Research Article

Maysaa Mohamed Al Qurashi*

Conserved vectors with conformable derivative for certain systems of partial differential equations with physical applications

<https://doi.org/10.1515/phys-2020-0127>

received March 24, 2020; accepted April 07, 2020

Abstract: In this paper, we examine conservation laws (Cls) with conformable derivative for certain nonlinear partial differential equations (PDEs). The new conservation theorem is used to the construction of nonlocal Cls for the governing systems of equation. It is worth noting that this paper introduces for the first time, to our knowledge, the analysis for Cls to systems of PDEs with a conformable derivative.

Keywords: systems of partial differential equations, Lie symmetry, conformable derivative, conservation laws

1 Introduction

Fractional calculus is an emerging field of mathematics having important real-world applications in various branches of science and engineering [1–10]. This fascinating field had many useful transformations during more than 300 years, and it appears within various representations, namely, several fractional operators were developed and successfully applied to solve complicated dynamical systems [1–10]. In [11], some researchers tried to introduce the conformable derivative to use the classical Leibniz rule and composition rule as in the classical case but they discovered a new local derivative containing a parameter. Within the literature, many authors used this conformable derivative in some useful applications [12–16].

We recall that conservation laws (Cls) stemmed from the pragmatic phenomena of energy, mass, and momentum [17]. The Cls were used to develop numerical methods, to demonstrate the nature and uniqueness of solutions [18], and to examine internal characteristics such as recurrence operators and bi-Hamiltonian

structures [19]. It should be noted that various generalizations of the Noether's theorem and Euler–Lagrange [20] along with various fractional derivatives to define Cls for fractional nonlinear partial differential equations (PDEs) with fractional Lagrangians [21–23] have been studied. The literature included several studies for Cls to various equations using fractional derivatives from the sense of Riemann–Liouville. In this research, we aim to present a technique using symmetries from Lie points to create Cls for nonlinear PDEs with physical applications with conformable derivatives using new conservation theorem.

This paper has been organized as follows: in Section 2, some basic properties of the conformable derivative are provided; Section 3 deals with the conformable symmetry analysis; Section 4 is focussing on the conformable dispersive long-wave system; the conformable Whitham–Broer–Kaup system is scrutinized in Section 5; and conclusions are presented in Section 6.

2 Basic properties of the conformable derivative

Conformable derivative (CD) is an extended classical derivative that was proposed in [11, 24–26]. This derivative has overcome the barriers with other derivatives. It is described as

Suppose $f: (0, \infty) \rightarrow \mathbb{R}$, then CD of f with order α is given by [11]

$$T_{\alpha}[f(t)] = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad (1)$$

for $t > 0$, $\alpha \in (0, 1)$. If $T_{\alpha}[f(t)]$ exists for t in some interval $(0, a)$ with $a > 0$, and $\lim_{t \rightarrow 0^+} T_{\alpha}[f(t)]$ also exists, then $T_{\alpha}[f(0)] = \lim_{t \rightarrow 0^+} T_{\alpha}[f(t)]$. Moreover, if $T_{\alpha}[f(t)]$ exists on $[0, \infty)$, then f is said to be α -differentiable at t .

The following properties are associated with the conformable derivative [11]:

* Corresponding author: Maysaa Mohamed Al Qurashi, Department of Mathematics, King Saud University, P.O.Box 22452, Riyadh, 11495, Saudi Arabia, e-mail: maysaa@ksu.edu.sa

- $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, $a, b \in \mathbb{R}$,
- $T_\alpha(t^\mu) = \mu t^{\mu-\alpha}$, $\mu \in \mathbb{R}$,
- $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$,
- $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$,
- If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dg}$.

The integral associated with the conformable derivative can be written as [11] $I^\alpha[f(t)] = I[t^{\alpha-1}f(t)] = \int_0^t \frac{f(\tau)}{\tau^{1-\alpha}} d\tau$, where the integral depicts the standard Riemann improper integral such that $\alpha \in (0, 1]$.

Besides, we can prove easily that [11]

$T^\alpha I^\alpha[f(t)] = f(t)$, for $t \geq 0$, where f depicts any function that is continuous within the domain of I^α . For every $t > 0$, we get $I^\alpha T^\alpha[f(t)] = f(t) - f(0)$. Also, we have $T_\alpha(fg)(t) = (f'(g(t)))(t)T_\alpha g(t)$ [11].

