Research Article

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Optimal control for generating excited state expansion in ring potential

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Abstract: We applied an optimal control algorithm to an ultra-cold atomic system for constructing an atomic Sagnac interferometer in a ring trap. We constructed a ring potential on an atom chip by using an RF-dressed potential. A field gradient along the radial direction in a ring trap known as the dimple-ring trap is generated by using an additional RF field. The position of the dimple is moved by changing the phase of the RF field [1]. For Sagnac interferometers, we suggest transferring Bose–Einstein condensates to a dimple-ring trap and shaking the dimple potential to excite atoms to the vibrational-excited state of the dimple-ring potential. The optimal control theory is used to find a way to shake the dimple-ring trap for an excitation. After excitation, atoms are released from the dimple-ring trap to a ring trap by adiabatically turning off the additional RF field, and this constructs a Sagnac interferometer when opposite momentum components are overlapped. We also describe the simulation to construct the interferometer.

Keywords: atomic Sagnac interferometer, atom chip, RF-dressed potential, optimal control algorithm

1 Introduction

Atom interferometers are dynamic tools applied for precision measurements and for studying the wave nature of interacting matter over the past few decades. Measurements of inertial effects such as acceleration and rotation [2–5], physical constants [6,7], gravitational constant [8,9], and dark energy [10,11] in free space have been demonstrated.

Interferometers in a confined trap using the wave nature of matter have also been constructed in various potentials such as multi-well [12,13], ring [14–18], optical lattice [19–22], and certain wave-guide traps [20–27]. Atom interferometers require splitting, reflecting, and recombining coherent sources just as an optical interferometer. Various ways have been adopted to realize interferometric components by using Raman pulse [2–5,7], Bragg diffraction [25–27], and adiabatically changing trapping potentials [11,12].

An optimal control algorithm is a process to find a way from the initial state to the desired state by perturbing systems. It was adopted to optimize atomic systems and manipulate the desired atomic states such as $\pm\hbar k$ momentum states or excited states [19,20,28–32]. It is also used for machine-learning when it works with closed-loop control and constructing atom interferometers to find ways to get interferometric components.

In this article, we describe a process to demonstrate an atomic Sagnac interferometer using ultra-cold atoms by using a vibrational-excited state in a ring trap potential. The RF-dressed trapping potentials that we experimentally demonstrated for a ring and a dimple-ring trap in ref. [1] are reviewed in Section 2. In Section 3, we introduce a way to construct a Sagnac interferometer. First, we adiabatically load ultra-cold atoms from the initial trapping potential to the ground state of the dimple-ring potential. And then, we shake the dimple-ring potential to excite atoms. The way to shake the potential is obtained from an optimal control algorithm. We describe the details of the optimal control algorithm we used in Section 3. The excited atoms are released to a ring potential, and they start to expand. We also simulated the Sagnac interferometry when opposite momentum components are overlapped.

2 RF-dressed dimple-ring potential

We demonstrated dimple-ring potential near an atom chip by using an RF-dressed potential [1,33]. Figure 1 shows the pattern of wires on our atom chip. We used

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Figure 1: (a) Wire pattern for the dimple-ring potential; (b) dimple-ring potential as δ_z , where $G = 50$ T/m, $B_I = B_{RF} = 1$ G, $B_{RF}^z = 100$ mG, $\omega_{RF} = 2\pi \times 650$ kHz, and $\delta = \pi/2$ [1].

the center wire to create an Ioffe magnetic trap with a bias magnetic field and the dimple wire for enhancing the longitudinal confinement of the trap. Each side wire, RF A and B, is used to create an RF-dressed potential, and the RF field on the center wire creates asymmetry in the potential such as the tilted double-well and dimple-ring potential [1]. The magnetic field is given by static, $\mathbf{B}_s$, and oscillating, $\mathbf{B}_{RF}$, fields such as

$$\mathbf{B}_s(\mathbf{r}) = Gx\mathbf{e}_x - Gy\mathbf{e}_y + B_I\mathbf{e}_z \quad (1)$$

$$\begin{aligned} \mathbf{B}_{RF} = & B_{RF}^A \mathbf{e}_x \cos[\omega_{RF}^A t] + B_{RF}^B \mathbf{e}_y \cos[\omega_{RF}^B t - \delta] \\ & + B_{RF}^z \mathbf{e}_z \cos[\omega_{RF}^z t - \delta_z], \end{aligned} \quad (2)$$

where G is the gradient of the static trap and B_I is the magnitude of the Ioffe trap-bottom. B_{RF}^A , B_{RF}^B , and B_{RF}^z

represent the amplitude of each RF field. δ and δ_z are the relative phases to the RF field on the A wire.

