Review Article

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Sine Topp-Leone-G family of distributions: Theory and applications

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Abstract: Recent studies have highlighted the statistical relevance and applicability of trigonometric distributions for the modeling of various phenomena. This paper contributes to the subject by investigating a new trigonometric family of distributions defined from the alliance of the families known as sine-G and Topp-Leone generated (TL-G), inspiring the name of sine TL-G family. The characteristics of this new family are studied through analytical, graphical and numerical approaches. Stochastic ordering and equivalence results, determination of the mode(s), some expansions of distributional functions, expressions of the quantile function and moments and basics on order statistics are discussed. In addition, we emphasize the fact that the sine TL-G family is able to generate original, simple and pliant trigonometric models for statistical purposes, beyond the capacity of the former sine-G models and other top models of the literature. This fact is revealed with the special three-parameter sine TL-G model based on the inverse Lomax model, through an efficient parametric estimation and the adjustment of two data sets of interest.

Keywords: Sine-G family, Topp-Leone-G family, moments, maximum likelihood estimation, real-life data

1 Introduction

In recent years, many authors have developed several ways to generate flexible continuous distributions from conventional continuous distributions. The related models have attracted statisticians of all stripes for various applications in physics, biology, medicine, finance, economy and engineering. From the mathematical point of view, these generalized distributions often belong to specific families defined from the parametric transformation of a parent cumulative distribution function (CDF). Then, the possible values of the new parameter(s) can significantly improve the statistical capabilities of the parent distribution, positively affecting the central and dispersion parameters, asymmetry, kurtosis and weight on the tails. Examples of such families are the exp-G family [35], Weibull-G family [20], Topp-Leone generated (TL-G) family [11], a new extended alpha power transformed-G [6], a new alpha power transformed-G [29], new power TL-G [17], type II general inverse exponential-G [40], truncated inverted Kumaraswamy-G [16], exponentiated truncated inverse Weibull-G [14], odd generalized NH-G [5], type II power TL-G [18] and others.

A modern approach consists in defining families of continuous distributions through the use of trigonometric transformation, parametric or not. This approach was launched by Kumar et al. [45] and Souza [67] with the use of the sine function, creating the sine-G family. On the basis of a parent continuous distribution with CDF and probability density function (PDF) given as $H(x)$ and $h(x)$, respectively, the sine-G family is obtained with the CDF and PDF defined by

$$\begin{aligned} F(x) &= \sin\left(\frac{\pi}{2}H(x)\right), \\ f(x) &= \frac{\pi}{2}h(x)\cos\left(\frac{\pi}{2}H(x)\right), \quad x \in \mathbb{R}, \end{aligned} \quad (1.1)$$

respectively. In addition, this trigonometric family possesses the following hazard rate function (HRF):

$$\kappa(x) = \frac{\pi}{2}h(x)\tan\left[\frac{\pi}{4}(1 + H(x))\right], \quad x \in \mathbb{R}.$$

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The advantages of the sine-G family are numerous (see [70]). One can evoke the mathematical simplicity of its probabilistic functions, its acceptable complexity of parameter(s) coinciding with that of the parent distribution, the use of the sine function makes it possible to define original trigonometric distributions through the “deformation” of any classic distribution and, in some ways, the sine-G family fills a certain lack of trigonometric distributions in the literature.

Motivated by the success of the sine-G family, several trigonometric families have seen the light, with different modeling aims. The most cited are the cos-G family by Souza et al. [67,71], sec-G family by Souza et al. [67,69], tan-G family by Souza et al. [67,68] and Ampadu [15], T-X-tan-G family by Al-Mofleh [10], N sine-G family by Mahmood et al. [50], cosine-sine-G family by Chesneau et al. [22], arcsine exponentiated-X family by Wenjing et al. [73], truncated Cauchy power-G family by Aldahlan et al. [12], sine Kumaraswamy-G family by Chesneau and Jamal [23] and TransSC-G family by Jamal and Chesneau [39].

More conventional and popular, the TL-G family studied by Al-Shomrani et al. [11] is based on a special power-polynomial transformation. From a parent continuous distribution with CDF and PDF specified by $G(x)$ and $g(x)$, respectively, it is defined by the following CDF and PDF:

$$\begin{aligned} H(x) &= (1 - \tilde{G}(x)^2)^\lambda, \\ h(x) &= 2\lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1}, \quad x \in \mathbb{R}, \end{aligned} \quad (1.2)$$

where $\lambda > 0$ refers to a shape parameter. The HRF of the TL-G family is obtained as

$$r(x) = \frac{2\lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1}}{1 - (1 - \tilde{G}(x)^2)^\lambda}, \quad x \in \mathbb{R}.$$

The advantages of the TL-G family include the simplicity and pliancy of its functions, the following notable first-order stochastic dominance property involving the CDF of the exp-G family: $H(x) \geq G(x)^\lambda$, $x \in \mathbb{R}$, and to have the ability to generate flexible models thanks to the additional parameter λ . Hence, it is motivated by offering a suitable alternative to the exp-G family, while keeping a similar degree of simplicity. Further discussions on the qualities of the TL-G family can be found in previous studies [64], [49], [30] and [37].

In this paper, we propose a quite logical alliance of the aforementioned two families; we develop a new family of sine generated distributions by considering the TL-G family as parent in the sine-G family. It refers to the sine Topp-Leone-G (STL-G) family. We show that the STL-G family can produce original trigonometric

distributions with attractive properties, in both theoretical and practical senses. Diverse mathematical results are proved, such as stochastic ordering properties, mathematical treatment of the mode(s), manageable expansions for the PDF, expressions of characteristic measures in a tractable way and the essential theory on the model parameters based on maximization of the likelihood function. Then, after a numerical check of the efficiency of certain STL-G models through simulated data, two different data sets are considered and analyzed. As a notable applied result, we show that the STL-G models can have significantly better fits to the sine-G models defined with the same parent, and so much more models. In particular, by considering the famous bladder cancer data reported by Lee and Wang [47], we show that the special STL-G model defined with the inverse Lomax model as parent is more adequate than 21 well-referenced competitors of the literature.

The rest of the paper is organized as follows. In Section 2, the functions involved in the STL-G family are described and some special distributions are discussed. Analytical, probabilistic and statistical results of the STL-G family are provided in Section 3. The parametric inference is performed in Section 4, illustrated by a numerical study. Practical applications are developed in Section 5. Some concluding notes are proposed in Section 6.

2 The STL-G family

The presentation of the STL-G family is the object of this section.

2.1 Definition

From a parent continuous distribution with CDF and PDF specified by $G(x)$ and $g(x)$, the STL-G family is designated by the CDF given as

$$F_{\text{STL-G}}(x) = \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right], \quad x \in \mathbb{R}. \quad (2.1)$$

Thus, it is constructed from the insertion of the CDF $H(x)$ in (1.2) into the CDF $F(x)$ given in (1.1). We have in mind to take advantage of the respective elasticity of the sine-G and TL-G families for the elaboration of new trigonometric statistical models. The notable features of the STL-G family will be discussed later.