3 Symmetry analysis

Consider the conformable system of PDEs [27]:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= F(x, t, u, u_x^{2\alpha}, u_x^{3\alpha}, \dots), \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= F(x, t, v, v_x^{2\alpha}, v_x^{3\alpha}, \dots), \end{aligned} \quad (2)$$

where $u = u(x, t)$, $F(x, t, u, u_{xx}, u_{xxx}, \dots)$ depicts a nonlinear function and $\frac{\partial^\alpha u}{\partial t^\alpha}$ and $\frac{\partial^\alpha v}{\partial t^\alpha}$ depict the conformable fractional derivative. Given a one-parameter Lie group of infinitesimal transformations of the form

$$\begin{aligned} \bar{t} &= t + \varepsilon \xi^1(t, x, u, v) + O(\varepsilon^2), \\ \bar{x} &= x + \varepsilon \xi^2(t, x, u, v) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon \eta_1(t, x, u, v) + O(\varepsilon^2), \\ \bar{v} &= v + \varepsilon \eta_2(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \eta_1^{a,t}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{v}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha v}{\partial t^\alpha} + \varepsilon \eta_2^{a,t}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{x}^\alpha} &= \frac{\partial u}{\partial x} + \varepsilon \eta_1^{a,x}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^\alpha \bar{v}}{\partial \bar{x}^\alpha} &= \frac{\partial v}{\partial x} + \varepsilon \eta_2^{a,x}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^{2\alpha} \bar{u}}{\partial \bar{x}^{2\alpha}} &= \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \varepsilon \eta_1^{a,xx}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^{2\alpha} \bar{v}}{\partial \bar{x}^{2\alpha}} &= \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}} + \varepsilon \eta_2^{a,xx}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^{3\alpha} \bar{u}}{\partial \bar{x}^{3\alpha}} &= \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} + \varepsilon \eta_1^{a,xxx}(t, x, u, v) + O(\varepsilon^2), \\ \frac{\partial^{3\alpha} \bar{v}}{\partial \bar{x}^{3\alpha}} &= \frac{\partial^{3\alpha} v}{\partial x^{3\alpha}} + \varepsilon \eta_2^{a,xxx}(t, x, u, v) + O(\varepsilon^2), \\ &\dots \end{aligned} \quad (3)$$

where

$$\begin{aligned} \eta_1^{a,t} &= t^{1-\alpha} \eta_1^t + (1-\alpha) \xi^1 t^{-\alpha} u_t, \\ \eta_2^{a,t} &= t^{1-\alpha} \eta_2^t + (1-\alpha) \xi^1 t^{-\alpha} v_t, \\ \eta_1^{a,x} &= x^{1-\alpha} \eta_1^x + (1-\alpha) \xi^2 x^{-\alpha} u_x, \\ \eta_2^{a,x} &= x^{1-\alpha} \eta_2^x + (1-\alpha) \xi^2 x^{-\alpha} v_x, \\ \eta_1^{a,xx} &= x^{2-2\alpha} \eta_1^{xx} + (1-\alpha) x^{1-2\alpha} \eta_1^x \\ &\quad + (2-2\alpha) x^{1-2\alpha} \xi^2 u_{xx} \\ &\quad + (1-\alpha)(1-2\alpha) x^{-2\alpha} \xi^2 u_x, \\ \eta_2^{a,xx} &= x^{2-2\alpha} \eta_2^{xx} + (1-\alpha) x^{1-2\alpha} \eta_2^x \\ &\quad + (2-2\alpha) x^{1-2\alpha} \xi^2 v_{xx} \\ &\quad + (1-\alpha)(1-2\alpha) x^{-2\alpha} \xi^2 v_x, \\ \eta_1^{a,xxx} &= x^{3-3\alpha} \eta_1^{xxx} + (3-3\alpha) x^{2-3\alpha} \eta_1^{xx} \\ &\quad + (1-\alpha)(1-2\alpha) x^{1-3\alpha} \eta_1^x \\ &\quad + (3-3\alpha) \xi^2 x^{2-3\alpha} u_{xxx} \\ &\quad + (3-3\alpha)(2-3\alpha) \xi^2 x^{1-3\alpha} u_{xx} \\ &\quad + (1-\alpha)(1-2\alpha)(1-3\alpha) \xi^2 x^{-3\alpha} u_x, \\ \eta_2^{a,xxx} &= x^{3-3\alpha} \eta_2^{xxx} + (3-3\alpha) x^{2-3\alpha} \eta_2^{xx} \\ &\quad + (1-\alpha)(1-2\alpha) x^{1-3\alpha} \eta_2^x \\ &\quad + (3-3\alpha) \xi^2 x^{2-3\alpha} v_{xxx} \\ &\quad + (3-3\alpha)(2-3\alpha) \xi^2 x^{1-3\alpha} v_{xx} \\ &\quad + (1-\alpha)(1-2\alpha)(1-3\alpha) \xi^2 x^{-3\alpha} v_x, \dots \end{aligned} \quad (4)$$