RF-induced potential can be written as the following equation:

$$V(\mathbf{r}) = m_F g_F \mu_B \sqrt{\Delta(\mathbf{r})^2 + \Omega(\mathbf{r})^2}. \quad (3)$$

In equation (3), m_F is the magnetic quantum number, g_F is the Lande g -factor of the hyperfine structure, and μ_B is the Bohr magneton [1,34]. The detuning term, $\Delta(\mathbf{r})$, and the coupling term between static and oscillating fields, $\Omega(\mathbf{r})$, can be given by the following equations:

$$\Delta(\mathbf{r}) = |B_s(\mathbf{r})| - \frac{\hbar\omega_{RF}}{|g_F\mu_B|}, \quad (4)$$

$$\Omega(\mathbf{r}) = \frac{|B_{eff}(\mathbf{r}) \times B_s(\mathbf{r})|}{|B_s(\mathbf{r})|}. \quad (5)$$

The effective field in equation (5) is calculated as follows:

$$\begin{aligned} B_{eff}(\mathbf{r}) = & \mathbb{R}_y[\beta(\mathbf{r})] \mathbb{R}_z[\alpha(\mathbf{r})] \begin{pmatrix} B_{RF}^A \\ 0 \\ 0 \end{pmatrix} \\ & + \mathbb{R}_z\left[-\frac{g_F}{|g_F|}\delta\right] \mathbb{R}_y[\beta(\mathbf{r})] \mathbb{R}_z[\alpha(\mathbf{r})] \begin{pmatrix} 0 \\ B_{RF}^B \\ 0 \end{pmatrix} \\ & + \mathbb{R}_z\left[-\frac{g_F}{|g_F|}\delta_z\right] \mathbb{R}_y[\beta(\mathbf{r})] \mathbb{R}_z[\alpha(\mathbf{r})] \begin{pmatrix} 0 \\ 0 \\ B_{RF}^z \end{pmatrix}, \end{aligned} \quad (6)$$

with a rotating matrix, $\mathbb{R}_i[\theta]$, along the i axis, where $\tan[\alpha(\mathbf{r})] = B_s(\mathbf{r})_y / B_s(\mathbf{r})_x$ and $\tan[\beta(\mathbf{r})] = \sqrt{B_s(\mathbf{r})_x^2 + B_s(\mathbf{r})_y^2} / B_s(\mathbf{r})_z$.

Since the detuning term is rotationally symmetric in the cylindrical coordinate, the coupling term determines the geometry of the RF-dressed potential. For simplicity, we assumed $\omega_{RF}^A = \omega_{RF}^B = \omega_{RF}^z = \omega_{RF}$ and $B_{RF}^A = B_{RF}^B = B_{RF}$. When B_{RF}^z is not zero and its amplitude is much less than B_{RF} , the coupling term difference between cases of $B_{RF}^z \neq 0$ and $B_{RF}^z = 0$ can be written as follows:

$$\begin{aligned} \Omega(\mathbf{r})^2|_{B_{RF}^z \neq 0} - \Omega(\mathbf{r})^2|_{B_{RF}^z = 0} = & -G \frac{(|B_s(\mathbf{r})| + B_I)}{2|B_s(\mathbf{r})|^2} B_{RF} \\ & \times B_{RF}^z (x \cos \delta_z - y \sin \delta_z), \end{aligned} \quad (7)$$

where $\delta = \pi/2$.

Figure 1 shows the wire pattern of our chip and the potential for the dimple-ring trap as δ_z , described by equations (6) and (7).