As a central function, the corresponding PDF is

$$f_{\text{STL-G}}(x) = \pi \lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \times \cos \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^{\lambda} \right], \quad x \in \mathbb{R}.$$

Among others, the characteristics and shape properties of this PDF are crucial to understand the capabilities of the related STL-G model for data fitting. As complementary reliability functions, the survival function, HRF and reversed HRF of the STL-G family are obtained as follows:

$$\begin{aligned} \bar{F}_{\text{STL-G}}(x) &= 1 - \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^{\lambda} \right], \\ h_{\text{STL-G}}(x) &= \frac{\pi \lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \cos \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^{\lambda} \right]}{1 - \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^{\lambda} \right]} \end{aligned}$$

and

$$\tau_{\text{STL-G}}(x) = \pi \lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \times \cot \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^{\lambda} \right], \quad x \in \mathbb{R}, \quad (2.2)$$

respectively. These functions can be involved in many mathematical treatments of the family. In particular, the shape properties of the HRF can inform on some features of the related STL-G model, as described in detail by Aarset [1].

2.2 Special three-parameter distributions of the STL-G Family

Now, we pay particular attention to the three-parameter distributions of the STL-G family defined with the following pliant parent distributions: the inverse Lomax distribution proposed by Kleiber and Kotz [43], exponentiated exponential distribution introduced by Gupta and Kundu [35] and exponentiated Lindley distribution developed by Nadarajah et al. [57]. These

parent distributions are described through their CDFs and PDFs in Table 1.

The special distributions of the STL-G family corresponding to the parent distributions are presented in Table 1.

First special distribution: the sine Topp-Leone inverse Lomax (STLIL) distribution with CDF and PDF expressed as

$$F_{\text{STLIL}}(x) = \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{\beta}{x} \right)^{-\alpha} \right)^2 \right]^{\lambda} \right\}, \quad x > 0$$

and

$$\begin{aligned} f_{\text{STLIL}}(x) &= \frac{\pi \beta \lambda \alpha}{x^2} \left(1 + \frac{\beta}{x} \right)^{-\alpha-1} \left(1 - \left(1 + \frac{\beta}{x} \right)^{-\alpha} \right) \\ &\times \left[1 - \left(1 - \left(1 + \frac{\beta}{x} \right)^{-\alpha} \right)^2 \right]^{\lambda-1} \\ &\times \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{\beta}{x} \right)^{-\alpha} \right)^2 \right]^{\lambda} \right\}, \\ &x > 0, \end{aligned} \quad (2.3)$$

respectively. The HRF can be expressed similarly via (2.2). The STLIL distribution can also be viewed as the special distribution of the sine family taking into account the modern Topp-Leone inverse Lomax distribution studied in ref. [38] as parent. To our knowledge, the STLIL distribution is the pioneer trigonometric version of this distribution. For an illustrated analysis, Figure 1 shows the diversity of the shapes presented by $f_{\text{STLIL}}(x)$ and $h_{\text{STLIL}}(x)$, varying the values of α , β and λ .

Figure 1 shows that the STLIL distribution is unimodal. Also, “decreasing,” “plate and spike bell” shapes for $f_{\text{STLIL}}(x)$, and “increasing,” “near constant,” “decreasing,” “reversed J” and “reverse U” shapes for $h_{\text{STLIL}}(x)$ are shown. These observations indicate that the STLIL distribution can be a good candidate to model various lifetime phenomena through the use of data.

Table 1: Examples of three pliant distributions than can be considered as parents in the STL-G family, all with two parameters and support $(0, +\infty)$

Parent distribution	Parameters	CDF ($G(x)$)	PDF ($g(x)$)
Inverse Lomax	(α, β)	$\left(1 + \frac{\beta}{x} \right)^{-\alpha}$	$\frac{\beta \alpha}{x^2} \left(1 + \frac{\beta}{x} \right)^{-\alpha-1}$
Exponentiated exponential	(δ, γ)	$(1 - e^{-x/\gamma})^{\delta}$	$\frac{\delta}{\gamma} e^{-x/\gamma} (1 - e^{-x/\gamma})^{\delta-1}$
Exponentiated Lindley	(a, b)	$\left[1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right]^b$	$\frac{ba^2(1+x)e^{-ax}}{a+1} \left[1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right]^{b-1}$

Figure 1: Selected curves of the PDF and HRF of the STLIL distribution.

Second special distribution: the sine Topp-Leone exponentiated exponential (STLEE) distribution with CDF and PDF specified by

$$F_{\text{STLEE}}(x) = \sin \left\{ \frac{\pi}{2} [1 - (1 - (1 - e^{-x/\gamma})^\delta)^2]^\lambda \right\}, \quad x > 0$$

and

$$f_{\text{STLEE}}(x) = \frac{\pi\lambda\delta}{\gamma} e^{-x/\gamma} (1 - e^{-x/\gamma})^{\delta-1} (1 - (1 - e^{-x/\gamma})^\delta) \times [1 - (1 - (1 - e^{-x/\gamma})^\delta)^2]^\lambda \times \cos \left\{ \frac{\pi}{2} [1 - (1 - (1 - e^{-x/\gamma})^\delta)^2]^\lambda \right\}, \quad x > 0,$$

respectively. The HRF can be expressed similarly via (2.2). One can note that, when $\delta = 1$, we get the sine

exponentiated exponential distribution with parameters $\gamma/2$ and λ . In addition, if $\lambda = 1$, we get the sine exponential distribution with parameter $\gamma/2$ as introduced by Kumar et al. [45]. Figure 2 shows the diverse possible shapes of $f_{\text{STLEE}}(x)$ and $h_{\text{STLEE}}(x)$, varying the values of δ , γ and λ .

Figure 2 shows that the STLEE distribution is unimodal. Also, this figure reveals “decreasing” and “more or less pronounced bell” shapes for $f_{\text{STLEE}}(x)$, and “increasing,” “near constant,” “decreasing,” “reversed J” and “U” shapes for $h_{\text{STLEE}}(x)$. This flexibility of shapes is clearly a plus for the STLEE distribution, for modeling purposes in particular.

Third special distribution: The sine Topp-Leone Lindley (STLELi) distribution with CDF and PDF defined by

Figure 2: Selected curves of the PDF and HRF of the STLEE distribution.

$$F_{\text{STLELi}}(x) = \sin \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right)^b \right)^{\lambda} \right] \right\}$$

$$x > 0$$

and

$$f_{\text{STLELi}}(x) = \frac{\pi b \lambda a^2 (1+x) e^{-ax}}{a+1}$$

$$\times \left(1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right)^{b-1}$$

$$\times \left(1 - \left(1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right)^{\beta} \right)$$

$$\times \left[1 - \left(1 - \left(1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right)^b \right)^{\lambda-1} \right]$$

$$\times \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 - \left(1 + \frac{ax}{a+1} \right) e^{-ax} \right)^b \right)^{\lambda} \right] \right\},$$

$$x > 0,$$

respectively. The HRF can be expressed similarly via (2.2). Figure 3 shows the variety of shapes presented by $f_{\text{STLELi}}(x)$ and $h_{\text{STLELi}}(x)$, varying the values of a , b and λ .