and

$$\begin{aligned} \eta_1^t &= D_t(\eta_1) - u_x D_t(\xi^2) - u_t D_t(\xi^1), \\ \eta_2^t &= D_t(\eta_2) - v_x D_t(\xi^2) - v_t D_t(\xi^1), \\ \eta_1^x &= D_x(\eta_1) - u_x D_x(\xi^2) - u_t D_x(\xi^1), \\ \eta_2^x &= D_x(\eta_2) - v_x D_x(\xi^2) - v_t D_x(\xi^1), \\ \eta_1^{xx} &= D_t(\eta_1^{xx}) - u_{xx} D_t(\xi^2) - u_{xt} D_t(\xi^1), \\ \eta_2^{xx} &= D_t(\eta_2^{xx}) - v_{xx} D_t(\xi^2) - v_{xt} D_t(\xi^1), \\ \eta_1^{xxx} &= D_t(\eta_1^{xxx}) - u_{xxx} D_t(\xi^2) - u_{xxt} D_t(\xi^1), \\ \eta_2^{xxx} &= D_t(\eta_2^{xxx}) - v_{xxx} D_t(\xi^2) - v_{xxt} D_t(\xi^1), \end{aligned} \quad (5)$$

The total differential operator D_x and D_t are defined by

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_x} + \dots + v_t \frac{\partial}{\partial v} \\ &\quad + v_{tt} \frac{\partial}{\partial v_t} + v_{xt} \frac{\partial}{\partial v_x} \dots \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \dots + v_x \frac{\partial}{\partial v} \\ &\quad + v_{xx} \frac{\partial}{\partial v_x} + v_{xt} \frac{\partial}{\partial v_t} \dots \end{aligned} \quad (6)$$

The associated Lie algebra of symmetries consists of a set of vector fields of the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta_1 \frac{\partial}{\partial u} + \eta_2 \frac{\partial}{\partial v}. \quad (7)$$

The vector field equation (7) is a Lie point symmetry of equation (2) provided that

$$P^{a,i} X(\Delta)|_{\Delta=0} = 0, \quad (8)$$

where P is the prolongation operator, Δ is the symbolize form of equation (2) and i is the order of the system in equation (2). Also, the invariance condition yields

$$\xi^1(x, t, u, v)|_{t=0} = 0. \quad (9)$$

4 The dispersive long-wave system

The conformable dispersive long-wave system is given by

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} &= \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right)^2 - \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + 2 \frac{\partial^\alpha v}{\partial x^\alpha}, \\ \frac{\partial^\alpha v}{\partial t^\alpha} &= 2u \frac{\partial^\alpha v}{\partial x^\alpha} + 2v \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^{2\alpha} v}{\partial x^{2\alpha}}, \end{aligned} \quad (10)$$

where $0 < \alpha \leq 1$ and α describes the order of the conformable derivative. If $\alpha = 1$, equation (10) reduces to the classical dispersive long-wave system which is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= (u^2 - u_x + 2v)_x, \\ \frac{\partial v}{\partial t} &= (2uv + v_x)_x. \end{aligned} \quad (11)$$

Equation (11) is the dispersive long-wave equation [28]. In hydrodynamics, it describes the evolution of the horizontal velocity portion of water waves, which propagates in both directions in an infinite narrow channel of constant depth.

