$$B = \psi + \delta_j (\exp(\tau^a) - 1) \sum_{i=r+1}^m (R_i + 1). \quad (26)$$

Hence, the Bayes estimator $\tilde{C}_L$ of C_L can be written by using (11) and (24) as

$$\tilde{C}_L = E(C_L | h(b|\underline{t})) = 1 - \frac{r + \varphi}{\eta^*}L. \quad (27)$$

Theorem 2: Let $T_1, T_2, \dots, T_r$ be an type-II progressive hybrid censored order statistic from two-parameter Chen distribution (3) and $Z = \exp(T^a) - 1$, with censored scheme $\mathbf{R}$, then

$$Y = 2\lambda\eta^* \sim \chi_{1-\alpha(2(r+\varphi))}^2, \quad (28)$$

where η^* given by (25).

Proof: Let $Y=2b\eta^*$ by using the change of variables see [20], then we obtain that the PDF of Y is given by

$$\begin{aligned} f_Y(y) &= h(2b\eta^* | \frac{y}{2b}) ||I_Y|| \\ &= \frac{y^{\frac{2(r+\varphi)}{2}-1}}{2^{\frac{2(r+\varphi)}{2}} \Gamma\left(\frac{2(r+\varphi)}{2}\right)} \exp\left(-\frac{y}{2}\right), \end{aligned} \quad (29)$$

hence, $2b\eta^* \sim \chi_{2(r+\varphi)}^2$.

Remark 2: The expectation of $\tilde{C}_L$ can be derived as follows

$$E(\tilde{C}_L) = 1 - 2(r+\varphi)bLE\left(\frac{1}{2b\eta^*}\right) = 1 - \frac{(r+\varphi)\lambda L}{(r+\varphi)-1}. \quad (30)$$

Hence, estimator $\tilde{C}_L$ is not an unbiased estimator of C_L . When $r \rightarrow \infty$, $E(\tilde{C}_L) \rightarrow C_L$, so Bayes estimator $\tilde{C}_L$ is asymptotic unbiased estimator, also $\tilde{C}_L$ is consistent.

4 Confidence interval for C_L

In this section, we construct $100(1-\alpha)\%$ lower bound for C_L with ML and Bayes methods to decide hypothesis testing procedure for determining that the lifetime performance index of products meets the predetermined level. For this purpose assume that target value c which the lifetime performance index is larger than it. Then, the null hypothesis $H_0: C_L < c$ is constructed against the alternative hypothesis $H_1: C_L > c$. The two methods is described as follows

1. In the classical approach, from Theorem (1) and the ML estimate $\hat{C}_L$ (17) the lower $(1-\alpha)$ percentile of $\chi_{1-\alpha(2r)}^2$ is given by

$$\begin{aligned} 1-\alpha &= P\left(2b\eta \leq \chi_{1-\alpha(2r)}^2\right) = P\left(b \leq \frac{\chi_{1-\alpha(2r)}^2}{2\eta}\right) \\ &= P\left(\frac{1-C_L}{L} \leq \frac{\chi_{1-\alpha(2r)}^2}{2\eta}\right) \\ &= P\left(C_L \geq 1 + \frac{(1-\hat{C}_L)L\chi_{1-\alpha(2r)}^2}{2[A(1-\hat{C}_L)-rL]}\right). \end{aligned} \quad (31)$$

For known α the $100(1-\alpha)\%$ lower bound for C_L is

$$L_{BML} = 1 + \frac{(\hat{C}_L - 1)L\chi_{1-\alpha(2r)}^2}{2[A(1-\hat{C}_L) + rL]}. \quad (32)$$

2. In the Bayesian approach, under consideration known α and $2\lambda\eta^* \sim \chi_{1-\alpha(2(r+\varphi))}^2$ the $100(1-\alpha)\%$ one-sided credible interval for C_L given by

$$\begin{aligned} 1-\alpha &= P\left(2b\eta^* \leq \chi_{1-\alpha(2(r+\varphi))}^2\right) \\ &= P\left(b \leq \frac{\chi_{1-\alpha(2(r+\varphi))}^2}{2\eta^*}\right) \\ &= P\left(\frac{1-C_L}{L} \leq \frac{\chi_{1-\alpha(2(r+\varphi))}^2}{2\eta^*}\right) \\ &= P\left(C_L \geq 1 + \frac{(\tilde{C}_L - 1)\chi_{1-\alpha(2(r+\varphi))}^2}{2(r+\varphi)}\right), \end{aligned} \quad (33)$$

then, the lower bound for C_L is given by

$$L_{BB} = 1 + \frac{(\tilde{C}_L - 1)\chi_{1-\alpha(2r)}^2}{2(r+\varphi)}, \quad (34)$$

where φ is a prior parameters, r is the number of failure and $\tilde{C}_L$ denotes the Bayes estimates of C_L .

