Research Article

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Floquet analysis of linear dynamic RLC circuits

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Abstract: In this article, the linear dynamic analysis of AC generators modeled as RLC circuits with periodically time-varying inductances via Floquet's theory is considered. Necessary conditions for the dynamic stability are derived. The harmonic balance method is employed to predict the transition curves and stability domains. An approximate expression for the Floquet form of solution is constructed using Whittaker's method in the neighborhood of transition curves. Numerical verifications for the obtained theoretical results are considered. In accordance with the experimental results, a satisfactory agreement is relatively achieved with the closed experimental literature of the problem.

Keywords: RLC circuits, linear differential equations, Floquet's theory, stability theory, periodic solutions

1 Introduction

In recent decades, there have been extensive studies to examine qualitative properties in problems of mathematical physics governed by linear differential equations that include the effects of periodic parametric excitations, time delays and fractional derivatives [2,3,31,33,39]. Specifically, this part of differential equations and its different variants of linear dynamical systems have mostly appeared in the description of numerous physical problems such as in the field of electrical circuits and small oscillation systems [5,8].

Modern analysis of such linear (nonlinear) dynamical systems becomes the main nerve of technological

machine development to improve the productivity as well as the efficiency of the mechanical and electrical equipment. One of these basic equipment, used in electrical power engineering, is AC generators in power generation systems [1,17,24]. The main principle in the theory generators is the change of mechanical energy into electrical energy depending upon the principle of magnetic induction [14].

The mathematical type of periodically time-varying problems was first studied by Mathieu [23] and followed up by extended applications for Mathieu's equation [12,40–42]. Consequently, Floquet [11] developed an interesting theory for the part of linear differential equations with periodic coefficients obtaining a general form of solution applied in every physical process [4,9,19,35,38].

The study of periodic solutions and stability problems has attracted the attention of many authors in particular for that type and its mathematical variant extensions [15,21,37,42,41]. Ignatyev [18] discussed the asymptotic stability of zero solution of second-order differential equation and Duc et al. [7] proved that the zero solution is also asymptotically stable if one of the hypotheses introduced in the study by Ignatyev [18] does not exist. Onitsuka [30] gave a natural generalization of results in [7,18] by proving that the zero solution is asymptotically stable if one of the hypotheses of [18] is fulfilled.

In general, many theorems and methods have constructed and applied second-order differential equations and their different variants to interpret different varieties of dynamical processes, which definitely possess the periodically parametric excitation nature [13,16,22,32,36].

This work is organized as follows. In section 2, the electrical circuit modeling for the AC generators is introduced and the governing equation of the system is deduced. In section 3, Floquet analysis for the governing equation yielding the stability conditions is presented and the transition curves are obtained by harmonic balance analysis. In section 4, an approximate form of Floquet solution near the transition curves using Whittaker's method and the stability of periodic solutions is constructed. In section 5, the conclusion is given.

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2 Modeling system

In general, AC generators are composed of two main mechanical parts: the rotor is the part that rotates and the stator is the part that remains stationary. The existence of relative movement between the stator and the rotor in AC generators makes inductances being time-varying coefficients [26,27]. Consequently, this system can be modeled as an equivalent RLC-series circuit, as shown in Figure 1. By using the basic principles of electrical circuits, the governing equation reads

$$L_t(\omega t) \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(t) dt = V_s(t), \quad (1)$$

where $L_t(\omega t)$ is the variable inductance, R is the resistance, C is the capacitance, $i(t)$ is the electric current in the circuit, ω is the frequency and $V_s(t)$ is the voltage source.

Electric current ($i(t)$) at a specified point and flowing in a specified direction is the instantaneous rate at which the net positive charge is moving past this point in that specified direction, then equation (1) is finally simplified to

$$L_t(\omega t) \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = V_s. \quad (2)$$

Let us consider the following parameters:

$$\begin{aligned} x &= \omega t, \quad L_t(x) = L_0(1 + h \cos 2x), \quad \omega_0^2 = \frac{1}{L_0 C}, \\ \alpha &= \frac{\omega}{\omega_0}, \quad Q = \frac{R}{\omega_0 L_0}, \end{aligned} \quad (3)$$

where h is a perturbation parameter arising from the relative movement between the rotor and the stator of AC generators. Then, the natural response of the circuit ($y(x)$) reads

$$(1 + h \cos 2x)y'' + \frac{Q}{\alpha}y' + \frac{1}{\alpha^2}y = 0. \quad (4)$$

Figure 1: The equivalent RLC circuit of AC generator.

Hence, the corresponding linear system of equation (4) reads

$$Y' = A(x)Y, \quad (5)$$

where

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\alpha^2(1 + h \cos 2x)} & -\frac{Q}{\alpha(1 + h \cos 2x)} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}. \quad (6)$$

If the system has no damping ($Q = 0$), according to Hamilton–Jacobi’s equations [10,20] and using Charpit’s equations, the following particular solution can be deduced:

$$\int_{y_0}^y \frac{1}{\sqrt{E(x) - \frac{1}{2\alpha^2}y^2}} dy = \int_0^x \frac{\sqrt{2}}{\sqrt{1 + h \cos 2x}} dx, \quad (7)$$

where $E(x)$ is the energy functional of the system. By considering that the total energy of the system (E) is constant, then the solution reads

$$y = \alpha \sqrt{2E} \sin \begin{cases} \frac{1}{\alpha}x, & \text{if } h = 0, \\ \frac{1}{\alpha} \sinh^{-1}(\tan x), & \text{if } h = 1, \\ \frac{a}{\alpha \sqrt{1+h}} \operatorname{sc}^{-1}\left(\tan x \left| \frac{a^2-1}{a^2} \right.\right), & \text{if } h < 1, a^2 > 1, \\ \frac{1}{\alpha \sqrt{1+h}} \frac{1}{\sqrt{1+a^2}} \operatorname{nc}^{-1}\left(\frac{\tan x}{a} \left| \frac{1}{1+a^2} \right.\right), & \text{if } h > 1, \end{cases} \quad (8)$$

where $a^2 = \frac{1+h}{1-h}$, nc and sc are the Jacobian elliptic functions [43].

Typically, this modeling step allows us to discuss the dynamic stability analysis of AC generators, which is considered the key step in understanding the changes in the system. Somehow, the parameters might have a big responsibility to cause a bounded or an unbounded response and the necessity for the design of stabilizing feedback controllers.

In this work, the main aim is centered on the study of existence and stability of periodic solutions in the (α, h) plane for the equivalent circuit represented by equation (4). Consequently, there is a need to obtain the predicted values for the main parameters of AC generators to guarantee stable motions or stable outputs free of resonances.