Figure 3 shows “decreasing” and various “bell” shapes for $f_{\text{STLELi}}(x)$, and “increasing,” “near constant,” “decreasing,” “reversed J” and “U” shapes for $h_{\text{STLELi}}(x)$, all of them being desirable for data analysis aims.

These three special distributions are evidence that the STL-G family can generate very flexible distributions, with great interest for statistical modeling. In our data analysis carried out in Section 5, we subjectively limit our attention to the STLIL model.

3 Some properties

Various properties of the STL-G family are now described, specifically stochastic ordering, equivalences, mode(s), quantile function with discussions on the skewness and kurtosis, expansion of the PDF, various moments and finally end with order statistics.

3.1 Stochastic ordering

The STL-G family satisfies the following first-order stochastic ordering properties.

- Let $F_{\text{STL-G}}(x; \lambda) = F_{\text{STL-G}}(x)$ be defined as (2.1). Then, when $\lambda_2 \geq \lambda_1$, it is immediate that, for any $x \in \mathbb{R}$,

$$F_{\text{STL-G}}(x; \lambda_2) \leq F_{\text{STL-G}}(x; \lambda_1).$$

- By applying the inequality $(2/\pi)y \leq \sin(y)$ for $y \in [0, \pi/2]$, we get, for any $x \in \mathbb{R}$,

$$F_{\text{TL-G}}(x) \leq F_{\text{STL-G}}(x),$$

where $F_{\text{TL-G}}(x) = H(x)$ is the CDF of the TL-G family as specified in (1.2).

- Since $H(x) = G(x)^\lambda (2 - G(x))^\lambda \geq G(x)^\lambda$, we also have the following inequalities:

$$F_{\text{E-G}}(x) \leq F_{\text{SE-G}}(x) \leq F_{\text{STL-G}}(x),$$

where $F_{\text{E-G}}(x) = G(x)^\lambda$ is the CDF of the exp-G family, and $F_{\text{SE-G}}(x) = \sin[(\pi/2)G(x)^\lambda]$ refers to the CDF of the sine exp-G distribution, which is a natural extension of the former sine-G family.

Figure 3: Selected curves of the PDF and HRF of the STLELi distribution.

- A refinement is given by noting that $2 - y \geq (1 + y)/2$ for $y \in [0, 1]$, implying that $H(x) = G(x)^\lambda(2 - G(x))^\lambda \geq G(x)^\lambda(1 + G(x))^\lambda/2^\lambda$ and

$$F_{\text{GNS-G}}(x) \leq F_{\text{STL-G}}(x),$$

where $F_{\text{GNS-G}}(x) = \sin[\pi 2^{\lambda-1} G(x)^\lambda(1 + G(x))^\lambda]$ is the CDF of an exponentiated version of the CDF of the N sine-G family, both coinciding with $\lambda = 1$.

Hence, for given parent distribution and λ , these inequalities illustrate the fact that a random variable (RV) having the CDF of the STL-G family is more probably less than a given value x , than any RV having the CDF of the other mentioned families. Thus, in this sense, the STL-G family offers an alternative to them.

3.2 Equivalences

The analysis of the equivalences of the functions of the STL-G family is of importance to understand the possibility of the related model in data fitting, mainly for the tails of the related PDF. When $G(x)$ tends to 0, the following equivalences hold:

$$\begin{aligned} F_{\text{STL-G}}(x) &\sim \pi 2^{\lambda-1} G(x)^\lambda, \\ f_{\text{STL-G}}(x) &\sim \pi \lambda 2^{\lambda-1} g(x) G(x)^{\lambda-1}, \\ h_{\text{STL-G}}(x) &\sim \pi \lambda 2^{\lambda-1} g(x) G(x)^{\lambda-1}. \end{aligned}$$

When $G(x)$ tends to 1, we have

$$\begin{aligned} F_{\text{STL-G}}(x) &\sim 1 - \frac{\pi^2}{8} \lambda^2 (1 - G(x))^4, \\ f_{\text{STL-G}}(x) &\sim \frac{\pi^2}{2} \lambda^2 g(x) (1 - G(x))^3, \\ h_{\text{STL-G}}(x) &\sim 4g(x) (1 - G(x))^{-1}. \end{aligned}$$

Naturally, we see that the choice of the parent distribution is determinant for the possible limits. Also, in this regard, the parameter λ is determinant when $G(x)$ tends to 0.

As an application, in the context of the STLIL distribution defined with the inverse Lomax as parent, the following equivalences hold. When x tends to 0, we have

$$\begin{aligned} F_{\text{STLIL}}(x) &\sim \pi 2^{\lambda-1} \beta^{-\alpha\lambda} x^{\alpha\lambda}, \\ f_{\text{STLIL}}(x) &\sim \pi 2^{\lambda-1} \beta^{-\alpha\lambda} \alpha \lambda x^{\alpha\lambda-1}, \\ h_{\text{STLIL}}(x) &\sim \pi 2^{\lambda-1} \beta^{-\alpha\lambda} \alpha \lambda x^{\alpha\lambda-1}. \end{aligned}$$

When x tends to $+\infty$, we get

$$\begin{aligned} F_{\text{STLIL}}(x) &\sim 1 - \frac{\pi^2}{8} \lambda^2 \alpha^4 \beta^4 x^{-4}, \\ f_{\text{STLIL}}(x) &\sim \frac{\pi^2}{2} \lambda^2 \alpha^4 \beta^4 x^{-3}, \\ h_{\text{STLIL}}(x) &\sim 4x^{-1}. \end{aligned}$$

In particular, when x tends to 0 and $\alpha\lambda < 1$, $f_{\text{STLIL}}(x)$ and $h_{\text{STLIL}}(x)$ tend to $+\infty$, when $\alpha\lambda = 1$, $f_{\text{STLIL}}(x)$ and $h_{\text{STLIL}}(x)$ tend to $\pi 2^{\lambda-1} \beta^{-1}$, and when $\alpha\lambda > 1$, $f_{\text{STLIL}}(x)$ and $h_{\text{STLIL}}(x)$ tend to 0. When x tends to $+\infty$, $f_{\text{STLIL}}(x)$ and $h_{\text{STLIL}}(x)$ tend to 0 with a polynomial decay for all the possible values of the parameters.