4.1 Symmetry analysis for equation (10)

In accordance with the invariance of equation (10) via equation (3), imposing equation (8) into equation (10), we have that

$$\begin{aligned} \eta_1^{\alpha,t} - 2u\eta_1^{\alpha,x} + 2\eta_2^{\alpha,x} + \eta_1^{\alpha,xx} &= 0, \\ \eta_1^{\alpha,t} - 2u\eta_1^{\alpha,x} + 2\eta_2^{\alpha,x} + \eta_2^{\alpha,xx} &= 0, \end{aligned} \quad (12)$$

which must hold whenever equation (10) holds. It is of great importance to note that, using the properties of conformable derivative, we get the following equivalent form of equation (10).

$$\begin{aligned} t^{1-\alpha} u_t - (x^{1-\alpha} u_x)^2 + (1-\alpha)x^{1-2\alpha} u_x + x^{2-2\alpha} u_{xx} \\ - 2x^{1-\alpha} v_x = 0, \\ t^{1-\alpha} v_t - 2x^{1-\alpha} u v_x - 2x^{1-\alpha} v u_x - (1-\alpha)x^{1-2\alpha} v_x \\ - x^{2-2\alpha} v_{xx} = 0. \end{aligned} \quad (13)$$

Plugging equations (4) and (5) into equation (12) and using equation (13) instead of u_{xx} and v_{xx} where ever they appear and equating the coefficients of the various monomials in partial derivatives of u and v , the determining equations is obtained. Solving the obtained determining equations, we acquire

$$\xi^1 = -\frac{2t^\alpha c_1}{\alpha} + c_2, \xi^2 = t^{1-\alpha} c_3, \eta_1 = c_1, \eta_2 = 0, \quad (14)$$

and the vector field is obtained as follows:

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = -\frac{2t^\alpha}{\alpha} \frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \quad (15)$$

4.2 Nonlocal conservation laws (CLs) for equation (10)

The new conservation theorem [29] will be used for building the non-local CLs of equation (10). The conservation equation

$$D_i(T^i) = 0, \quad (16)$$

needs to be satisfied by the obtained conserved vectors. Where

$$\begin{aligned} T^i &= \xi^i \mathcal{L} + \mathcal{W}^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \dots \right] \\ &+ D_j (\mathcal{W}^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \dots \right] \\ &+ D_j D_k (\mathcal{W}^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right] + \dots \end{aligned} \quad (17)$$

where $\mathcal{W}^\alpha$ is the characteristics function given by $\mathcal{W}^\alpha = \eta - \xi^i u_i$.

The nonlocal CLs for the governing equation will now be presented. We begin with the Lagrangian of equation (10) as

$$\begin{aligned} \mathcal{L} &= p(t^{1-\alpha} u_t - (x^{1-\alpha} u_x)^2 + (1-\alpha)x^{1-2\alpha} u_x + x^{2-2\alpha} u_{xx} \\ &- 2x^{1-\alpha} v_x) + q(t^{1-\alpha} v_t - 2x^{1-\alpha} u v_x - 2x^{1-\alpha} v u_x \\ &- (1-\alpha)x^{1-2\alpha} v_x - x^{2-2\alpha} v_{xx}) \end{aligned} \quad (18)$$

The adjoint equations can be presented as follows:

$$\begin{aligned} \left\{ \begin{aligned} \frac{\delta \mathcal{L}}{\delta u} &\equiv t^{-\alpha} x^{-2\alpha} (-2t^{\alpha} x^{\alpha} (-1 + \alpha) qv - tx^{2\alpha} p_t \\ &\quad + 2t^{\alpha} x^2 p_t u_t + p((-1 + \alpha) \\ &\quad (x^{2\alpha} + t^{\alpha} (-1 + 2\alpha)) + 2t^{\alpha} x^2 u_{tt}) + 3t^{\alpha} x p_x \\ &\quad - 3t^{\alpha} x a p_x + 2t^{\alpha} x^{1+\alpha} v q_x + t^{\alpha} x^2 p_{xx}) = 0, \\ \frac{\delta \mathcal{L}}{\delta v} &\equiv t^{-\alpha} x^{-2\alpha} (-2t^{\alpha} x^{\alpha} (-1 + \alpha) p \\ &\quad - (-1 + \alpha) q (-x^{2\alpha} + t^{\alpha} (-1 + 2\alpha) + 2t^{\alpha} x^{\alpha} u) \\ &\quad - tx^{2\alpha} q_t + 2t^{\alpha} x^{1+\alpha} p_x - 3t^{\alpha} x q_x + 3t^{\alpha} x a q_x \\ &\quad + 2t^{\alpha} x^{1+\alpha} u q_x - t^{\alpha} x^2 q_{xx}) = 0 \}. \end{aligned} \right. \quad (19) \end{aligned}$$

Now, with the help of the obtained point symmetries equation (15), we use the Noether operator $\mathcal{N}$ [29–30] to obtain conserved vectors, (T^1, T^2) as follows:

1. For the symmetry $X_1 = \frac{\partial}{\partial x}$, we obtain

$$\begin{aligned} T_1^t &= t^{-\alpha} x^{-2\alpha} (p(tx^{2\alpha} u_t - t^{\alpha} x^2 u_t^2 - 2t^{\alpha} x(-1 + \alpha) u_x) \\ &\quad + q(tx^{2\alpha} v_t + 2t^{\alpha} x(-1 + \alpha) v_x) \\ &\quad + t^{\alpha} x^2 (p_x u_x - q_x v_x)), \\ T_1^x &= p(-t^{1-\alpha} + 2x^{2-2\alpha} u_t) u_x - t^{1-\alpha} q(x, t) v_x \end{aligned} \quad (20)$$

2. For the symmetry $X_2 = \frac{\partial}{\partial t}$, we obtain

$$\begin{aligned} T_2^t &= t^{1-\alpha} x^{1-2\alpha} (x(u_t p_x - v_t q_x) + p(-(-1 + \alpha) u_t \\ &\quad + 2x^{\alpha} v_t - x u_{xt}) + q(2x^{\alpha} v(-1 + \alpha) u_t \\ &\quad + (-1 + \alpha + 2x^{\alpha} u(x, t)) v_t + x v_{xt})), \\ T_2^x &= t^{1-\alpha} x^{1-2\alpha} (p(x u_t^2 - (-1 + \alpha) u_x - 2x^{\alpha} v_x + x u_{xx}) \\ &\quad - q(2x^{\alpha} v u_x + (1 - \alpha + 2x^{\alpha} u) v_x + x v_{xx})) \end{aligned} \quad (21)$$

3. For the symmetry $X_3 = -\frac{2t^{\alpha}}{\alpha} \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$, we obtain

$$\begin{aligned} T_3^t &= \frac{1}{\alpha} (x^{-2\alpha} (p(-2tx^{2\alpha} u_t + 2t^{\alpha} x^2 u_t^2 \\ &\quad + x(-1 + \alpha)(\alpha + 4t^{\alpha} u_x)) \\ &\quad - 2q(x^{1+\alpha} a v + tx^{2\alpha} v_t + 2t^{\alpha} x(-1 + \alpha) v_x) \\ &\quad - x^2 (p_x(\alpha + 2t^{\alpha} u_x) - 2t^{\alpha} q_x v_x))), \\ T_3^x &= p(t^{1-\alpha} - 2x^{2-2\alpha} u_t) \left(1 + \frac{2t^{\alpha} u_x}{\alpha}\right) + \frac{2t q v_x}{\alpha} \end{aligned} \quad (22)$$

5 The Whitham–Broer–Kaup system

The conformable Whitham–Broer–Kaup Wilson system is given by

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} + u \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \frac{\partial^{\alpha} v}{\partial x^{\alpha}} + \mu \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} &= 0, \\ \frac{\partial^{\alpha} v}{\partial t^{\alpha}} + u \frac{\partial^{\alpha} v}{\partial x^{\alpha}} + v \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \beta \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} - \mu \frac{\partial^{2\alpha} v}{\partial t^{2\alpha}} &= 0 \end{aligned} \quad (23)$$

If $\alpha = 1$, equation (23) becomes the classical Whitham–Broer–Kaup system given by

$$\begin{aligned} \frac{\partial u}{\partial t} + uu_x + v_x + \mu u_{xx} &= 0, \\ \frac{\partial v}{\partial t} + (uv)_x + \beta u_{xxx} - \mu v_{xx} &= 0. \end{aligned} \quad (24)$$

Equation (24) is a completely integrable model that describes a dispersive long wave in shallow water. In equation (23), β and μ are real constants that represent different dispersive powers.

5.1 Symmetry analysis for equation (28)

Considering the invariance of equation (23) under the group of transformations equation (3), applying equation (8) into equation (23), we obtain

$$\begin{aligned} \eta_1^{\alpha, t} + u \eta_1^x + \eta_2^x + \mu \eta_1^{xx} &= 0, \\ \eta_2^{\alpha, t} + v \eta_1^x + u \eta_2^x - \mu \eta_2^{xx} + \beta \eta_1^{xxx} &= 0, \end{aligned} \quad (25)$$

which must hold whenever equation (23) holds. It is of great importance to note that, using the properties of conformable derivative, we get the following equivalent form of equation (23) as