3 Floquet analysis

In accordance with Floquet’s theory, the general solution of equation (4) admits solutions of period π or 2π and has the form

$$y(x) = c_1 \phi_1(x) e^{\gamma_1 x} + c_2 \phi_2(x) e^{\gamma_2 x}, \quad (9)$$

where γ_i is the characteristic exponent, $\rho_i = e^{\pi\gamma_i}$ is the characteristic multiplier, c_i is a constant and $\phi_i(x + \pi) = \phi_i(x)$ ($i = 1, 2$). Clearly from equation (5), $z(x)$ can be obtained from $z = y'$. The boundedness of the solution $Y = 0$ depends on the values of $x = \pi$ of the 2×2 matrix $W(x)$ that satisfies the system

$$W' = A(x)W, \quad W(0) = I, \quad (10)$$

where I is the 2×2 identity matrix. Specifically, if and only if the distinct eigenvalues of $W(\pi)$ have a modulus less than or equal to unity, and if, for those eigenvalues ρ_i with $|\rho_i| = 1$, the multiplicity μ_i of ρ_i equals nullity ν_i of $W(\pi) - \rho_i I$, then for every $\varepsilon > 0$ there exists an $\varepsilon > 0$ such that the initial condition $\|Y(0)\| < \varepsilon$ guarantees $\|Y\| < \varepsilon$. The stability of the solution $Y = 0$ thus depends on the roots of

$$\det[W(\pi) - \rho I] = 0, \quad (11)$$

or more explicitly,

$$\rho^2 - (W_{11} + W_{22})\rho + (W_{11}W_{22} - W_{12}W_{21}) = 0, \quad (12)$$

where W_{ij} is the element in the i th row and j th column of $W(\pi)$. The constant term in equation (12) can be recognized as $\det[W(\pi)]$ and may be shown to be obtained by the following equation:

$$\det[W(\pi)] = \det[W(0)] \exp\left(\int_0^\pi \operatorname{tr}\{A(x)\} dx\right). \quad (13)$$

Now, $\det[W(0)] = 1$ and

$$\int_0^\pi \operatorname{tr}\{A(x)\} dx = -2\pi k, \quad k = \frac{Q}{2\alpha} \frac{1}{\sqrt{1-h^2}}. \quad (14)$$

Consequently,

$$\det[W(\pi)] = \lambda^2, \quad \lambda = \exp(-\pi k). \quad (15)$$

Since $Q > 0$ and $\alpha > 0$, then $k \geq 0$. Hence, the parameter k measures the dynamic state of the system and it might be considered as a control parameter. Eventually, it might be shown that k has a big influence on the state of the output.

The characteristic multipliers are obtained by

$$\rho_1 \rho_2 = \exp\left(\int_0^\pi \operatorname{tr}\{A(x)\} dx\right) = \lambda^2. \quad (16)$$

Thus, equation (12) can be expressed as

$$\rho^2 - 2\psi\rho + \lambda^2 = 0, \quad (17)$$

where the value of ψ reads

$$\psi = \frac{1}{2}(W_{11}(\pi) + W_{22}(\pi)). \quad (18)$$

Then, the eigenvalues of $W(\pi)$ are given by

$$\rho_1, \rho_2 = \psi \pm \sqrt{\psi^2 - \lambda^2}. \quad (19)$$

Consequently, based on equation (19), there are three cases for considering the stability of the system depending on the value of ψ which are summarized in the following theorem.

Theorem 1. *Stability regions of the governing system (equation (5)) can be determined by the following conditions:*

$$|h| < 1, \quad Q \geq h\alpha, \quad (20)$$

$$-\frac{1}{2}(1 + \lambda^2) < \psi < \frac{1}{2}(1 + \lambda^2), \quad \lambda < \psi < -\lambda. \quad (21)$$

Consequently, the resonance case is guaranteed if

$$1 + \lambda^2 < \psi < -(1 + \lambda^2), \quad |\rho_1| < 1, \quad \text{and} \quad |\rho_2| > 1, \quad (22)$$

and the periodic solutions are predicted if

$$\psi = 1 + \lambda^2, \quad \text{or} \quad \psi = -(1 + \lambda^2). \quad (23)$$

Proof. Hence, steps of the proof depend on the value of ψ as illustrated in the following. $\square$

Case 1: $|\psi| < \lambda$

In this case, the eigenvalues of equation (19) are a pair of complex conjugate values of ρ with $|\rho| = \lambda$, then $\mu = \nu = 1$. Thus, $\rho = \lambda e^{\pm i\theta}$ then $\gamma = -\pi k \pm i\theta$, so that in both cases the solutions are asymptotically stable. In general, the system admits the asymptotic stability if $|\psi| < 1$ since $\Re\{\rho\} < 1$ (where $\Re$ is the real part and $\Im$ is the imaginary part), as shown in Figure 2 by the grey region. This means that $k > 0$ or in general

$$\frac{1}{\sqrt{1-h^2}} > 0, \quad \text{or} \quad |h| < 1. \quad (24)$$

Case 2: $|\psi| = \lambda$

In this case, the eigenvalues of equation (19) are real repeated with $|\rho| = \lambda$ so that $\mu_1 = 2$, $\nu_1 = 1$ and $\mu_1 \neq \nu_1$. If $\rho = \lambda$, then $\gamma = -k$ and if $\rho = -\lambda$, then $\Re\{\gamma\} = -k$, so that in both cases the solution is asymptotic stable, as shown in Figure 2 by the grey region, under the condition:

$$Q \geq \frac{\alpha\sqrt{1-h^2}}{\pi}. \quad (25)$$

For the small values of k where $k \approx 0$ is considered, it reaches

$$Q \geq h\alpha. \quad (26)$$

Case 3: $|\psi| > \lambda$

In this case, the eigenvalues are real and distinct with one root less than λ and the other is greater. According to the sign of ψ , there are distinct real and positive or both distinct real and negative, therefore we have the following:

When they are both positive, this assures that there is a stable periodic solution if

$$\psi < \frac{1}{2}(1 + \lambda^2), \quad |\rho_2| < |\rho_1| < 1, \quad \forall \quad k > 0. \quad (27)$$

When they are both negative, this assures that there is a stable periodic solution if

$$\psi > -\frac{1}{2}(1 + \lambda^2), \quad |\rho_2| < |\rho_1| < 1, \quad \forall \quad k > 0. \quad (28)$$

When they are real and distinct with one root less than unity and the other greater than unity, the instability of the solution $Y = 0$ is assured if

$$1 + \lambda^2 \leq \psi \leq -(1 + \lambda^2), \quad |\rho_1| < 1, \quad \text{and} \quad |\rho_2| > 1, \quad (29)$$

as shown in Figure 2 by the white region.

Case 4: $|\psi| = 1 + \lambda^2$

In this case, the eigenvalues are equal unity. Thus, $|\rho_1| = |\rho_2| = 1$; this is the case of periodic solutions which are represented by red lines in Figure 2.

From these cases, the regions of stability are accurately predicted for the governing system and all of these results are summarized in Figure 2.