In the following, we employ the one-sided hypothesis testing to test if the lifetime performance index reduces to the required level. The testing steps can be described as follows

- Step 1.** Gini statistic is employed to estimate the shape Chen parameter a as given in [21] and [22], as follows the Gini statistic is defined by

$$G_r = \frac{\sum_{i=1}^{r-1} iD_{i+1}}{(r-1)\sum_{i=1}^r D_i}, \quad (35)$$

where $D_i = (r-1)(Y_{i;m,n} - Y_{i-1;m,n})$ for $i = 2, 3, \dots, r$ and $D_1 = rY_{1;m,n}$ while $T_{i;m,n} = \exp(T^a) - 1$. For $r > 20$, $12\sqrt{r-1}(G_r - 0.5)$ tends to the standard normal distribution $N(0, 1)$. Hence, the P -value = $P(|z| > |12\sqrt{r-1}(g_r - 0.5)|)$, where g_r is the observed value of G_m and Z has an approximation of $N(0, 1)$. So, by using the maximum P -value method, the optimum value of a is selected and then we suppose a is known.

- Step 2.** From the observed original data set $(t_1, t_2, \dots, t_n)$ present $(y_{1;m,n}, y_{2;m,n}, \dots, y_{m;m,n})$ with transform given by (8).

- Step 3.** The index value c and the corresponding lower lifetime limit L is determine, then the null hypothesis $H_0: C_L < c$ against alternative $H_1: C_L > c$ is constructed.

Table 2: A real data set originally reported in Hand *et al.* [12]

0.035	0.068	0.100	0.101	0.167	0.168	0.197	0.213	0.233	0.234
0.508	0.508	0.533	0.633	0.767	0.768	0.770	1.066	1.267	1.300
1.600	1.639	1.803	1.867	2.180	2.667	2.967	3.328	3.393	3.700
3.803	4.311	4.867	5.180	6.233	6.367	6.600	6.600	7.180	7.667
7.733	7.800	7.933	7.967	8.016	8.300	8.410	8.607	8.667	8.800
9.100	9.233	10.541	10.607	10.633	10.667	10.869	11.067	11.180	11.443
12.213	12.508	12.533	13.467	13.800	14.267	14.475	14.500	15.213	15.333
15.525	15.533	15.541	15.934	16.200	16.300	16.344	16.600	16.700	16.933
17.033	17.067	17.475	17.667	17.700	17.967	18.115	18.115	18.933	18.934
19.508	19.574	19.733	20.148	20.180	20.900	21.167	21.233	21.600	22.100
22.148	22.180	22.180	22.267	22.300	22.500	22.533	22.867	23.738	24.082,
24.180	24.705	25.213	25.705	29.705	30.443	31.667	31.934	32.180	32.367
32.672	32.705	33.148	33.567	33.770	33.869	34.836	34.869	34.934	35.738
36.180	36.213	39.410	39.433	39.672	40.001	41.733	41.734	42.311	42.869
43.180	43.279	43.902	44.267	44.475	44.900	45.148	46.451		

Table 3: The random generated type-II progressive hybrid censored order statistic

t_i	0.035	0.068	0.100	0.101	0.167	0.168	0.197	0.213	0.233	0.234
	0.508	0.508	0.633	0.768	1.267	1.600	1.639	1.867	3.328	3.393
	3.803	4.311	4.867	6.233	6.600	7.180	8.016	8.300	8.667	9.100
z_i	0.323	0.433	0.517	0.520	0.660	0.662	0.715	0.743	0.777	0.779
	1.166	1.166	1.318	1.471	1.987	2.305	2.342	2.553	3.851	3.908
	4.266	4.711	5.200	6.422	6.756	7.290	8.074	8.345	8.698	9.119