3.3 Mode(s)

A mode of the STL-G family, say x_m , belongs to the arguments of the maxima of $f_{\text{STL-G}}(x)$. Among others, it satisfies $d \log [f_{\text{STL-G}}(x)]/dx|_{x=x_m} = 0$, where

$$\begin{aligned} \frac{d}{dx} \log [f_{\text{STL-G}}(x)] &= \frac{dg(x)/dx}{g(x)} - \frac{g(x)}{\tilde{G}(x)} \\ &\quad + 2(\lambda - 1) \frac{g(x)\tilde{G}(x)}{1 - \tilde{G}(x)^2} \\ &\quad - \pi \lambda g(x)\tilde{G}(x)(1 - \tilde{G}(x)^2)^{\lambda-1} \\ &\quad \times \tan \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \end{aligned}$$

and $d^2 \log [f_{\text{STL-G}}(x)]/dx^2|_{x=x_m} < 0$. The considered equations are complex from an analytical point of view; the mathematical expression of x_m is not available but can be calculated with a high degree of precision using any mathematical software when all the quantities involved are known.

3.4 Quantile function

The inversion of $F_{\text{STL-G}}(x)$ yields the quantile function of the STL-G family. After some developments, we arrive at

$$\begin{aligned} Q_{\text{STL-G}}(u) &= Q_G \left\{ 1 - \sqrt{1 - \left(\frac{2}{\pi} \arcsin(u) \right)^{1/\lambda}} \right\}, \quad (3.1) \\ u &\in (0, 1), \end{aligned}$$

where $Q_G(u)$ denotes the quantile function of the parent distribution. From this expression, one can derive the quartiles: $Q_1 = Q_{\text{STL-G}}(1/4)$, $Q_2 = Q_{\text{STL-G}}(1/2)$ and $Q_3 = Q_{\text{STL-G}}(3/4)$, the octiles and several important measures of skewness/asymmetry and kurtosis of the

STL-G family. For instance, one can present the MacGillivray skewness introduced by [48], and given as

$$\rho(u) = \frac{Q_{\text{STL-G}}(1-u) + Q_{\text{STL-G}}(u) - 2Q_2}{Q_{\text{STL-G}}(1-u) - Q_{\text{STL-G}}(u)}, \quad u \in (0, 1).$$

It is useful to understand the role of the involved parameters in the skewness; the more the shapes of $\rho(u)$ differ, the more flexible the asymmetry. Alternatively, one can consider the Galton skewness introduced by [33] and defined by $S = \rho(1/4)$. The related distribution is left-skewed, symmetrical or right-skewed depending on $S < 0$, $S = 0$ or $S > 0$, respectively. As kurtosis measure, one can consider the Moors kurtosis proposed in ref. [55], and expressed as

$$K = \frac{Q_{\text{STL-G}}(7/8) - Q_{\text{STL-G}}(5/8) + Q_{\text{STL-G}}(3/8) - Q_{\text{STL-G}}(1/8)}{Q_3 - Q_1},$$

where K measures the weight of the tails; a high value for K implies heavy tails, and a small value for K implies light tails. Also, one can use $Q_{\text{STL-G}}(u)$ to generate values for special distributions of the family.

In the setting of the STLIL distribution, the quantile function can be expressed as

$$Q_{\text{STLIL}}(u) = \beta \left[\left\{ 1 - \sqrt{1 - \left(\frac{2}{\pi} \arcsin(u) \right)^{1/\lambda}} \right\}^{-1/\alpha} - 1 \right]^{-1}, \quad (3.2)$$

$$u \in (0, 1).$$

As illustrative examples, the Galton skewness and Moors kurtosis of the STLIL distribution are plotted in Figures 4–6.

Figures 4–6 show that the STLIL distribution is quite right-skewed, with a varying kurtosis. For the considered

values, certain monotonic structures appear, showing the versatility of these measures.

3.5 Expanded forms

Here, we derive an expansion of the PDF of the STL-G family involving transformations of the parent function. We assume that integration and differentiation term by term under the infinite sum are mathematically possible. The power series expansion for the sine function gives

$$F_{\text{STL-G}}(x) = \sum_{i=0}^{+\infty} \frac{(-1)^i}{(2i+1)!} \left(\frac{\pi}{2} \right)^{2i+1} (1 - \tilde{G}(x)^2)^{\lambda(2i+1)}.$$

By noting that $G(x)/2 \in (0, 1)$, the binomial formula (generalized version) gives

$$\begin{aligned} (1 - \tilde{G}(x)^2)^{\lambda(2i+1)} &= G(x)^{\lambda(2i+1)} (2 - G(x))^{\lambda(2i+1)} \\ &= \sum_{j=0}^{+\infty} \binom{\lambda(2i+1)}{j} 2^{\lambda(2i+1)-j} (-1)^j G(x)^{\lambda(2i+1)+j}. \end{aligned}$$

Therefore, by putting the aforementioned expansions together, we arrive at

$$F_{\text{STL-G}}(x) = \sum_{i,j=0}^{+\infty} v_{i,j} F_{\text{E-G}}(x; \lambda(2i+1) + j),$$

where

$$v_{i,j} = \frac{(-1)^i}{(2i+1)!} \left(\frac{\pi}{2} \right)^{2i+1} \binom{\lambda(2i+1)}{j} 2^{\lambda(2i+1)-j} (-1)^j$$

and $F_{\text{E-G}}(x; \zeta) = G(x)^\zeta$. Note that $F_{\text{E-G}}(x; \zeta)$ is the CDF of the exp-G family with parameter ζ .

Figure 4: Two dimensional plots of S and K of the STLIL distribution for $\beta = 5$ and $\alpha, \lambda \in (0, 4)$.

Figure 5: Two dimensional plots of S and K of the STLIL distribution for $\lambda = 5$ and $\alpha, \beta \in (0, 4)$.

Figure 6: Two dimensional plots of S and K of the STLIL distribution for $\alpha = 4$ and $\lambda, \beta \in (0, 4)$.

Therefore, by differentiation with respect to x , we get

$$f_{\text{STL-G}}(x) = \sum_{i,j=0}^{+\infty} v_{i,j} f_{\text{E-G}}(x; \lambda(2i+1) + j), \quad (3.3)$$

where $f_{\text{E-G}}(x; \zeta) = \zeta g(x) G(x)^{\zeta-1}$. Note that $f_{\text{E-G}}(x; \zeta)$ corresponds to the PDF of the exp-G family with parameter ζ .

As a result of this manageable extended form, many mathematical properties of the STL-G family involving its PDF come directly from those of the exp-G family.

3.6 Some moments

Now, we consider an RV X having the PDF $f_{\text{STL-G}}(x)$. A plethora of measures and functions related to X can be expressed under the following form: $E[\Phi(X)]$, where E

denotes the expectation and $\Phi(x)$ denotes a certain function, provided that $E[\Phi(X)]$ exists. We may refer to the book by Cordeiro et al. [26] in this regard. Some essentials of them are described in 2.