3.1 Prediction of transition curves

Many analytical approaches can be employed to figure out the stability domains of linear dynamical systems

Figure 2: Stability and periodic solution states of the governing system.

with periodic coefficients. However, here we consider the harmonic balance method, which relatively seems to be an appropriate approach for the studied system. The transition curves can be predicted for equation (4) with periods π and 2π according to the Floquet analysis, since the minimal period of the parametric excitation is π . If the periodic solution of a system exists, then it is readily searched in the form of Fourier series whose coefficients are determined by requiring the series to satisfy the equation of motion. However, in order to avoid solving an infinite system, we will approximate the periodic solution by finite sums (up to $M < \infty$) of trigonometric functions, i.e.,

$$y(x) = \sum_{m=0}^M (A_m \cos mx + B_m \sin mx), \quad (30)$$

$$m = 0, 1, 2, \dots, M.$$

By substitution into equation (4), it yields

$$\begin{aligned} & \frac{A_0}{\alpha^2} - 4A_2 \frac{h}{2} + \left(A_1 \left(\frac{1}{\alpha^2} - 1 \right) + B_1 \frac{Q}{\alpha} - A_1 \frac{h}{2} - 9A_3 \frac{h}{2} \right) \cos x \\ & + \left(A_2 \left(\frac{1}{\alpha^2} - 4 \right) + B_2 \frac{2Q}{\alpha} - 8A_4 h \right) \cos 2x \\ & + \left(B_1 \left(\frac{1}{\alpha^2} - 1 \right) + A_1 \frac{Q}{\alpha} + B_1 \frac{h}{2} - 9B_3 \frac{h}{2} \right) \sin x \\ & + \left(B_2 \left(\frac{1}{\alpha^2} - 4 \right) - A_2 \frac{2Q}{\alpha} - 8B_4 h \right) \sin 2x \\ & + \sum_{m=3}^M \left(A_m \left(\frac{1}{\alpha^2} - m^2 \right) + B_m \frac{mQ}{\alpha} - A_{m-2} \left(\frac{h}{2} (m-2)^2 \right) \right. \\ & \left. - A_{m+2} \left(\frac{h}{2} (m+2)^2 \right) \right) \cos mx \\ & + \sum_{m=3}^M \left(B_m \left(\frac{1}{\alpha^2} - m^2 \right) - A_m \frac{mQ}{\alpha} - B_{m-2} \left(\frac{h}{2} (m-2)^2 \right) \right. \\ & \left. - B_{m+2} \left(\frac{h}{2} (m+2)^2 \right) \right) \sin mx = 0. \end{aligned} \quad (31)$$

Then after simplifying, we obtain

$$\frac{A_0}{\alpha^2} - 2hA_2 = 0, \quad (32a)$$

$$\left(\frac{1}{\alpha^2} - 1 - \frac{h}{2} \right) A_1 + \frac{Q}{\alpha} B_1 - \frac{9}{2} h A_3 = 0, \quad (32b)$$

$$\left(\frac{1}{\alpha^2} - 1 + \frac{h}{2} \right) B_1 - \frac{Q}{\alpha} A_1 - \frac{9}{2} h B_3 = 0, \quad (32c)$$

$$\left(\frac{1}{\alpha^2} - 4 \right) A_2 - 8hA_4 + \frac{2Q}{\alpha} B_2 = 0, \quad (32d)$$

$$\left(\frac{1}{\alpha^2} - 4 \right) B_2 - \frac{2Q}{\alpha} A_2 - 8hB_4 = 0. \quad (32e)$$

For $3 \leq m \leq M$,

$$\left(\frac{1}{\alpha^2} - m^2\right)A_m + \frac{mQ}{\alpha}B_m - \frac{h}{2}(m-2)^2A_{m-2} - \frac{h}{2}(m+2)^2A_{m+2} = 0, \quad (32f)$$

$$\left(\frac{1}{\alpha^2} - m^2\right)B_m - \frac{mQ}{\alpha}A_m - \frac{h}{2}(m-2)^2B_{m-2} - \frac{h}{2}(m+2)^2B_{m+2} = 0. \quad (32g)$$

In accordance with the harmonic balance procedure, it gives two sets of algebraic equations on the coefficients A_m and B_m . Each set deals with even and odd harmonics giving all sets of harmonic balances of the system.

Thus, the odd and even sets of harmonic balances of equation (32) have non-trivial solution. Then, we will find the functional relationship between α and h by solving the determinants of matrices which are called Hill's determinants [6,25]. Consequently, we will plot them as a set of transition curves in the (α, h) plane.

This is one of the main results of this work, which considers the transition curves of the governing system as the same as the stability of it. This means that there exist critical boundary curves which divide the space (α, h) into regions where the number of unstable characteristic exponents might be constant. Therefore, the change of these numbers along the boundary curves can be determined by the analysis of the exponent-crossing direction for the functional relationship between α and h using equation (32). Hence, it yields two regions separated by such transition curves: stable and unstable.

Figures 3 and 4, drawn at $M = 30$, illustrate the transition curves for each value of Q as a barrier between

Figure 3: Transition curves at different values of Q and $\alpha = \frac{\omega}{\omega_0}$.

Figure 4: Transition curves at different values of Q and $\alpha = 1$.

the stability and instability regions. Practically, in accordance with the Floquet analysis, the region of parameter h lies on $(-1, 1)$ for stable motion coinciding with the machine operation points in that region. As shown in Figure 4, it explains the stable and unstable regions around $\left(\frac{\omega}{\omega_0} = 1\right)$ and the set of transition curves separate the domain for $h \in [0, 1)$ to stability and instability regions according to the value of Q . As it is clearly noted, when Q is increased the region of stability becomes larger until the instability region completely disappears. Clearly indeed, if the fraction $\left(\frac{\omega}{\omega_0}\right)$ is odd the curves show that the periodic solutions have 2π -periods, and if it is even the curves show that the periodic solutions have π -periods.

4 Construction of normal Floquet and periodic solutions

In this section, an approximate form of Floquet solution and the transition curves for the governing equation is constructed by using semi-analytical methods based on the perturbation techniques. Since the necessary condition for asymptotic stability $|h| < 1$ allows us to construct a convergent solution as a function of $y = y(x, h)$, [29,34] the perturbed periodic solutions are constructed as a special case from the Floquet solution along the transition curves. The perturbed general solution based on the Floquet analysis of the governing equation can be constructed by combining Whittaker's method and another perturbed one yielding the solutions in the neighborhood transition curves. Through the used

methods, frequencies that produce secular terms are eliminated to avoid resonances. In order to facilitate the perturbation idea, the coefficient Q is scaled by hQ_0 . Hence, equation (4) reads

$$y'' + \delta^2 y = -h \cos(2x)y'' - hQ_0 \delta y', \quad (33)$$

where

$$\delta = \frac{1}{\alpha} = \frac{\omega_0}{\omega}. \quad (34)$$

4.1 Construction of normal Floquet solutions

Perturbation techniques that are based on a periodic generating solution yield merely an approximate form of periodic solution along the transition curves. But, if it is needed to construct solutions in the neighborhood of transition curves, then we use a different method to look for them based on their normal forms. The method that possesses this property is Whittaker's method. Thus, in accordance with Floquet's theory, the normal form of solution in Whittaker's method reads

$$y(x) = e^{\gamma x} \phi(x), \quad (35)$$

where γ is called the characteristic exponent and $\phi(x)$ is a periodic function such that $\phi(x + \pi) = \phi(x)$.