Table 4: Numerical values of P -values for Real data given in Table 3

0.255	0.0088	0.330	0.3226	0.405	0.6318	0.480	0.0725
0.260	0.0123	0.335	0.3754	0.410	0.5676	0.485	0.0603
0.265	0.0168	0.340	0.4329	0.415	0.5071	0.490	0.0499
0.270	0.0226	0.345	0.4947	0.42	0.4506	0.495	0.0412
0.275	0.0302	0.350	0.5604	0.425	0.3983	0.500	0.0338
0.280	0.0397	0.355	0.6297	0.430	0.3501	0.505	0.0276
0.285	0.0515	0.360	0.7018	0.435	0.3061	0.510	0.0225
0.290	0.0661	0.365	0.7761	0.440	0.2662	0.515	0.0182
0.295	0.0838	0.370	0.8520	0.445	0.2303	0.520	0.0147
0.300	0.1050	0.375	0.9288	0.450	0.1982	0.525	0.0118
0.305	0.1300	0.380	0.9943	0.455	0.1697	0.530	0.0095
0.310	0.1592	0.385	0.9180	0.460	0.1446	0.535	0.0076
0.315	0.1929	0.390	0.8430	0.465	0.1225	0.540	0.0060
0.320	0.2313	0.395	0.7699	0.470	0.1034	0.545	0.0048
0.325	0.2745	0.400	0.6994	0.475	0.0868	0.550	0.0038

Step 4. Two values $\hat{C}_L$ and $\tilde{C}_L$ are calculated.**Step 5.** The two lower bound L_{BML} and L_{BB} for C_L are calculated.**Step 6.** Specify a significance level α .**Step 7.** Statistical test measured as, if $c \notin [L_{BB}, \infty)$ or $c \notin [L_{BML}, \infty)$, the null hypothesis is rejected and say that lifetime performance index of product meets the required level.

Table 5: AV(Er) of empirical confidence level $(1 - \alpha)$ for C_L when $\alpha = 0.05$ and $(a, b) = (1.2, 1.2)$

τ	n	m	C.S.	MLE	Bayes		
					$\varphi = 1, \psi = 1$	$\varphi = 2, \psi = 3$	$\varphi = 3, \psi = 2$
0.4	30	15	$(15, 0^{14})$	0.93(0.0010)	0.92(0.0007)	0.97(0.0008)	0.94(0.0009)
			$(0^{14}, 15)$	0.95(0.0015)	0.933(0.008)	0.955(0.0008)	0.95(0.0009)
			$(0^7, 15, 0^7)$	0.95(0.0013)	0.95(0.0007)	0.94(0.0009)	0.95(0.0009)
			(1^{15})	0.95(0.0012)	0.94(0.0007)	0.94(0.0009)	0.95(0.0010)
0.7	30	15	$(15, 0^{14})$	0.95(0.0009)	0.94(0.0006)	0.97(0.0006)	0.95(0.0008)
			$(0^{14}, 15)$	0.94(0.0020)	0.93(0.0008)	0.96(0.0006)	0.95(0.0007)
			$(0^7, 15, 0^7)$	0.95(0.0010)	0.96(0.0007)	0.96(0.0006)	0.95(0.0006)
			(1^{15})	0.93(0.0011)	0.95(0.0006)	0.96(0.0007)	0.94(0.0007)
0.4	30	25	$(5, 0^{24})$	0.94(0.0009)	0.95(0.0005)	0.95(0.0006)	0.95(0.0006)
			$(0^{24}, 5)$	0.94(0.0010)	0.94(0.0006)	0.94(0.0007)	0.95(0.0008)
			$(0^{12}, 5, 0^{12})$	0.93(0.0011)	0.94(0.0006)	0.96(0.0007)	0.93(0.0007)
			$((1, 0^4)^5)$	0.93(0.0011)	0.95(0.0007)	0.96(0.0008)	0.96(0.0007)
0.7	30	25	$(5, 0^{24})$	0.95(0.0008)	0.95(0.0004)	0.94(0.0006)	0.94(0.0006)
			$(0^{24}, 5)$	0.94(0.0009)	0.95(0.0005)	0.96(0.0006)	0.94(0.0007)
			$(0^{12}, 5, 0^{12})$	0.95(0.0010)	0.97(0.0006)	0.95(0.0006)	0.94(0.0006)
			$((1, 0^4)^5)$	0.96(0.0010)	0.96(0.0005)	0.95(0.0007)	0.95(0.0006)
0.4	50	25	$(25, 0^{24})$	0.95(0.0007)	0.96(0.0004)	0.95(0.0005)	0.95(0.0005)
			$(0^{24}, 25)$	0.94(0.0007)	0.94(0.0005)	0.95(0.0006)	0.95(0.0006)
			$(0^{12}, 25, 0^{12})$	0.94(0.0007)	0.95(0.0005)	0.94(0.0005)	0.95(0.0006)
			(1^{25})	0.96(0.0007)	0.95(0.0004)	0.96(0.0005)	0.94(0.0007)
0.7	50	25	$(25, 0^{24})$	0.94(0.0006)	0.95(0.0004)	0.94(0.0005)	0.95(0.0005)
			$(0^{24}, 25)$	0.94(0.0006)	0.94(0.0004)	0.94(0.0005)	0.94(0.0005)
			$(0^{12}, 25, 0^{12})$	0.93(0.0007)	0.94(0.0004)	0.94(0.0006)	0.95(0.0006)
			(1^{25})	0.96(0.0007)	0.95(0.0004)	0.95(0.0005)	0.96(0.0006)
0.4	50	40	$(10, 0^{39})$	0.94(0.0005)	0.95(0.0003)	0.95(0.0004)	0.95(0.0004)
			$(0^{39}, 10)$	0.94(0.0006)	0.95(0.0005)	0.96(0.0004)	0.94(0.0004)
			$(0^{19}, 10, 0^{20})$	0.95(0.0006)	0.95(0.0004)	0.95(0.0004)	0.95(0.0005)
			$((1, 0^3)^{10})$	0.95(0.0005)	0.95(0.0004)	0.96(0.0004)	0.95(0.0004)
0.7	50	40	$(10, 0^{39})$	0.95(0.0004)	0.95(0.0002)	0.95(0.0003)	0.95(0.0004)
			$(0^{39}, 10)$	0.94(0.0005)	0.96(0.0004)	0.94(0.0004)	0.94(0.0004)
			$(0^{19}, 10, 0^{20})$	0.94(0.0005)	0.94(0.0004)	0.94(0.0005)	0.94(0.0004)
			$((1, 0^3)^{10})$	0.95(0.0005)	0.95(0.0005)	0.96(0.0004)	0.95(0.0004)