When B_{RF}^z is zero, the effective potential has a ring shape on the XY plane. We can also generate a field gradient along the radial direction of the ring trap called the dimple-ring potential by using I_{RF}^D to generate B_{RF}^z ,

and the position of the trap minimum can be controlled by changing the phase difference between I_{RF}^A and I_{RF}^D . Experimental demonstration to generate condensation in the ring trap and the dimple-ring trap is described in ref. [1] and [33].

3 Optimal control algorithms for exciting atoms in a dimple-ring trap

We suggest the following method for constructing a Sagnac-type atom interferometer using a ring potential. First, we generated condensed atoms in an Ioffe trap, and then we transferred the atoms to a dimple-ring trap, i.e., to one of the dimples shown in Figure 1(b). We were able to add momentum to atoms by shaking the dimple-ring potential along the tangential direction to excite atoms in the trap. In order to make the excitation, we applied an optimal control algorithm to find a way of shaking the dimple-ring potential by changing δ_z . After excitation, we transferred the excited atoms to a ring potential by adiabatically ramping down the AC current of the dimple wire, I_{RF}^z . Since there is no confinement along the radial direction in a ring trap as shown in Figure 1(b), atoms will freely expand in a ring trap along clock-wise and counter clock-wise directions. Atoms would overlap similar to an optical Sagnac interferometer finally.

For simplicity, we assume that the trap and the perturbation are in one dimension to apply an optimal algorithm. In Figure 2, blue shows the initial state as the ground state in a dimple-ring trap, and red shows the desired state, which is the first excited state. When the 1D trapping potential is in black (Figure 2), we solved the Gross–Pitaevskii equation (GPE) to find the ground state of condensed atoms, and the excited state is calculated by optimizing GPE orthogonality with the ground state.

The goal of the optimal algorithm is to find the appropriate shaking function of the dimple-ring trap to excite the atoms, and it runs by the following way [28,30].

First, the algorithm calculates the time evolution of the initial state with a given initial shaking, which can be any function. The initial function in our case is a sinusoidal function, which has the same initial and final conditions, $\lambda(0) = \lambda(T) = 0$. After shaking, in a second, we calculate the cost function using equation (8), which says how the result is close to the desired state and how smooth the shaking is.

Figure 2: Dimple-ring potential given in black, where $\delta_z = 4\pi/5$, blue is the ground state as the initial state.

$$J(\psi(T), \lambda) = \frac{1}{2} [1 - |\langle \psi_D | \psi(T) \rangle|^2] + \frac{\gamma}{2} \int_0^T [\dot{\lambda}(t)]^2 dt. \quad (8)$$

In the above equation, ψ_D is the desired state, $\psi(T)$ is the wave function at the final time of shaking, γ is a weight function showing how smooth the shaking is, and λ is a control parameter related to shaking, which is the displacement of the dimple-ring trap or δ_z in this case. The optimal algorithm seeks the control, λ , that minimizes the cost function, $J(\psi, \lambda)$. In order to minimize the cost function, ref. [28] and [30] introduce the Lagrange function with the constrain that the wave function has to fulfill GPE as shown in equation (9)

$$L(\psi, p, \lambda) = J(\psi, \lambda) + \text{Re} \left[\int_0^T \langle p(r, t) | i\dot{\psi}(t) - (H_\lambda + g |\psi(t)|^2)\psi(t) \rangle dt \right], \quad (9)$$

where $p(r, t)$ is the adjoint function of Lagrangian, H_λ is the Hamiltonian of the shaken dimple-ring potential as a function of λ , and g is the coupling constant of GPE related to atom–atom interaction. At the minimum of $J(\psi, \lambda)$, $L(\psi, p, \lambda)$ must fulfill equation (10),

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} = 0, \quad (10a)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{p}} \right) - \frac{\partial L}{\partial p} = 0, \quad (10b)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{\lambda}} \right) - \frac{\partial L}{\partial \lambda} = 0. \quad (10c)$$