By substituting equation (35) and its derivatives into equation (33), we obtain

$$\begin{aligned} \phi'' + 2\gamma\phi' + \gamma^2\phi &= -h \cos 2x (\phi'' + 2\gamma\phi' + \gamma^2\phi) \\ &\quad - hQ_0\delta(\phi' + \gamma\phi). \end{aligned} \quad (36)$$

Thus, the problem is transformed into determining γ , the periodic solution of equation (36) for given δ which can be taken as a constant or from another perturbation technique based on a generating periodic solution. The solution will be constructed near the transition curves, where γ is small ($|\gamma| < 1$), using the parameter h as a small parameter ($|h| < 1$). Hence, we can seek the solution using the following perturbed expansions:

$$\phi(x, h) = \phi_0 + h\phi_1 + h^2\phi_2 + h^3\phi_3 + O(h^4), \quad (37)$$

$$\delta = n^2 + h\delta_1 + h^2\delta_2 + h^3\delta_3 + O(h^4), \quad (38)$$

$$\delta^2 = n^4 + h(2n^2\delta_1) + h^2(\delta_1^2 + 2n^2\delta_2) + O(h^3), \quad (39)$$

$$\gamma = h\gamma_1 + h^2\gamma_2 + h^3\gamma_3 + O(h^4), \quad (40)$$

$$\gamma^2 = h^2\gamma_1^2 + 2h^3\gamma_1\gamma_2 + O(h^4). \quad (41)$$

Thus, after substitution into equation (36) and upon equating the coefficients of each power of h , it yields

$$\phi''_0 + n^4\phi_0 = 0, \quad (42)$$

$$\begin{aligned} \phi''_1 + n^4\phi_1 &= -(2n^2\delta_1\phi_0 + (2\gamma_1 + Q_0n^2)\phi'_0 \\ &\quad + \phi''_0 \cos 2x), \end{aligned} \quad (43)$$

$$\begin{aligned} \phi''_2 + n^4\phi_2 &= -(A_1\phi_0 + (A_2 + 2\gamma_1 \cos 2x)\phi'_0 \\ &\quad + (2n^2\delta_1\phi_1 + A_3)\phi'_1 + \phi''_1 \cos 2x), \end{aligned} \quad (44)$$

$$\begin{aligned} \phi''_3 + n^4\phi_3 &= -((A_4 + \gamma_1^2 \cos 2x)\phi_0 \\ &\quad + (A_5 + 2\gamma_2 \cos 2x)\phi'_0 \\ &\quad + A_6\phi_1 + (A_7 + 2\gamma_1 \cos 2x)\phi'_1 + 2n^2\delta_1\phi_2 \\ &\quad + A_8\phi'_2 + \phi''_2 \cos 2x), \end{aligned} \quad (45)$$

where the coefficients are

$$\begin{aligned} A_1 &= 2n^2\delta_2 + \delta_1^2 + \gamma_1^2 + Q_0n^2\gamma_1 + \gamma_1^2, \\ A_2 &= 2\gamma_2 + Q_0\delta_1, \\ A_3 &= 2\gamma_1 + Q_0n^2, \\ A_4 &= 2\gamma_1\gamma_2 + 2\delta_1\delta_2 + 2n^2\delta_3 + Q_0\delta_1\gamma_1 + Q_0n^2\gamma_2, \\ A_5 &= 2\gamma_3 + Q_0\delta_2, \\ A_6 &= \gamma_1^2 + \delta_1^2 + 2n^2\delta_2 + Q_0\gamma_1n^2, \\ A_7 &= 2\gamma_2 + Q_0\delta_1, \\ A_8 &= 2\gamma_1 + Q_0n^2. \end{aligned}$$

To look for the normal forms at different n and the characteristic exponent, clearly if the real part of the characteristic exponent is negative, then the solution is asymptotically stable; if it is positive, then the solution is unstable; and if it equals zero, then the solution is periodic with period 2π when n is odd or π when n is even. Therefore, in what follows, we consider some cases of n obtaining positive and negative values of γ as follows.

Case $n = 1$

The solution is as equation (35) such that

$$\begin{aligned} \phi(x, h) &= (a \cos x + b \sin x) \\ &\quad + h \left[-\frac{1}{16}(a \cos 3x + b \sin 3x) \right] \\ &\quad + h^2 \left[\frac{1}{8}(B_1 \cos 3x + B_2 \sin 3x) \right] \\ &\quad + \frac{3}{256}(a \cos 5x + b \sin 5x) \\ &\quad + h^3 \left[\frac{1}{8}(B_3 \cos 3x + B_4 \sin 3x) \right] \\ &\quad + \frac{1}{24}(B_5 \cos 5x + B_6 \sin 5x) \\ &\quad - \frac{5}{1,696}(a \cos 7x + b \sin 7x) \Big] + O(h^4), \end{aligned} \quad (46)$$

where

$$a = -\left(\frac{2\gamma_1 + Q_0}{2\delta_1 - \frac{1}{2}}\right)b \text{ for } \gamma^+, a = \left(\frac{2\gamma_1 + Q_0}{2\delta_1 + \frac{1}{2}}\right)b \text{ for } \gamma^-,$$

where a and b are arbitrary constants to be determined from the initial conditions, and

$$\begin{aligned} B_1 &= -\frac{1}{8}\delta_1 a + \left(\gamma_1 - \frac{3}{16}A_3\right)b, \\ B_2 &= -\frac{1}{8}\delta_1 b - \left(\gamma_1 - \frac{3}{16}A_3\right)a, \\ B_3 &= \left(\frac{\gamma_1^2}{2} - \gamma_2 - \frac{A_6}{16} + \frac{15}{106}\right)a \\ &\quad - \frac{3}{16}A_7 b + \frac{\delta_1}{16}B_1 + \frac{3}{8}(2\gamma_1 + Q_0)B_2, \\ B_4 &= \left(\frac{\gamma_1^2}{2} - \gamma_2 - \frac{A_6}{16} + \frac{15}{106}\right)b \\ &\quad + \frac{3}{16}A_7 a + \frac{\delta_1}{16}B_2 - \frac{3}{8}(2\gamma_1 + Q_0)B_1, \\ B_5 &= \frac{6}{265}\delta_1 a - \frac{3}{16}\left[\gamma_1 + \frac{1}{16}(2\gamma_1 + Q_0)\right]b, \\ B_6 &= \frac{6}{265}\delta_1 b + \frac{3}{16}\left[\gamma_1 + \frac{1}{16}(2\gamma_1 + Q_0)\right]a. \end{aligned}$$

The corresponding positive and negative values of γ read

$$\begin{aligned} \gamma^+ &= h\left[-\frac{1}{2}\left(2\delta_1 + Q_0 - \frac{1}{2}\right)\right] \\ &\quad + h^2\left[\frac{1}{2}\left(\frac{\left(A_1 + \frac{9}{32}\right)(2\gamma_1 + Q_0)}{2\delta_1 - \frac{1}{2}} - (Q_0\delta_1 + \gamma_1)\right)\right] \\ &\quad + h^3\left[-\frac{1}{2}\left(\frac{2\gamma_1 + Q_0}{2\delta_1 - \frac{1}{2}}\right)\left(A_4 + \frac{\gamma_1^2}{2} + \frac{9}{128}\delta_1\right)\right. \\ &\quad \left. + \frac{1}{2}\left(Q_0\delta_2 + \gamma_2 + \frac{3}{16}\gamma_1 + \frac{9}{16}\left(\gamma_1 - \frac{3}{16}A_3\right)\right)\right] + O(h^4), \end{aligned} \quad (47)$$

$$\begin{aligned} \gamma^- &= h\left[\frac{1}{2}\left(2\delta_1 - Q_0 - \frac{1}{2}\right)\right] \\ &\quad + h^2\left[\frac{1}{2}\left(\frac{\left(A_1 + \frac{9}{32}\right)(2\gamma_1 + Q_0)}{2\delta_1 - \frac{1}{2}} - (Q_0\delta_1 - \gamma_1)\right)\right] \\ &\quad + h^3\left[-\frac{1}{2}\left(\frac{2\gamma_1 + Q_0}{2\delta_1 + \frac{1}{2}}\right)\left(A_4 + \frac{\gamma_1^2}{2} + \frac{9}{128}\delta_1\right)\right. \\ &\quad \left. + \frac{1}{2}\left(-Q_0\delta_2 + \gamma_2 + \frac{3}{16}\gamma_1 + \frac{9}{16}\left(\gamma_1 - \frac{3}{16}A_3\right)\right)\right] + O(h^4). \end{aligned} \quad (48)$$