5 Illustrative examples

In the real life problem, we consider a real data presented in [23]. The data is listed in Table 3 to represent graft survival times in months of 148 renal transplant patients. This data is fitted with Chen distribution see [15]. The observed type-II progressive hybrid censored order statistic is generated from the data in Table 2, with censoring scheme $\mathbf{R} = \{5^{(5)}, 0^{(5)}, 5^{(5)}, 0^{(5)}, 4^{(5)}, 0^{(5)}, 4^{(5)}, 0^{(3)}, 1^{(10)}\}$, $n = 148$, $m = 48$ and $\tau = 10$ this data are summarized in Table 3. The estimate value of shape parameter α of Chen distribution is calculated by using the concept of Gini statistic

with maximum P -value method as given in Table 4. The optimum values a that maximize P -value as given in Table 4 is $\alpha = 0.38$ and P -value = 0.9943 hence the parameter a is known. The observed exponential type-II progressive hybrid censored order statistics obtained with transformation $z_{i,m,n} = \exp(t_i^{0.38}) - 1$ are reported also in Table 4.

The lower lifetime limit and transformed limit L_t and L_y , respectively given by 0.314 and $\exp(0.314^{0.38}) - 1 = 0.90393$. For consideration of product managers' the P_r of operational performances is required to exceed 80%. With results in Table 1, the C_L value operational performances are required to exceed 0.80. Hence, the target value of performance index is taken as $c = 0.80$. The testing hypoth-

Table 6: AV(Er) of empirical confidence level $(1 - \alpha)$ for C_L when $\alpha = 0.05$ (a, b) = (1.7, 0.2)