In full generality, $E[\Phi(X)]$ can be expressed by the transfer theorem as

$$E[\Phi(X)] = \int_{\mathbb{R}} \Phi(x) f_{\text{STL-G}}(x) dx$$

or, equivalently,

$$E[\Phi(X)] = \int_0^1 \Phi[Q_{\text{STL-G}}(u)] du.$$

Such integrals can be determined by applying standard integration techniques, depending on the tractability of $\Phi(x)$ and $f_{\text{STL-G}}(x)$, or $Q_{\text{STL-G}}(x)$. A more optimistic

Table 2: Standard measures and functions that are written under the form $E[\Phi(X)]$

$\Phi(x)$	Name of $E[\Phi(X)]$	Usual notation
x	Mean	μ
$x^2 - \mu^2$	Variance	σ^2
x^r	r th raw moment	μ_r'
$x^r 1_{x \leq t}$	r th incomplete moment at t	$\mu_r'(t)$
e^{tx}	Moment generating function at t	$M(t)$
e^{itx}	Characteristic function at t	$\varphi(t)$
$\left(\frac{x-\mu}{\sigma}\right)^3$	(Moment) skewness	C_3
$\left(\frac{x-\mu}{\sigma}\right)^4$	(Moment) kurtosis	C_4
$ x - \mu $	Mean deviation about μ	δ_1
$ x - Q_2 $	Mean deviation about Q_2	δ_2
$[1 - f_{\text{STL-G}}(x)^{q-1}]/(q-1)$	Tsallis entropy	T_q

approach is to use the expansion of $f_{\text{STL-G}}(x)$ presented in (3.3), which gives

$$E[\Phi(X)] = \sum_{i,j=0}^{+\infty} v_{i,j} \int_{\mathbb{R}} \Phi(x) f_{\text{E-G}}(x; \lambda(2i+1) + j) dx, \quad (3.4)$$

knowing that the integral $\int_{\mathbb{R}} \Phi(x) f_{\text{E-G}}(x; \lambda(2i+1) + j) dx$ has already been expressed for various $\Phi(x)$ and a large class of parent distributions in the literature. For practical manipulations of $E[\Phi(X)]$, one can take into account the finite sum expansion given as

$$E[\Phi(X)] \approx \sum_{i,j=0}^J v_{i,j} \int_{\mathbb{R}} \Phi(x) f_{\text{E-G}}(x; \lambda(2i+1) + j) dx, \quad (3.5)$$

where J denotes a large integer.

As application, let us consider the framework of the STLIL distribution and discuss the r th incomplete moments of X at $t \in \mathbb{R}$ given as $\mu_r'(t) = E(X^r 1_{X \leq t})$ with possibly negative r , referring to the “negative” $(-r)$ th incomplete moment in this case. By using (2.3), we get

$$\begin{aligned} \mu_r'(t) &= \int_{-\infty}^t x^r f_{\text{STLIL}}(x) dx = \pi \beta \lambda \alpha \int_0^t x^{r-2} \left(1 + \frac{\beta}{x}\right)^{-\alpha-1} \\ &\quad \times \left(1 - \left(1 + \frac{\beta}{x}\right)^{-\alpha}\right) \times \left[1 - \left(1 - \left(1 + \frac{\beta}{x}\right)^{-\alpha}\right)^2\right]^{\lambda-1} \\ &\quad \times \cos \left\{ \frac{\pi}{2} \left[1 - \left(1 - \left(1 + \frac{\beta}{x}\right)^{-\alpha}\right)^2\right]^{\lambda} \right\} dx. \end{aligned}$$

Obviously, the analytical calculation of this integral is unexpected. One can however envisage a numerical study if all the quantities involved are fixed. Another solution, by applying the expansion approach, is

$$\mu_r'(t) = \sum_{i,j=0}^{+\infty} v_{i,j} U_{i,j}^{[r]}(t),$$

where $U_{i,j}^{[r]}(t) = \int_0^t x^r f_{\text{EIL}}(x; \lambda(2i+1) + j) dx$, where $f_{\text{EIL}}(x; \lambda(2i+1) + j)$ denotes the PDF of the EIL distribution with power parameter $\lambda(2i+1) + j$, also corresponding to the IL distribution with parameters $\alpha[\lambda(2i+1) + j]$ and β . By performing the changes of variables $y = \beta/x$ and $z = (1+y)^{-1}$ and assuming that $r < 1$ and $\lambda\alpha + r > 0$ (r being possibly negative), we get

$$\begin{aligned} U_{i,j}^{[r]}(t) &= \beta \alpha [\lambda(2i+1) + j] \int_0^t x^{r-2} \left(1 + \frac{\beta}{x}\right)^{-\alpha[\lambda(2i+1)+j]-1} dx \\ &= \alpha [\lambda(2i+1) + j] \beta^r \int_{\beta/t}^{+\infty} y^{-r} (1+y)^{-\alpha[\lambda(2i+1)+j]-1} dy \\ &= \alpha [\lambda(2i+1) + j] \beta^r \\ &\quad \times B_{(1+\beta/t)^{-1}}(1-r, \alpha[\lambda(2i+1) + j] + r), \end{aligned}$$

where $B_x(u, v) = \int_0^x t^{u-1} (1-t)^{v-1} dt$, $x \in [0, 1]$, $u, v > 0$, denotes the incomplete beta function. That is,

$$\begin{aligned} \mu_r'(t) &= \sum_{i,j=0}^{+\infty} v_{i,j} \alpha [\lambda(2i+1) + j] \beta^r \\ &\quad \times B_{(1+\beta/t)^{-1}}(1-r, \alpha[\lambda(2i+1) + j] + r). \end{aligned}$$

One can derive an approximation of $\mu_r'(t)$, a plethora of functions depending on $\mu_r'(t)$ (mean residual function, mean waiting time and curve of Lorenz), as well as (negative) moments of X by $t \rightarrow +\infty$, and so on. Let us notice that standard skewness and kurtosis measures defined with moments are not well defined for the STLIL distribution. Thus, for such an investigation, the MacGillivray or Galton skewness and Moors kurtosis measures as applied in Section 3.4 are essential.

3.7 Order statistics

Some basic notions on the distributional properties of the order statistics of the STL-G family are presented below. Let $X_1, \dots, X_n$ be n independent RVs having the CDF and PDF given as $F_{\text{STL-G}}(x)$ and $f_{\text{STL-G}}(x)$, respectively. Now, let us arrange these RVs in an ascending order, say $X_{1:n}, X_{2:n}, \dots, X_{n:n}$, satisfying $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ almost surely. Then, these RVs are called ordered

statistics. They find many applications in survival and reliability analyses. We may refer to ref. [27] for theory and practice of these mathematical objects. As the alpha distributional result, the i th order statistic has the CDF denoted by $F_{i:n}(x)$, which can be written as

$$\begin{aligned} F_{i:n}(x) &= \frac{1}{B_1(i, n-i+1)} \sum_{j=0}^{n-i} \frac{(-1)^j}{i+j} \binom{n-i}{j} F_{\text{STL-G}}(x)^{i+j} \\ &= \frac{1}{B_1(i, n-i+1)} \sum_{j=0}^{n-i} \frac{(-1)^j}{i+j} \binom{n-i}{j} \\ &\quad \times \left\{ \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \right\}^{i+j}, \quad x \in \mathbb{R}, \end{aligned}$$

where $B_1(i, n-i+1)$ is the complete beta function $B_1(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt$ taken at $u=i$ and $v=n-i+1$, which can be reduced to $(i-1)!(n-i)!/n!$.