Case $n = \sqrt{2}$

The solution is as equation (35) such that

$$\begin{aligned} \phi(x, h) &= (a \cos 2x + b \sin 2x) \\ &\quad + h\left[-\frac{1}{16}(a \cos 4x + b \sin 4x) + \frac{a}{2}\right] \\ &\quad + h^2\left[\frac{1}{12}(B_1 \cos 4x + B_2 \sin 4x) \right. \\ &\quad \left. + \frac{1}{24}(a \cos 6x + b \sin 6x) \right. \\ &\quad \left. - \frac{1}{4}(2\delta_1 a + \gamma_1 b)\right] \\ &\quad + h^3\left[\frac{1}{12}(B_3 \cos 4x + B_4 \sin 4x) \right. \\ &\quad \left. + \frac{1}{32}(B_5 \cos 6x + B_6 \sin 6x) \right. \\ &\quad \left. - \frac{1}{80}(a \cos 8x + b \sin 8x) \right. \\ &\quad \left. - \frac{1}{4}\left(\left(\frac{1}{2}A_6 - 2\delta_1^2\right)a + (\gamma_2 - \delta_1\gamma_1)b\right)\right] \\ &\quad + O(h^4), \end{aligned} \quad (49)$$

where

$$\begin{aligned} a &= -\left(\frac{\gamma_1 + Q_0}{\delta_1}\right)b \text{ for } \gamma^+, b = \left(\frac{\gamma_1 + Q_0}{\delta_1}\right)a \text{ for } \gamma^-, \\ B_1 &= -\frac{2}{3}\delta_1 a + \left(\gamma_1 - \frac{2}{3}A_3\right)b, \\ B_2 &= -\frac{2}{3}\delta_1 b - \left(\gamma_1 - \frac{2}{3}A_3\right)a, \\ B_3 &= \left(\frac{\gamma_1^2}{2} - \frac{A_6}{6} - \frac{3}{4}\right)a + \left(\frac{-2}{3}A_7 + \gamma_2\right)b \\ &\quad + \frac{\delta_1}{3}B_1 + \frac{1}{3}(2\gamma_1 + 2Q_0)B_2, \\ B_4 &= \left(\frac{\gamma_1^2}{2} - \frac{A_6}{6} - \frac{3}{4}\right)b - \left(\frac{-2}{3}A_7 + \gamma_2\right)a \\ &\quad + \frac{\delta_1}{3}B_2 + \frac{1}{3}(2\gamma_1 + 2Q_0)B_1, \\ B_5 &= \frac{1}{6}\delta_1 a - \frac{2}{3}B_1, \\ B_6 &= \frac{1}{6}\delta_1 b - \frac{2}{3}B_2. \end{aligned}$$

The corresponding positive and negative values of γ are

$$\begin{aligned} \gamma^+ &= h[\delta_1 - Q_0] + \frac{1}{4}h^2\left[\frac{\left(A_1 + \frac{4}{3}\right)(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2}Q_0\delta_1\right] \\ &\quad + \frac{1}{4}h^3\left[\frac{\left(A_4 + \frac{4}{9}\delta_1\right)(\gamma_1 + Q_0)}{\delta_1} \right. \\ &\quad \left. - \left(2Q_0\delta_2 - \frac{2}{3}\gamma_1 + \frac{4}{9}A_3\right)\right] + O(h^4), \end{aligned} \quad (50)$$

$$\begin{aligned}
\gamma^- = & -h[\delta_1 + Q_0] + \frac{1}{4}h^2 \left[\frac{(A_1 + \frac{4}{3})(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2}Q_0\delta_1 \right] \\
& + \frac{1}{4}h^3 \left[\frac{(A_4 + \frac{4}{9}\delta_1)(\gamma_1 + Q_0)}{\delta_1} \right. \\
& \left. - \left(2Q_0\delta_2 - \frac{2}{3}\gamma_1 + \frac{4}{9}A_3 \right) \right] + O(h^4).
\end{aligned} \quad (51)$$

Case $n = \sqrt{3}$

The solution is as equation (35) such that

$$\begin{aligned}
\phi(x, h) = & a \cos 3x + b \sin 3x \\
& + h \left[\frac{9}{16}(a \cos x + b \sin x) \right. \\
& \left. - \frac{1}{2}(a \cos 5x + b \sin 5x) \right] \\
& + h^2 \left[-\frac{1}{8}(B_1 \cos x + B_2 \sin x) \right. \\
& + \frac{1}{16}(B_3 \cos 5x + B_4 \sin 5x) \\
& \left. - \frac{4}{25}(a \cos 7x + b \sin 7x) \right] \\
& + h^3 \left[-\frac{1}{12}(B_5 \cos x + B_6 \sin x) \right. \\
& + \frac{1}{16}(B_7 \cos 5x + B_8 \sin 5x) \\
& + \frac{1}{40}(B_9 \cos 7x + B_{10} \sin 7x) \\
& \left. - \frac{245}{4069}(a \cos 9x + b \sin 9x) \right] \\
& + O(h^4),
\end{aligned}$$

where

$$a = -\frac{\gamma_1 + Q_0}{\delta_1}b \text{ for } \gamma^+, \quad b = -\frac{\gamma_1 + Q_0}{\delta_1}a \text{ for } \gamma^-,$$

$$B_1 = \left(\delta_1 + \frac{9}{32} \right) a + \left(3\gamma_1 + \frac{9}{16}A_3 \right) b,$$

$$B_2 = \left(\delta_1 + \frac{9}{32} \right) b - \left(3\gamma_1 + \frac{9}{16}A_3 \right) a,$$