τ	n	m	C.S.	MLE	Bayes		
					$\varphi = 1, \psi = 1$	$\varphi = 2, \psi = 3$	$\varphi = 3, \psi = 2$
0.4	30	15	(15, 0 ¹⁴)	0.91(0.0025)	0.93(0.0014)	0.92(0.0017)	0.96(0.0018)
			(0 ¹⁴ , 15)	0.90(0.0030)	0.95(0.0022)	0.94(0.0023)	0.92(0.0019)
			(0 ⁷ , 15, 0 ⁷)	0.92(0.0027)	0.94(0.0020)	0.97(0.0020)	0.94(0.0021)
			(1 ¹⁵)	0.91(0.0025)	0.92(0.0019)	0.93(0.0018)	0.96(0.0020)
0.7	30	15	(15, 0 ¹⁴)	0.92(0.0023)	0.94(0.0012)	0.95(0.0014)	0.92(0.0013)
			(0 ¹⁴ , 15)	0.93(0.0032)	0.93(0.0019)	0.95(0.0018)	0.94(0.0020)
			(0 ⁷ , 15, 0 ⁷)	0.94(0.0028)	0.93(0.0017)	0.96(0.0015)	0.94(0.0014)
			(1 ¹⁵)	0.92(0.0024)	0.94(0.0016)	0.97(0.0017)	0.96(0.0017)
0.4	30	25	(5, 0 ²⁴)	0.92(0.0018)	0.94(0.0011)	0.95(0.0012)	0.95(0.0015)
			(0 ²⁴ , 5)	0.93(0.0019)	0.95(0.0013)	0.95(0.0015)	0.94(0.0015)
			(0 ¹² , 5, 0 ¹²)	0.92(0.0020)	0.94(0.0015)	0.97(0.0017)	0.94(0.0018)
			((1, 0 ⁴) ⁵)	0.93(0.0019)	0.94(0.0014)	0.96(0.0018)	0.95(0.0019)
0.7	30	25	(5, 0 ²⁴)	0.93(0.0015)	0.94(0.0012)	0.95(0.0014)	0.92(0.0013)
			(0 ²⁴ , 5)	0.92(0.0017)	0.93(0.0013)	0.95(0.0018)	0.94(0.0020)
			(0 ¹² , 5, 0 ¹²)	0.95(0.0016)	0.93(0.0012)	0.96(0.0015)	0.94(0.0014)
			((1, 0 ⁴) ⁵)	0.92(0.0016)	0.94(0.0013)	0.97(0.0014)	0.96(0.0013)
0.4	50	25	(25, 0 ²⁴)	0.93(0.0013)	0.94(0.0010)	0.95(0.0010)	0.95(0.0009)
			(0 ²⁴ , 25)	0.93(0.0018)	0.95(0.0012)	0.95(0.0012)	0.94(0.0011)
			(0 ¹² , 25, 0 ¹²)	0.94(0.0014)	0.93(0.0013)	0.95(0.0011)	0.94(0.0010)
			(1 ²⁵)	0.94(0.0015)	0.94(0.0012)	0.94(0.0011)	0.95(0.0010)
0.7	50	25	(25, 0 ²⁴)	0.94(0.0012)	0.94(0.0007)	0.94(0.0007)	0.93(0.0008)
			(0 ²⁴ , 25)	0.92(0.0014)	0.95(0.0009)	0.95(0.0009)	0.94(0.0009)
			(0 ¹² , 25, 0 ¹²)	0.94(0.0010)	0.95(0.0008)	0.94(0.0009)	0.95(0.0007)
			(1 ²⁵)	0.93(0.0010)	0.94(0.0009)	0.93(0.0009)	0.95(0.0010)
0.4	50	40	(10, 0 ³⁹)	0.94(0.0008)	0.94(0.0005)	0.94(0.0004)	0.96(0.0006)
			(0 ³⁹ , 10)	0.93(0.0009)	0.95(0.0009)	0.95(0.0006)	0.94(0.0007)
			(0 ¹⁹ , 10, 0 ²⁰)	0.95(0.0008)	0.94(0.0008)	0.96(0.0006)	0.95(0.0007)
			((1, 0 ³) ¹⁰)	0.93(0.0007)	0.94(0.0009)	0.93(0.0007)	0.96(0.0007)
0.7	50	40	(10, 0 ³⁹)	0.95(0.0006)	0.94(0.0004)	0.94(0.0004)	0.96(0.0004)
			(0 ³⁹ , 10)	0.94(0.0007)	0.95(0.0005)	0.95(0.0005)	0.94(0.0005)
			(0 ¹⁹ , 10, 0 ²⁰)	0.93(0.0006)	0.94(0.0006)	0.96(0.0004)	0.95(0.0004)
			((1, 0 ³) ¹⁰)	0.93(0.0007)	0.94(0.0005)	0.93(0.0005)	0.96(0.0004)

esis $H_0: C_L < 0.80$ against the alternative $H_1: C_L > 0.80$ is constructed.