From equations (8) and (9), we can write equation (10) as equation (11):

$$i \frac{\partial p}{\partial t} = (H_\lambda + 2g |\psi|^2) p + g \psi^2 p^*, \quad (11a)$$

$$i \frac{\partial \psi}{\partial t} = (H_\lambda + g |\psi|^2) \psi, \quad (11b)$$

$$\gamma \frac{\partial^2 \lambda}{\partial t^2} = -\text{Re} \left\langle \psi \left| \frac{\partial H_\lambda}{\partial \lambda} \right| p \right\rangle, \quad (11c)$$

at initial conditions $\psi(0) = \psi_0$, $\lambda(0) = \lambda_0$, and $\lambda(T) = \lambda_T$, where ψ_0 is the ground state in the dimple-ring trap, and λ_0 and λ_T are the first and the final control parameters. We can also find the final adjoint, p , by integrating equation (11a) with time such as $i_p = -\langle \psi_D | \psi(T) \rangle \psi_D$.

It is difficult for the initial guess of λ to fulfill equation (11) simultaneously. However, equation (11c) can tell us how the control goes to minimize $J(\psi, \lambda)$ by adding a fraction term given as equation (12) to previous control when equations (11a) and (11b) are satisfied.

$$\frac{\partial L}{\partial \lambda} = -\gamma \ddot{\lambda} - \text{Re} \left\langle \psi \left| \frac{\partial H_\lambda}{\partial \lambda} \right| p \right\rangle. \quad (12)$$

Therefore, it calculates $p(t)$ from equation (11a) and $p(T)$ as reverse in time for the third step. And then, a new $\lambda(t)$ can be calculated from equation (12). By keeping these iterations, we could optimize the control to get to the desired state.

Besides calculating the time-evolution of the wave function and the way to minimize the cost function we used as described in ref. [30], there are other mathematical ways about fractional derivatives to calculate differential equations [36–45].

Figure 3(a) shows the atomic wave function as time when the dimple-ring potential is shaken, resulting in the optimized control to excite the ground state of the dimple-ring trap for 20 ms and hold the excited state in the dimple-ring trap for 20 ms. Figure 3(b) shows the optimized control in black and the initial guess in red. The cost function as a function of the iteration of the optimal algorithm is shown in the box. A smaller cost function means that the result is closer to the desired state and the control is smoother. We can see that the cost function is exponentially decreased as the iteration increased, and the decay time to the minimum depends on the initial guess of control. The time step of our optimal algorithm is set to 25 μs .

The range of phase for the optimized control is less than 3 deg, and the main frequency component is about

Figure 3: (a) Excitation of the ground state in the dimple-ring trap for 20 ms and holding it for 20 ms. (b) Initial guess of control in red and the optimized control in black, and cost function as a function of iteration is given in the box.

2.4 kHz in our case. Therefore, it requires a phase shifter that has 650 kHz in the main frequency, ± 1.5 deg in the bandwidth, and 2.4 kHz in a dynamic range at least when we experimentally apply the result.

Figure 4 shows the next step of constructing the atomic Sagnac interferometer in a ring trap. After excitation, we suggest adiabatically transferring atoms from the dimple-ring trap to a ring trap by ramping down the RF field in the center wire. After that, we let atoms expand in a ring potential. Finally, we construct an atomic Sagnac interferometer in the ring trap when opposite momentum components are overlapped. The interferometric phase is obtained from the position of each atomic cloud and we expect that the system rotation changes the phase [35].

Figure 4: Expansion of the excited atoms in a ring trap. Time step is $800 \mu\text{s}$.

4 Conclusion

We simulated a method to construct an atomic Sagnac interferometer in a ring trap. In order to add a bi-directional momentum, we suggest using the excited state. First, we load condensed atoms in a dimple-ring trap. After loading, we shake the trap to excite the atoms by changing the center wire RF field. The shaking function is obtained by the optimal control algorithm. The advantage of the optimal algorithm is that we are able to shake the atoms to find a way and just use it experimentally even though we do not know the physics of the result. After excitation, atoms are adiabatically transferred to a ring trap, and they expand in the ring trap similar to an optical Sagnac interferometer. Our main idea is to apply an optimal algorithm to our trapping system for constructing an atomic Sagnac interferometer. This does not require other experimental devices such as lasers for Bragg diffractions or Raman pulses and perturbing tapping potentials during interferometric integration time. We expect that this type of interferometer measures system rotation.

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