$$B_3 = -\frac{1}{2}\delta_1 a + \left(3\gamma_1 - \frac{45}{32}A_3 \right) b,$$

$$B_4 = -\frac{1}{2}\delta_1 b - \left(3\gamma_1 - \frac{45}{32}A_3 \right) a,$$

$$\begin{aligned}
B_5 = & \left(\frac{\gamma_1^2}{2} - \frac{9}{16}A_6 \right) a + \left(\frac{9}{16}A_7 + \gamma_2 \right) b \\
& - \frac{1}{8}B_1 \left(6\delta_1 - \frac{1}{2} \right) - \frac{1}{8}A_8B_2, \\
B_6 = & \left(\frac{\gamma_1^2}{2} - \frac{9}{16}A_6 \right) b - \left(\frac{9}{16}A_7 + \gamma_2 \right) a \\
& - \frac{1}{8}B_2 \left(6\delta_1 + \frac{1}{2} \right) + \frac{1}{8}A_8B_1, \\
B_7 = & \left(\frac{1}{2}\gamma_1^2 - \frac{9}{32}A_6 - \frac{2205}{512} \right) a \\
& + \left(\gamma_2 - \frac{45}{32}A_7 + \frac{9}{16}\gamma_1 \right) b + \frac{1}{16}(6\delta_1B_3 + 5A_8B_4), \\
B_8 = & \left(\frac{1}{2}\gamma_1^2 - \frac{9}{32}A_6 - \frac{2205}{512} \right) b \\
& - \left(\gamma_2 - \frac{45}{32}A_7 + \frac{9}{16}\gamma_1 \right) a - \frac{1}{16}(6\delta_1B_4 + 5A_8B_3), \\
B_9 = & \frac{135}{256}\delta_1 a + \frac{45}{32} \left(\gamma_1 + \frac{7}{16} \right) b - \frac{25}{32}B_3, \\
B_{10} = & \frac{135}{256}\delta_1 b - \frac{45}{32} \left(\gamma_1 + \frac{7}{16} \right) a - \frac{25}{32}B_4.
\end{aligned}$$

(52) The corresponding positive and negative values of γ are as follows:

$$\begin{aligned}
\gamma^+ = & h[\delta_1 - Q_0] + \frac{1}{6}h^2 \left[\frac{(A_1 - \frac{207}{64})(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2}Q_0\delta_1 \right] \\
& + \frac{1}{6}h^3 \left[\frac{A_4 + \frac{29}{64}\delta_1 - \frac{9}{512}}{\delta_1} \right. \\
& \left. - \left(3Q_0\delta_2 - \frac{45}{32}A_7 - \frac{69}{32}\gamma_1 + \frac{1161}{1024}A_3 \right) \right] + O(h^4),
\end{aligned} \quad (53)$$

$$\begin{aligned}
\gamma^- = & -h[\delta_1 + Q_0] + \frac{1}{6}h^2 \left[\frac{(A_1 - \frac{207}{64})(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2}Q_0\delta_1 \right] \\
& + \frac{1}{6}h^3 \left[\frac{A_4 + \frac{29}{64}\delta_1 - \frac{9}{512}}{\delta_1} \right. \\
& \left. - \left(3Q_0\delta_2 - \frac{45}{32}A_7 - \frac{69}{32}\gamma_1 + \frac{1161}{1024}A_3 \right) \right] + O(h^4).
\end{aligned} \quad (54)$$

Case $n = 2$

The solution is as equation (35) such that

$$\begin{aligned}
\phi(x, h) = & (a \cos 4x + b \sin 4x) \\
& + h \left[\frac{1}{6} (a \cos 2x + b \sin 2x) \right. \\
& \left. - \frac{1}{10} (a \cos 6x + b \sin 6x) \right] \\
& + h^2 \left[-\frac{1}{12} (B_1 \cos 2x + B_2 \sin 2x) \right. \\
& + \frac{1}{20} (B_3 \cos 6x + B_4 \sin 6x) \\
& + \frac{3}{80} (a \cos 8x + b \sin 8x) + \frac{1}{48} a \Big] \\
& + h^3 \left[-\frac{1}{12} (B_5 \cos 2x + B_6 \sin 2x) \right. \\
& + \frac{1}{20} (B_7 \cos 8x + B_8 \sin 8x) \\
& + \frac{1}{48} (B_9 \cos 8x + B_{10} \sin 8x) \\
& - \frac{1}{70} (a \cos 10x + b \sin 10x) \\
& \left. - \frac{1}{96} (\delta_1 a + 2\gamma_1 b + B_1) \right] + O(h^4),
\end{aligned} \quad (55)$$

where

$$\begin{aligned}
a &= -\frac{\gamma_1 + 2Q_0}{\delta_1} b \text{ for } \gamma^+, \\
b &= \frac{\gamma_1 + 2Q_0}{\delta_1} a \text{ for } \gamma^-, \\
B_1 &= -\frac{4}{3} \delta_1 a + \left(\gamma_1 + \frac{1}{3} (A_3 - 1) \right) b, \\
B_2 &= -\frac{4}{3} \delta_1 b - \left(\gamma_1 + \frac{1}{3} (A_3 - 1) \right) a, \\
B_3 &= \left(-\frac{4}{5} \delta_1 + \frac{9}{5} \right) a + \left(\gamma_1 - \frac{3}{5} A_3 \right) b, \\
B_4 &= \left(-\frac{4}{5} \delta_1 + \frac{9}{5} \right) b - \left(\gamma_1 - \frac{3}{5} A_3 \right) a, \\
B_5 &= \frac{\gamma_1^2}{2} a + \left(\frac{1}{3} A_7 + \gamma_2 + \frac{3}{5} \gamma_1 \right) b \\
&\quad - \left(\frac{1}{12} A_6 + \frac{2}{3} \delta_1 \right) B_1 - \frac{1}{6} A_8 B_2, \\
B_6 &= \frac{\gamma_1^2}{2} b - \left(\frac{1}{3} A_7 + \gamma_2 + \frac{3}{5} \gamma_1 \right) a \\
&\quad - \left(\frac{1}{12} A_6 + \frac{2}{3} \delta_1 \right) B_2 + \frac{1}{6} A_8 B_1, \\
B_7 &= \left(\frac{1}{2} \gamma_1^2 - \frac{6}{5} \right) a + \left(\gamma_2 - \frac{3}{5} A_7 \right) b + \frac{2}{5} \delta_1 B_3 + \frac{3}{10} A_8 B_4, \\
B_8 &= \left(\frac{1}{2} \gamma_1^2 - \frac{6}{5} \right) b - \left(\gamma_2 - \frac{3}{5} A_7 \right) a + \frac{2}{5} \delta_1 B_4 - \frac{3}{10} A_8 B_3, \\
B_9 &= \frac{3}{10} (a + A_8 b + 3B_3), \\
B_{10} &= \frac{3}{10} (b - A_8 a + 3B_4).
\end{aligned}$$

The corresponding positive and negative values of γ are as follows:

$$\begin{aligned}
\gamma^+ = & h [\delta_1 - 2Q_0] + \frac{1}{2} h^2 \left[\frac{\left(A_1 + \frac{22}{5} \right) (\gamma_1 + 2Q_0)}{4\delta_1} - Q_0 \delta_1 \right] \\
& + \frac{1}{8} h^3 \left[\frac{\left(A_4 + \frac{212}{225} \delta_1 - \frac{81}{50} \right) (\gamma_1 + 2Q_0)}{\delta_1} \right. \\
& \left. - \left(4Q_0 \delta_2 + \frac{1}{5} \gamma_1 + \frac{134}{225} A_3 - \frac{1}{18} \right) \right] + O(h^4),
\end{aligned} \quad (56)$$

$$\begin{aligned}
\gamma^- = & -h [\delta_1 - 2Q_0] + \frac{1}{2} h^2 \left[\frac{\left(A_1 + \frac{22}{5} \right) (\gamma_1 + 2Q_0)}{4\delta_1} - Q_0 \delta_1 \right] \\
& + \frac{1}{8} h^3 \left[\frac{\left(A_4 + \frac{212}{225} \delta_1 - \frac{81}{50} \right) (\gamma_1 + 2Q_0)}{\delta_1} \right. \\
& \left. - \left(4Q_0 \delta_2 + \frac{1}{5} \gamma_1 + \frac{134}{225} A_3 - \frac{1}{18} \right) \right] + O(h^4).
\end{aligned} \quad (57)$$

Hence, by plugging terms of δ into Whittaker's method to obtain the terms of γ as follows.