The Bayes estimates which do not have any prior informations small values are selected to the gamma hyper parameters to reflect vague prior informations. So, φ and ψ are assumed to be the same value 0.0001. Hence the results in the Bayesian and non-Bayesian are conforming. Calculate the 95% one-sided confidence and credible intervals of $[L_{BML}, \infty)$ and $[L_{BB}, \infty)$ from (32) and (34), respectively.

1. ML approach the 95% lower bound for C_L is obtained as $[L_{BML}, \infty) = [0.9675, \infty)$, since the target

value $c = 0.80 \notin [0.9675, \infty)$, so the null hypothesis $H_0: C_L \leq 0.8$ is rejected.

2. In Bayesian approach the 95% lower bound for C_L is obtained as $[L_{BB}, \infty) = [0.9513, \infty)$, since the target value $c = 0.80 \notin [0.9513, \infty)$, so the null hypothesis $H_0: C_L \leq 0.8$ is rejected.

Then, in the two approaches we can say the lifetime performance index of the products does meet the required level.

6 Simulation study

Over the one sided confidence or credible intervals of C_L the simulation study is constructed to study the confidence level $(1 - \alpha)$ of the lifetime performance index C_L . In our studying consider confidence level $\alpha = 0.05$ and for the Chen parameters are chosen to be two sets $(a, b) = \{(1.2, 1.2), (1.7, 0.2)\}$. This study is discussed for different size (n, m) , different priors (φ, ψ) and different sampling schemes. The lower lifetime limit L_x respected to model parameters is assumed to be 0.2 and 0.4, respectively. This problem is constructed under the following steps

1. The type-II progressive hybrid censored order statistic $(t_1, t_2, \dots, t_r)$ is generated from Chen distribution as proposed in [15]. The observed exponential data $(z_1, z_2, \dots, z_r)$ is obtained by transformation $z = \exp(t^a) - 1$.
2. From Eqs (32) and (34) 95% lower bounds L_{BML} and L_{BB} are calculated.
3. Put $\Omega_1 = 1$ or $\Omega_2 = 1$ If $C_L > L_{BML}$ or $C_L > L_{BB}$, respectively elsewhere $\Omega_1 = 0$ or $\Omega_2 = 0$.
4. For each data generation from Steps 2 and 3 are repeated to 100 times.
5. The total count $\frac{\text{TotalCount } \Omega_1}{100}$ or $\frac{\text{TotalCount } \Omega_2}{100}$ are used as the estimate of confidence level $(1 - \hat{\alpha})$ respectively.
6. The last steps are repeated 1000 times, we get $(1 - \hat{\alpha})^{(1)}, (1 - \hat{\alpha})^{(2)}, \dots, (1 - \hat{\alpha})^{(1000)}$.
7. The average empirical confidence level is calculated by

$$AV(1 - \hat{\alpha}) = \frac{1}{1000} \sum_{i=1}^{1000} (1 - \hat{\alpha})^{(i)},$$

with error Er (mean square error) given by

$$Er(1 - \hat{\alpha}) = \frac{1}{1000} \sum_{i=1}^{1000} \left[(1 - \hat{\alpha})^{(i)} - (1 - \alpha) \right]^2.$$

The results of simulation study are reported in Tables 5 and 6.

7 Conclusions and recommendations

The problem of determine the quality of the industrial products requires some tests on the lifetime of the products. The calculation of some statistical measures which are used to prove whether the product quality meets the re-

quired level. Capability index C_L which can be used in measuring process performance and potential capability. This paper, consider this problem when the lifetime of products follows Chen distribution and the experiment running under general hybrid censoring scheme. Then, we constructed the ML and Bayes stimators for C_L and developed a testing procedure for the lifetime performance index of the products with Chen distribution. We reported some comments observed from numerical results as follows.

1. Form Tables 5 and 6 all empirical results are close to confidence level $(1 - \alpha)$.
2. The value of Er is smaller over all of the average empirical confidence levels.
3. The results in Bayesian approach are smaller than the ML approach through Er for one-sided credible and confidence intervals.
4. The results are improved in terms of AV and (Er) when τ is increasing, this means that when $\tau \rightarrow \infty$.

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