$$\begin{aligned} t^{1-\alpha} u_t + x^{1-\alpha} u u_x + x^{1-\alpha} v_x + \mu (1 - \alpha) x^{1-2\alpha} u_x \\ + u x^{2-2\alpha} u_{xx} &= 0, \\ t^{1-\alpha} v_t + x^{1-\alpha} u v_x x^{1-\alpha} v u_x - \mu (1 - \alpha) x^{1-2\alpha} u_x \\ - \mu x^{2-2\alpha} u_{xx} \\ + \lambda (1 - \alpha) (1 - 2\alpha) x^{1-3\alpha} u_x \\ + 3\lambda (1 - \alpha) x^{2-3\alpha} u_{xx} + \lambda x^{3-3\alpha} u_{xxx} &= 0. \end{aligned} \quad (26)$$

Plugging equations (4) and (5) into equation (23) and using equation (26) and equating the coefficients of the various monomials in partial derivatives of u and v , the determining equations are obtained. Solving the obtained determining equations, we acquire

$$\xi^1 = \frac{t^{\alpha} c_1}{\alpha} + c_2, \xi^2 = t^{1-\alpha} c_3, \eta_1 = c_1, \eta_2 = 0, \quad (27)$$

and the associated Lie algebra is generated by the vector field as follows:

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{t^{\alpha}}{\alpha} \frac{\partial}{\partial x} + \frac{\partial}{\partial u}. \quad (28)$$

5.2 Nonlocal Cls for equation (28)

Now we present the nonlocal Cls for the governing equation. We now begin with the Lagrangian of equation (23) as

$$\begin{aligned}\mathcal{L} = & p(t^{1-\alpha}u_t + x^{1-\alpha}uu_x + x^{1-\alpha}v_x \\ & + \mu(1-\alpha)x^{1-2\alpha}u_x + ux^{2-2\alpha}u_{xx}) \\ & + q(t^{1-\alpha}v_t + x^{1-\alpha}uv_x x^{1-\alpha}vu_x \\ & - \mu(1-\alpha)x^{1-2\alpha}u_x - \mu x^{2-2\alpha}u_{xx} \\ & + \lambda(1-\alpha)(1-2\alpha)x^{1-3\alpha}u_x \\ & + 3\lambda(1-\alpha)x^{2-3\alpha}u_{xx} + \lambda x^{3-3\alpha}u_{xxx}).\end{aligned}\quad (29)$$

The adjoint equations can be presented as follows:

$$\begin{aligned}\left\{\frac{\delta \mathcal{L}}{\delta u}\right\} \equiv & t^{-\alpha}x^{-3\beta}(x^\beta p(x^{2\beta}(-1+\alpha) \\ & + t^\alpha(1-3\beta+2\beta^2)\mu + t^\alpha x^\beta(-1+\beta)u) \\ & + t^\alpha(-1+\beta)q((-1+2\beta)((-1+3\beta)\lambda - x^\beta\mu) \\ & + x^{2\beta}v) - tx^{3\beta}p_t + 3t^\alpha x^{1+\beta}\mu p_x \\ & - 3t^\alpha x^{1+\beta}\beta\mu p_x t^\alpha x^{1+2\beta}u p_x - 7t^\alpha x\lambda q_x \\ & + 18t^\alpha x\beta\lambda q_x - 11t^\alpha x\beta^2\lambda q_x - 3t^\alpha x^{1+\beta}\mu q_x \\ & + 3t^\alpha x^{1+\beta}\beta\mu q_x - t^\alpha x^{1+2\beta}v q_x + t^\alpha x^{2+\beta}\mu p_{xx} \\ & - 6t^\alpha x^2\lambda q_{xx} + 6t^\alpha x^2\beta\lambda q_{xx} \\ & - t^\alpha x^{2+\beta}\mu q_{xx} - t^\alpha x^3\lambda q_{xxx}) = 0, \\ \frac{\delta \mathcal{L}}{\delta v} \equiv & t^{-\alpha}x^{-\beta}(t^\alpha(-1+\beta)p[x, t] \\ & + q(x^\beta(-1+\alpha) + t^\alpha(-1+\beta)u) \\ & - tx^\beta q_t - t^\alpha x p_x - t^\alpha x u q_x) = 0.\end{aligned}\quad (30)$$

Now, with the help of the obtained point symmetries equation (28), we use the Noether operator $\mathcal{N}$ [29] to obtain conserved vectors, (T^1, T^2) .