The corresponding PDF is easily deduced as follows:

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B_1(i, n-i+1)} f_{\text{STL-G}}(x) \\ &\quad \times \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F_{\text{STL-G}}(x)^{i+j-1} \\ &= \frac{\pi \lambda}{B_1(i, n-i+1)} g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \\ &\quad \times \cos \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \\ &\quad \times \left\{ \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \right\}^{i+j-1}, \quad x \in \mathbb{R}. \end{aligned}$$

In particular, by taking $i=1$, we get the PDF of the first-order statistics specified by

$$\begin{aligned} f_{1:n}(x) &= n\pi \lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \\ &\quad \times \cos \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \\ &\quad \times \left\{ 1 - \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \right\}, \quad x \in \mathbb{R}^{n-1}. \end{aligned}$$

Also, by taking $i=n$, the PDF of the largest order statistics is given by

$$\begin{aligned} f_{n:n}(x) &= n\pi \lambda g(x) \tilde{G}(x) (1 - \tilde{G}(x)^2)^{\lambda-1} \\ &\quad \times \cos \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \\ &\quad \times \left\{ \sin \left[\frac{\pi}{2} (1 - \tilde{G}(x)^2)^\lambda \right] \right\}^{n-1}, \quad x \in \mathbb{R}. \end{aligned}$$

From these expressions, one can envisage the determination of some characteristics of $X_{i:n}$. Alternatively, as for the former PDF, series expansions are possible. Indeed, by applying [36, Item 0.314], for any integer s and $x \in (0, \pi/2)$, the following expansion holds:

$[\sin(x)]^s = \sum_{k=0}^{+\infty} c_{s,k} x^{2k+s}$, where $c_{s,0} = 1$ and, for any $m \geq 1$,

$$c_{s,m} = (1/m) \sum_{\ell=1}^m (\ell(s+1) - m) (-1)^\ell [(2\ell+1)!]^{-1} c_{s,m-\ell}.$$

Therefore, following the same development of those to obtain (3.3), we have

$$F_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \eta_{j,k,\ell}^{[i]} F_{\text{EG}}(x; \lambda(2k+i+j) + \ell), \quad x \in \mathbb{R},$$

where

$$\begin{aligned} \eta_{j,k,\ell}^{[i]} &= \frac{1}{B_1(i, n-i+1)} \frac{(-1)^j}{i+j} \binom{n-i}{j} c_{i+j,k} \left(\frac{\pi}{2} \right)^{2k+i+j} \\ &\quad \times \binom{\lambda(2k+i+j)}{\ell} 2^{\lambda(2k+i+j)-\ell} (-1)^\ell. \end{aligned}$$

Then, we have

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \eta_{j,k,\ell}^{[i]} f_{\text{EG}}(x; \lambda(2k+i+j) + \ell), \quad x \in \mathbb{R}.$$

Thus, the PDF of $X_{i:n}$ can be written as an infinite mixture of PDFs of the exp-G family. We can use this property to derive various moment measures and functions, as performed in Section 3.6. For instance, the r th incomplete moment of $X_{i:n}$ at $t \in \mathbb{R}$ can be expressed as

$$\mu'_{r,i:n}(t) = E(X_{i:n}^r \mathbf{1}_{X_{i:n} \leq t}) = \sum_{j=0}^{n-i} \sum_{k=0}^{+\infty} \sum_{\ell=0}^{+\infty} \eta_{j,k,\ell}^{[i]} V_{j,k,\ell}^{[r,i:n]}(t),$$

where $V_{j,k,\ell}^{[r,i:n]}(t) = \int_{-\infty}^t x^r f_{\text{EG}}(x; \lambda(2k+i+j) + \ell) dx$, which can be determined for most of the conventional parent distributions, as already done before with the inverse Lomax distribution.

4 Inference

Here, an inferential study of the STL-G models is proposed, including a simulation work.

4.1 Method

The inference on the parameters of the STL-G models can be carried out by using different methods. Here, we adopt the most used of all: the maximum likelihood (ML) method. The resulting estimators enjoy attractive asymptotic properties from which we can derive diverse

statistical objects, such as confidence regions (including intervals) and statistical tests. The basics of the method in the setting of the STL-G models are developed below. Based on a simple random sample (SRS), let $x_1, \dots, x_n$ be n independent observations from the STL-G family and $\delta = (\delta_1, \dots, \delta_q)$ be a vector with q components, with $\delta_1 = \lambda$ and the other components are supposed to be the parameter(s) of the parent distribution. For the sake of precision, let us set $\delta_* = (\delta_2, \dots, \delta_q)$ such that $\delta = (\delta_1, \delta_*)$, $g(x; \delta_*) = g(x)$, $\bar{G}(x; \delta_*) = \bar{G}(x)$ and $G(x; \delta_*) = G(x)$. Then, the total log-likelihood function for the vector δ is given by

$$\begin{aligned} L_n(\delta) = & n \log(\pi) + n \log(\lambda) + \sum_{i=1}^n \log [g(x_i; \delta_*)] \\ & + \sum_{i=1}^n \log [\bar{G}(x_i; \delta_*)] \\ & + (\lambda - 1) \sum_{i=1}^n \log(1 - \bar{G}(x_i; \delta_*)^2) \\ & + \sum_{i=1}^n \log \left\{ \cos \left[\frac{\pi}{2} (1 - \bar{G}(x_i; \delta_*)^2)^\lambda \right] \right\}. \end{aligned}$$

Then, assuming that they are unique, the ML estimates are given under a vector form as $\hat{\delta} = \arg \max_{\delta} L_n(\delta)$. The maximization of the function $L_n(\delta)$ with respect to δ can be done either directly by using any mathematical software or by solving the following system: $\partial L_n(\delta) / \partial \delta = 0$, with

$$\begin{aligned} \frac{\partial L_n(\delta)}{\partial \delta_1} = & \frac{n}{\lambda} + \sum_{i=1}^n \log(1 - \bar{G}(x_i; \delta_*)^2) \\ & - \frac{\pi}{2} \sum_{i=1}^n (1 - \bar{G}(x_i; \delta_*)^2)^\lambda \log(1 - \bar{G}(x_i; \delta_*)^2) \\ & \times \tan \left[\frac{\pi}{2} (1 - \bar{G}(x_i; \delta_*)^2)^\lambda \right] \end{aligned}$$

and, for any $k = 2, \dots, q$,

$$\begin{aligned} \frac{\partial L_n(\delta)}{\partial \delta_k} = & \sum_{i=1}^n \frac{\partial g(x_i; \delta_*) / \partial \delta_k}{g(x_i; \delta_*)} - \sum_{i=1}^n \frac{g(x_i; \delta_*)}{\bar{G}(x_i; \delta_*)} \\ & + 2(\lambda - 1) \sum_{i=1}^n \frac{g(x_i; \delta_*) \bar{G}(x_i; \delta_*)}{1 - \bar{G}(x_i; \delta_*)^2} \\ & - \pi \lambda \sum_{i=1}^n \frac{\partial G(x_i; \delta_*)}{\partial \delta_k} \bar{G}(x_i; \delta_*) (1 - \bar{G}(x_i; \delta_*)^2)^{\lambda-1} \\ & \times \tan \left[\frac{\pi}{2} (1 - \bar{G}(x_i; \delta_*)^2)^\lambda \right]. \end{aligned}$$