Case $n = 1$

The positive and negative terms of γ are as follows:

$$\gamma_1^+ = -\frac{1}{2} \left(2\delta_1 + Q_0 - \frac{1}{2} \right), \quad \gamma_1^- = \frac{1}{2} \left(2\delta_1 - Q_0 - \frac{1}{2} \right), \quad (58)$$

$$\gamma_2^+ = \frac{1}{2} \left(\frac{\left(A_1 + \frac{9}{32} \right) (2\gamma_1 + Q_0)}{2\delta_1 - \frac{1}{2}} - (Q_0 \delta_1 + \gamma_1) \right), \quad (59)$$

$$\gamma_2^- = \frac{1}{2} \left(\frac{\left(A_1 + \frac{9}{32} \right) (2\gamma_1 + Q_0)}{2\delta_1 - \frac{1}{2}} - (Q_0 \delta_1 - \gamma_1) \right), \quad (60)$$

$$\gamma_3^+ = -\frac{1}{2} \left(\frac{2\gamma_1 + Q_0}{2\delta_1 - \frac{1}{2}} \right) \left(A_4 + \frac{\gamma_1^2}{2} + \frac{9}{128} \delta_1 \right) \quad (61)$$

$$+ \frac{1}{2} \left(Q_0 \delta_2 + \gamma_2 + \frac{3}{16} \gamma_1 + \frac{9}{16} \left(\gamma_1 - \frac{3}{16} A_3 \right) \right), \quad (62)$$

$$\gamma_3^- = -\frac{1}{2} \left(\frac{2\gamma_1 + Q_0}{2\delta_1 + \frac{1}{2}} \right) \left(A_4 + \frac{\gamma_1^2}{2} + \frac{9}{128} \delta_1 \right) \quad (63)$$

$$+ \frac{1}{2} \left(-Q_0 \delta_2 + \gamma_2 + \frac{3}{16} \gamma_1 + \frac{9}{16} \left(\gamma_1 - \frac{3}{16} A_3 \right) \right). \quad (64)$$

Case $n = \sqrt{2}$

The positive and negative terms of γ are as follows:

$$\gamma_1^+ = \delta_1 - Q_0, \quad \gamma_1^- = -(\delta_1 + Q_0), \quad (65)$$

$$\gamma_2^+ = \frac{1}{4} \left(\frac{\left(A_1 + \frac{4}{3}\right)(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2} Q_0 \delta_1 \right), \quad (66)$$

$$\gamma_2^- = \frac{1}{4} \left(\frac{\left(A_1 + \frac{4}{3}\right)(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2} Q_0 \delta_1 \right), \quad (67)$$

$$\gamma_3^+ = \frac{1}{4} \left(\frac{\left(A_4 + \frac{4}{9} \delta_1\right)(\gamma_1 + Q_0)}{\delta_1} - \left(2Q_0 \delta_2 - \frac{2}{3} \gamma_1 + \frac{4}{9} A_3\right) \right), \quad (68)$$

$$\gamma_3^- = \frac{1}{4} \left(\frac{\left(A_4 + \frac{4}{9} \delta_1\right)(\gamma_1 + Q_0)}{\delta_1} - \left(2Q_0 \delta_2 - \frac{2}{3} \gamma_1 + \frac{4}{9} A_3\right) \right). \quad (69)$$

Case $n = \sqrt{3}$

The positive and negative terms of γ are as follows:

$$\gamma_1^+ = \delta_1 - Q_0, \quad \gamma_1^- = -(\delta_1 + Q_0), \quad (70)$$

$$\gamma_2^+ = \frac{1}{6} \left(\frac{\left(A_1 - \frac{207}{64}\right)(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2} Q_0 \delta_1 \right), \quad (71)$$

$$\gamma_2^- = \frac{1}{6} \left(\frac{\left(A_1 - \frac{207}{64}\right)(\gamma_1 + Q_0)}{\delta_1} - \frac{1}{2} Q_0 \delta_1 \right), \quad (72)$$

$$\gamma_3^+ = \frac{1}{6} \left(\frac{A_4 + \frac{29}{64} \delta_1 - \frac{9}{512}}{\delta_1} - \right. \quad (73)$$

$$\left. \left(3Q_0 \delta_2 - \frac{45}{32} A_7 - \frac{69}{32} \gamma_1 + \frac{1161}{1024} A_3\right) \right), \quad (74)$$

$$\gamma_3^- = \frac{1}{6} \left(\frac{A_4 + \frac{29}{64} \delta_1 - \frac{9}{512}}{\delta_1} - \right. \quad (75)$$

$$\left. \left(3Q_0 \delta_2 - \frac{45}{32} A_7 - \frac{69}{32} \gamma_1 + \frac{1161}{1024} A_3\right) \right). \quad (76)$$

Case $n = 2$

The positive and negative terms of γ are as follows:

$$\gamma_1^+ = \delta_1 - 2Q_0, \quad \gamma_1^- = -(\delta_1 - 2Q_0), \quad (77)$$

$$\gamma_2^+ = \frac{1}{2} \left[\frac{\left(A_1 + \frac{22}{5}\right)(\gamma_1 + 2Q_0)}{4\delta_1} - Q_0 \delta_1 \right], \quad (78)$$

$$\gamma_2^- = \frac{1}{2} \left[\frac{\left(A_1 + \frac{22}{5}\right)(\gamma_1 + 2Q_0)}{4\delta_1} - Q_0 \delta_1 \right], \quad (79)$$

$$\gamma_3^+ = \frac{1}{8} \left(\frac{\left(A_4 + \frac{212}{225} \delta_1 - \frac{81}{50}\right)(\gamma_1 + 2Q_0)}{\delta_1} - \right. \quad (80)$$

$$\left. \left(4Q_0 \delta_2 + \frac{1}{5} \gamma_1 + \frac{134}{225} A_3 - \frac{1}{18}\right) \right), \quad (81)$$

$$\gamma_3^- = \frac{1}{8} \left(\frac{\left(A_4 + \frac{212}{225} \delta_1 - \frac{81}{50}\right)(\gamma_1 + 2Q_0)}{\delta_1} - \right. \quad (82)$$

$$\left. \left(4Q_0 \delta_2 + \frac{1}{5} \gamma_1 + \frac{134}{225} A_3 - \frac{1}{18}\right) \right). \quad (83)$$

Numerical solutions are considered to verify the corresponding derived analytical ones. For instance, in the case of $n = 1$, the solution $(\gamma(x))$ is represented by equations (46), (47) and (48) to provide an approximation near the transition curves emanating from $\delta = 1$. Consequently, based on γ , the state of the system can be bounded, unbounded or periodic when $\gamma = 0$. Based on the following suggested approximation of δ emanating from $\delta = 1$,

$$\delta = 1 + 0.1406h^2 + 0.123h^3 + O(h^4), \quad (84)$$

the numerical verifications of the derived approximate forms are done. It is vividly shown in Figures 5, 6 and 7 that the deduced results are comparable with the corresponding numerical ones.