1. For the symmetry $X_1 = \frac{\partial}{\partial x}$, we get

$$\begin{aligned}T_1^t = & t^{-\alpha}x^{-3\alpha}(x^\alpha p(tx^{2\alpha}u_t - 2t^\alpha x(-1+\alpha)\mu u_x) \\ & + q(tx^{3\alpha}v_t + t^\alpha x(-1+\alpha)(2x^\alpha\mu u_x - 3x\lambda u_{xx})) \\ & + t^\alpha x^2(x^\alpha\mu p_x u_x - x\lambda u_x q_{xx} \\ & + q_x((3(-1+\alpha)\lambda - x^\alpha\mu)u_x + x\lambda u_{xx}))), \\ T_1^x = & -t^{1-\alpha}(p(x, t)u_x + q(x, t)v_x)\end{aligned}\quad (31)$$

2. For the symmetry $X_2 = \frac{\partial}{\partial t}$, we have

$$\begin{aligned}T_2^t = & -t^{1-\alpha}x^{1-3\alpha}(x^\alpha p(((1-\alpha)\mu \\ & + x^\alpha u(x, t))u_t + x^\alpha v_t + x\mu u_{xt}) \\ & + x(-x\lambda q_x u_{xt} + u_t(-x^\alpha\mu p_x \\ & + (3\lambda - 3\beta\lambda + x^\alpha\mu)q_x + x\lambda q_{xx})) \\ & + q(((1-\alpha)(-\lambda + 2\alpha\lambda \\ & - x^\alpha\mu) + x^{2\alpha}v)u_t + x^{2\alpha}uv_t \\ & + x(-x^\alpha\mu u_{xt} + x\lambda u_{xxt}))),\end{aligned}\quad (32)$$

$$\begin{aligned}T_2^x = & t^{1-\alpha}x^{1-3\alpha}(x^\alpha((\mu - \alpha\mu \\ & + x^\alpha u(x, t))u_x + x^\alpha v_x + x\mu u_{xx}) \\ & + q(((1-\alpha)(-\lambda + 2\alpha\lambda \\ & + x^\alpha\mu) + x^{2\alpha}v)u_x + x^{2\alpha}uv_x \\ & + x(-(3(-1+\alpha)\lambda + x^\alpha\mu)_{xx} + x\lambda u_{xxx})))\end{aligned}$$

3. For the symmetry $X_3 = \frac{t^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}$, we acquire

$$\begin{aligned}T_3^t = & \frac{1}{\alpha}x^{-3\alpha}(x^\alpha p(x^{1+\alpha}au(x, t) \\ & + tx^{2\alpha}u_t + x(-1+\alpha)\mu(\alpha - 2t^\alpha u_x)) \\ & + x^2(-x^\alpha\mu p_x(\alpha - t^\alpha u_x) + x\lambda(\alpha - t^\alpha u_x)q_{xx} \\ & + q_x(3\alpha\lambda - 3\alpha^2\lambda + x^\alpha\alpha\mu \\ & + t^\alpha(3(-1+\alpha)\lambda - x^\alpha\mu)u_x + t^\alpha x\lambda u_{xx})) \\ & + q(x^{1+2\alpha}av + tx^{3\alpha}v_t - x(-1+\alpha) \\ & (\alpha(\lambda - 2\alpha\lambda + x^\alpha\mu) - 2t^\alpha x^\alpha\mu u_x + 3t^\alpha x\lambda u_{xx}))), \\ T_3^x = & \frac{t(p(t^{-\alpha}\alpha - u_x) - qv_x)}{\alpha}\end{aligned}\quad (33)$$

6 Conclusion

Finding a suitable operator for a better description of the dynamical complex systems is a big issue nowadays for researchers from various fields of science and engineering. In this paper, we investigated Cls for some nonlinear PDEs with conformable derivative. The new conservation theorem is applied to construct nonlocal Cls for nonlinear PDEs possessing conformable derivative. It is worth noting that, the construction of Cls to nonlinear PDEs with conformable derivative is presented for the first time, to our knowledge, in this paper.

References

- [1] Caputo M, Mainardi F. A new dissipation model based on memory mechanism. *Pure Appl Geophys.* 1971;91:134–47.
- [2] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. *Prog Fract Differ Appl.* 2015;1(2):73–85.
- [3] Atangana A, Baleanu D. New fractional derivatives with nonlocal and non-singular kernel. *Theory application heat Transf model, Therm Sci.* 2016;20(2):763–9.
- [4] Baleanu D, Avkar T. Lagrangians with linear velocities within Riemann-Liouville fractional derivatives. *Nuovo Cim B.* 2004;119(1):73–9.
- [5] Yildiz TA, Jajarmi A, Yildiz B, Baleanu D. New aspects of time fractional optimal control problems within operators with nonsingular kernel. *Discr Cont Dynam Syst S.* 2020;13(3):407–28.