Under some regularity assumptions, when n tends to $+\infty$, one can show that the random sequence version of

the deterministic vector $\hat{\delta} - \delta$ tends in distribution to the multivariate normal distribution $N_q(0_q, I^{-1})$, where

$$I = \left\{ -\frac{\partial^2 L_n(\delta)}{\partial \delta_k \partial \delta_\ell} \right\}_{k, \ell=1, \dots, q} \bigg|_{\delta=\hat{\delta}}.$$

This result makes the construction of asymptotic confidence intervals (CIs) of the parameters possible. For any $k = 1, \dots, q$, the CI of δ_k at the level $100(1 - \nu)\%$ with $\nu \in (0, 1)$ is constructed with the lower bound (LB) and upper bound (UB) defined by $LB = \hat{\delta}_k - z_{1-\nu/2} \sqrt{I_{k,k}^{-1}}$ and $UB = \hat{\delta}_k + z_{1-\nu/2} \sqrt{I_{k,k}^{-1}}$, respectively, where $I_{k,k}^{-1}$ refers to the k th component in the diagonal of I^{-1} and $z_{1-\nu/2}$ is the quantile of the standard normal distribution of the value $1 - \nu/2$. For the theoretical details, see [20].

4.2 Simulation

This section is devoted to a numerical study on the convergence of the ML estimates of the parameters of both the three-parameter STLIL and STLEE models based on SRSs. The following criteria are used: mean square errors (MSEs) and (average) CIs at the level 90% or 95%, defined with their corresponding LBs and UBs, as well as their corresponding average length (AL). The software Mathematica 9 is used. The simulation procedure takes into account the following items.

- (i): Chose the STLIL or STLEE models with selected values for α , β and λ .
- (ii): A random sample of values of size $n = 50, 100, 500$ and $1,000$ is generated from the considered model.
- (iii): For each sample, the ML estimates are computed.
- (iv): Repeat items (ii) and (iii) $N = 5,000$ times. Then, the average ML estimates and corresponding MSEs are computed.
- (v): The CI object estimates, i.e., LBs, UBs and ALs, for the chosen parameters are also calculated with the same approach.
- (vi): Numerical results are given in Tables 3 and 4 for the STLIL model and Tables 5–8 for the STLEE model.

From the above tables, for both the STLIL and STLEE models, as n increases, we see that the ML estimates tend to the corresponding parameter values, and the MSEs and ALs decrease quickly to 0. These observations are coherent with the theoretical convergence properties of the ML estimators and provide guaranty on the asymptotic efficiency of the method.

Table 3: ML estimates, MSEs and CI object estimates for the STLIL model with ($\alpha = 0.5, \beta = 0.5, \lambda = 0.5$)

n	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	0.491	0.005	0.211	0.771	0.560	0.157	0.825	0.667
	0.543	0.166	−0.292	1.378	1.670	−0.452	1.538	1.990
	0.583	0.066	0.323	0.844	0.521	0.273	0.893	0.620
100	0.593	0.018	0.339	0.847	0.507	0.291	0.895	0.605
	0.669	0.099	0.040	1.298	1.258	−0.080	1.418	1.498
	0.489	0.002	0.366	0.613	0.247	0.342	0.637	0.295
500	0.472	0.009	0.336	0.608	0.273	0.310	0.635	0.325
	0.447	0.042	0.093	0.801	0.708	0.025	0.869	0.844
	0.525	0.004	0.415	0.634	0.219	0.394	0.655	0.261
1,000	0.480	0.002	0.368	0.593	0.226	0.346	0.615	0.269
	0.496	0.008	0.196	0.796	0.600	0.138	0.854	0.715
	0.491	0.002	0.414	0.569	0.155	0.399	0.583	0.184

Table 4: ML estimates, MSEs and CI object estimates for the STLIL model with ($\alpha = 0.8, \beta = 0.5, \lambda = 0.8$)

n	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	1.176	0.373	0.147	2.206	2.059	−0.050	2.403	2.453
	1.208	1.444	−0.834	3.249	4.083	−1.225	3.640	4.865
	0.974	0.340	0.140	1.807	1.667	−0.019	1.967	1.986
100	0.924	0.103	0.620	1.228	0.608	0.562	1.286	0.724
	0.647	0.145	0.179	1.115	0.936	0.090	1.205	1.115
	0.829	0.010	0.639	1.018	0.379	0.603	1.055	0.452
500	0.829	0.007	0.624	1.033	0.409	0.585	1.073	0.488
	0.554	0.039	0.226	0.881	0.655	0.164	0.944	0.780
	0.810	0.008	0.658	0.962	0.304	0.629	0.991	0.362
1,000	0.821	0.003	0.702	0.939	0.238	0.679	0.962	0.283
	0.529	0.008	0.347	0.710	0.362	0.313	0.745	0.432
	0.803	0.001	0.715	0.890	0.175	0.699	0.907	0.208

Table 5: ML estimates, MSEs and CI object estimates for the STLEE model with ($\delta = 0.5, \gamma = 0.5, \lambda = 0.5$)

n	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	0.547	0.021	0.325	0.768	0.443	0.283	0.811	0.528
	0.513	0.112	0.011	1.015	1.004	−0.085	1.111	1.197
	0.543	0.007	0.395	0.691	0.297	0.366	0.720	0.353
100	0.570	0.014	0.395	0.745	0.350	0.362	0.779	0.417
	0.639	0.133	0.171	1.107	0.936	0.082	1.197	1.115
	0.506	0.004	0.410	0.603	0.193	0.391	0.621	0.230
500	0.524	0.003	0.398	0.650	0.252	0.374	0.674	0.300
	0.559	0.026	0.226	0.893	0.667	0.162	0.956	0.794
	0.495	0.002	0.419	0.571	0.152	0.404	0.585	0.181
1,000	0.479	0.002	0.394	0.564	0.170	0.378	0.580	0.203
	0.474	0.012	0.256	0.692	0.436	0.214	0.734	0.519
	0.499	0.001	0.439	0.560	0.121	0.427	0.572	0.145

Table 6: ML estimates, MSEs and CI object estimates for the STLEE model with ($\delta = 0.9, \gamma = 0.5, \lambda = 0.5$)