In Figure 5, the periodic solution of the system is obtained when the characteristic exponent is equal to zero and the approximate solution is very close to the numerical one. In Figure 6, it is clearly noted that the stable solution of the system is achieved when the characteristic exponent is negative and the system is quickly damped by increasing the value of the characteristic exponent. In contrast, as shown in Figure 7, the solution of the system is unbounded when the characteristic exponent is positive.

The transition curves, shown in Figures 3 and 4 using the harmonic balance method, might also be obtained by the analytical ones by using the relations of $\delta(h)$ in equation (38). Since, by reversing the results as $\delta = \frac{1}{\alpha}$, the same behavior might be obtained.

4.2 Stability of periodic solutions

By rewriting the perturbed form of the governing equation (equation (4)) as follows:

$$y'' + \delta^2 y = hF(y, y', y'', x), \quad (85)$$

where

$$F = -\cos(2x)y'' - Q_0\delta y'. \quad (86)$$

Hence, the solution of the perturbed problem has the periodic form

$$y(x) = \Lambda(x)\sin(\delta x + \Theta(x)), \quad (87)$$

where $\Lambda(x)$ is the amplitude and $\Theta(x)$ is the phase angle. Hence, the necessary condition required for the stability of perturbed periodic solution is proved in the following theorem.

Theorem 2. *The perturbed periodic solution of the governing equation (4) is asymptotically stable if the following condition holds:*

$$\frac{1}{\delta-1} \cos 4\pi \frac{\delta-1}{\delta} + \frac{1}{\delta+1} \cos 4\pi \frac{\delta+1}{\delta} - \frac{2\delta}{\delta^2-1} < \frac{8\pi Q}{h\delta}. \quad (88)$$

Proof. Following Krylov and Bogoliubov approximations in [28,29], Λ and Θ are replaced with their average values over the period $T = \frac{2\pi}{\delta}$. By considering that Λ and Θ are constants in taking the values of average and apply the method of averaging, we obtain $\square$

$$\dot{\Lambda} = -h \frac{1}{2\pi} \int_0^T \cos \psi F(\Lambda \sin \psi, \Lambda \delta \cos \psi, -\Lambda \delta^2 \sin \psi, x) dx, \quad (89a)$$

$$\dot{\Theta} = h \frac{1}{2\pi \Lambda} \int_0^T \sin \psi F(\Lambda \sin \psi, \Lambda \delta \cos \psi, -\Lambda \delta^2 \sin \psi, x) dx, \quad (89b)$$

where $\psi = \delta x + \Theta$.

Once the integrals have been evaluated, we have first-order differential equations to obtain the amplitude and the phase angle, then an approximate periodic solution is obtained. The constant amplitudes of the obtained periodic solutions can be measured from

$$\mathbb{G}(\Lambda) = -\frac{h}{2\pi} \int_0^T \cos \psi F(\Lambda \sin \psi, \Lambda \delta \cos \psi, -\Lambda \delta^2 \sin \psi, x) dx, \quad (90a)$$

$$\mathbb{G}(\Lambda) = 0. \quad (90b)$$

By considering that the roots ($\Lambda = \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3, \dots$) are real, we obtain simply the amplitudes of them. Consequently, to get the condition of stability of the periodic solutions, the following condition has to be satisfied:

$$\mathbb{G}'(\Lambda) = \frac{d\mathbb{G}(\Lambda)}{d\Lambda} \bigg|_{\Lambda=\Lambda_i} < 0, \quad i = 0, 1, 2, 3, \dots \quad (91)$$

Then, we have

$$\mathbb{G}(\Lambda) = \frac{h\Lambda\delta^2}{2\pi} \mathbf{G}, \quad (92)$$

where

$$\mathbf{G} = \frac{1}{8(\delta-1)} \cos 4\pi \frac{\delta-1}{\delta} + \frac{1}{8(\delta+1)} \cos 4\pi \frac{\delta+1}{\delta} - \frac{\delta}{4(\delta^2-1)} - \frac{\pi Q_0}{\delta}. \quad (93)$$

Then, we obtain

$$\Lambda = \Lambda_0 \neq 0 \text{ at } \Theta = \text{const. and } \delta \neq 0. \quad (94)$$

Apply the stability condition of equation (91), $\mathbf{G}$ should satisfy

$$\mathbf{G} < 0. \quad (95)$$

Then, the conclusion holds.

Figures 8 and 9 vividly illustrate the stability regions of the theoretical condition of equation (88). As shown, for specific values of δ or α , a stable periodic solution can be obtained depending on the value of the parameter Q with radius Λ_0 if the curve lies under the critical line. As it is clearly noted before, when Q is increased the system enters the region of stability until the instability region completely disappears. Additionally, around $\alpha = 1$, the operation point of the dynamic machine is stable since it lies under the critical line but if α is slightly increased or decreased the dynamic operation point touches the trouble regions getting resonances. Indeed, it roughly fits the previously derived theoretical results and the experimental ones in [26,27].

Figure 5: The periodic solution at $\gamma = 0$. (a) $h = 0.1, Q_0 = 0.075, \alpha = 0.98$ and $\gamma = 0$. (b) $h = 0.1954, Q_0 = 0.1, \alpha = 1.032$ and $\gamma = 0$.

Figure 6: The stable solution for negative γ . (a) $h = 0.3, Q_0 = 0.5, \alpha = 0.98$ and $\gamma = -0.116$. (b) $h = 0.2, Q_0 = 0.3, \alpha = 1.012$ and $\gamma = -0.09$.

Figure 7: The unstable solution for positive γ . (a) $h = 0.3, Q_0 = 0.1, \alpha = 1.043$ and $\gamma = 0.028$. (b) $h = 0.2, Q_0 = 0.3, \alpha = 1.043$ and $\gamma = 0.028$.

5 Conclusion

This work is concerned with the mathematical analysis of AC dynamic machines modeled as RLC series circuits with periodically time-varying inductances. The governing

equation is described by a linear second-order differential equation with periodic coefficients due to the periodic variation in inductance. Floquet's theory is applied to obtain the necessary conditions for the dynamic stability of the modeled dynamic machines and approximate normal form of

Figure 8: Regions of stable periodic solutions according to δ .

Figure 9: Regions of stable periodic solutions according to α .

solution is constructed via Whittaker's method to obtain the natural response. Harmonic balance analysis is employed to predict the stability regions separated by the transition curves showing that the dynamic operation region has been limited for the perturbing parameter (h) into the interval $(-1, 1)$. Moreover, approximated values for the frequency of AC machines to run on with the sustained stable periodic modes are obtained. The derived results are mostly comparable with the corresponding experimental ones in the closed problem literature to realize a satisfactory agreement specially at the small values of the perturbation parameter (h).

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