- [6] Baleanu D, Jajarmi A, Mohammadi H, Rezapour S. A new study on the mathematical modelling of human liver with Caputo-Fabrizio fractional derivative. *Chaos Soliton Fract.* 2020;134:109705.
- [7] Arshad S, Deftlerli O, Baleanu D. A second order accurate approximation for fractional derivatives with singular and non-singular kernel applied to a HIV model. *Appl Math Comput.* 2020;374:125061.
- [8] Samko SG, Kilbas AA, Marichev OL. *Fractional integrals and derivatives: theory and applications.* New York: Gordon and Breach; 1993.
- [9] Jajarmi A, Baleanu D, Sajjadi SS, Asad JH. A new feature of the fractional Euler-Lagrange equations for a coupled oscillator using a nonsingular operator approach. *Front Phys* 2019;7:196.
- [10] Baleanu D, Jajarmi A, Sajjadi SS, Mozyrska D. A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator. *Chaos.* 2019;29(8):083127.
- [11] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. *J Comput Appl Math.* 2014;264:65–70.
- [12] Atangana A, Baleanu D, Alsaedi A. New properties of conformable derivative. *Open Math.* 2015;13:889–98.
- [13] Zheng A, Feng Y, Wang W. The Hyers-Ulam stability of the conformable fractional differential equation. *Math Aeterna.* 2015;5:485–92.
- [14] Iyiola OS, Nwaeze ER. Some new results on the new conformable fractional calculus with application sing D'Alambert approach. *Progr Frac Differ Appl.* 2016;2:115–22.
- [15] Michal P, Skripkov LP. Sturm's theorems for conformable fractional differential equations. *Math Commun.* 2016;21:273–281.
- [16] Fuat U, Mehmet ZS. Explicit bounds on certain integral inequalities via conformable fractional calculus. *Cogent Math.* 2017;4:1277505.
- [17] Bluman GW, Cheviakov AF, Anco SC. *Applications of symmetry methods to partial differential equations.* New York: Springer; 2010.
- [18] Leveque RJ. *Numerical methods for conservation laws. Lectures in mathematics.* ETH Zurich: Birkhauser Verlag; 1992.
- [19] Naz R. Conservation laws for some systems of nonlinear partial differential equations via multiplier approach. *J Appl Math.* 2012;2012:1–13.
- [20] Noether E. Invariante variations probleme. *Nachr Kn Gesell Wissen, Gttingen, Math-Phys Kl, Heft.* 1918;2:235–57, English translation in *Transp Theor. Stat. Phys.* 1971;1:186–207.
- [21] Agrawal OP. Formulation of Euler-Lagrange equations for fractional variational problems. *J Math Appl.* 2002;272:368–79.
- [22] Frederico GST, Delfim T. A formulation of Noether's theorem for fractional problems of the calculus of variations. *J Math Anal.* 2007;334:834–46.
- [23] Baleanu D. About fractional quantization and fractional variational principles. *Commun Nonlin Sci Numer Simul.* 2009;14:2520–3.
- [24] Abdeljawad T. On conformable fractional calculus. *J Comput Appl Math.* 2015;279:57–66.
- [25] Chung WS. Fractional Newton mechanics with conformable fractional derivative. *J Comput Appl Math.* 2015;290:150–8.
- [26] Cenesiz Y, Kurt A. The new solution of time fractional wave equation with conformable fractional derivative definition. *J N Theor.* 2015;7:79–85.
- [27] Tayyan BA, Sakka AH. Lie symmetry analysis of some conformable fractional partial differential equations. *Arab J Math.* 2020;9:201–12.
- [28] Kaup DJ. A higher-order water-wave equation and the method for solving it. *Prog Theor Phys* 1975;54:396–408.
- [29] Ibragimov NH. A new conservation theorem. *J Math Anal Appl.* 2007;333:311–28.
- [30] Yomba E. The Extended Fan Sub-Equation Method and its Application to the (2+1)-Dimensional Dispersive Long Wave and Whitham-Broer-Kaup Equations. *Chin J Phys.* 2005;43(4):789–805.