<i>n</i>	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	0.893	0.181	0.331	1.455	1.124	0.223	1.563	1.339
	0.533	0.243	−0.134	1.201	1.335	−0.262	1.329	1.590
	0.586	0.040	0.407	0.765	0.358	0.372	0.799	0.427
100	0.956	0.038	0.578	1.333	0.755	0.506	1.405	0.900
	0.573	0.057	0.141	1.004	0.863	0.059	1.087	1.028
	0.516	0.003	0.426	0.606	0.181	0.408	0.624	0.215
500	0.916	0.020	0.637	1.195	0.559	0.583	1.249	0.665
	0.494	0.024	0.206	0.782	0.577	0.150	0.837	0.687
	0.513	0.002	0.448	0.598	0.150	0.434	0.612	0.178
1,000	0.958	0.019	0.720	1.196	0.475	0.675	1.241	0.567
	0.543	0.014	0.288	0.799	0.512	0.239	0.848	0.610
	0.503	0.001	0.448	0.558	0.110	0.438	0.569	0.131

Table 7: ML estimates, MSEs and CI object estimates for the STLEE model with ($\delta = 0.5, \gamma = 0.1, \lambda = 1.2$)

<i>n</i>	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	0.487	0.008	0.355	0.620	0.264	0.330	0.645	0.315
	0.156	0.027	−0.041	0.353	0.394	−0.079	0.391	0.469
	1.382	0.231	0.359	2.406	2.046	0.163	2.602	2.438
100	0.500	0.001	0.407	0.593	0.186	0.389	0.611	0.222
	0.119	0.003	0.003	0.234	0.231	−0.019	0.256	0.275
	1.415	0.136	0.756	2.074	1.319	0.629	2.201	1.571
500	0.481	0.002	0.409	0.554	0.145	0.395	0.568	0.173
	0.094	0.003	0.009	0.180	0.172	−0.008	0.197	0.204
	1.422	0.089	0.857	1.988	1.130	0.749	2.096	1.347
1,000	0.497	0.000	0.438	0.557	0.120	0.426	0.569	0.143
	0.106	0.001	0.076	0.235	0.159	0.061	0.250	0.189
	1.158	0.009	0.920	1.396	0.476	0.874	1.441	0.567

Table 8: ML estimates, MSEs and CI object estimates for the STLEE model with ($\delta = 1.5, \gamma = 0.5, \lambda = 1.5$)

<i>n</i>	ML estimate	MSE	90%			95%		
			LB	UB	AL	LB	UB	AL
50	1.771	0.523	0.711	2.831	2.120	0.508	3.034	2.526
	0.642	0.198	−0.072	1.356	1.428	−0.209	1.493	1.702
	1.715	0.654	0.676	2.755	2.080	0.476	2.954	2.478
100	1.685	0.289	0.999	2.371	1.372	0.868	2.502	1.635
	0.616	0.132	0.117	1.114	0.997	0.022	1.210	1.188
	1.614	0.274	1.132	2.096	0.963	1.040	2.188	1.148
500	1.477	0.033	1.049	1.906	0.857	0.967	1.988	1.021
	0.504	0.029	0.187	0.822	0.635	0.126	0.882	0.757
	1.562	0.101	1.174	1.949	0.775	1.100	2.023	0.924
1,000	1.522	0.019	1.170	1.875	0.704	1.103	1.942	0.839
	0.502	0.011	0.276	0.794	0.518	0.226	0.843	0.617
	1.510	0.025	1.239	1.780	0.542	1.187	1.832	0.645

5 Applications

Thanks to its remarkable mathematical and statistical properties highlighted in the previous sections, the three-parameter STLIL model is adopted as a paragon of the possible models derived for the STL-G family. Two real data sets are now considered. These data have been already analyzed in other studies, showing the applicability of various models with diverse number of parameters. We also evaluate the quality of the adjustments of the models considered and make comparisons between them. The measures of goodness-of-fits include the very standard ones: Akaike information criterion (AIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) and Cramer-von Mises (W^*). It is recognized that the smaller the values of these criteria, the more the corresponding model fits the data.

The data and considered competing models are described below.

- The first data set: survival times data. The first data set was studied by Bjerkedal [19]. It contains the survival times (in days) of 72 guinea pigs being infected with pathogenic bacteria. For this survival times data set, we compare the fit of the STLIL model with the fits of the following competitors: sine inverse Lomax (SIL) model derived by Souza [67], Topp Leone Weibull Lomax (TLWL) model derived by Jamal et al. [41], Topp-Leone Lomax (TLL) model derived by Al-Shomrani et al. [11], Weibull Lomax (WL) model derived by Tahir et al. [72], Kumaraswamy Lomax (KwL) model derived by Shams [66], beta Lomax (BL) model derived by Eugene et al. [32], exponentiated Lomax (EL) model derived by Abdul-Moniem and Abdel-Hameed [2], Gompertz Lomax (GzL) model derived by Oguntunde et al. [62], Marshall-Olkin exponential (MOE) model derived by Alice and Jose [13], Burr X-exponential (BrXE) model derived by Oguntunde et al. [61], Kumaraswamy exponential (KwE) model derived by Cordeiro and de Castro [24], beta exponential (BE) model derived by Nadarajah and Kotz [59], and Kumaraswamy Marshall-Olkin

exponential (KwMOE) model derived by George and Thobias [34].

- The second data set: Cancer patient data. The second data set describes the remission times (in months) of 128 bladder cancer patients. It was extracted from the study by Lee and Wang [47]. For this cancer patient data set, we compare the fit of the STLIL model with the fits of the following competitors: SIL model derived by Souza [67], sine Weibull (SW) model derived by Souza [67], Kumaraswamy-log-logistic (KwLL) model derived by Santana et al. [65], transmuted complementary Weibull geometric (TCWG) model derived by Afify et al. [3], Kumaraswamy exponentiated BurrXII (KwEBXII) model derived by Mead and Afify [54], beta exponentiated Burr XII (BEBXII) model derived by Mead [52], generalized inverse gamma (GIG) model derived by Mead [53], beta Fréchet (BFR) model derived by Nadarajah and Gupta [58], exponentiated transmuted generalized Rayleigh (ETGR) model derived by Afify et al. [4], transmuted modified Weibull (TMW) model derived by Khan and King [42], transmuted additive Weibull (TAW) model derived by Elbatal and Aryal [30], generalized transmuted Weibull (GTW) model derived by Nofal et al. [60], beta extended Pareto (BEP) model derived by Mahmoudi [51], exponentiated Weibull (EW) model derived by Mudholkar and Srivastava [56], exponentiated exponential (EE) model derived by Gupta and Kundu [36], Kumaraswamy Weibull (KwW) model derived by Cordeiro et al. [25], beta Weibull (BW) model derived by Lee et al. [46], gamma Weibull (GW) model derived by Provost et al. [63], Kumaraswamy gamma (KwG) model derived by de Castro et al. [28], beta gamma (BG) model derived by Kong et al. [44], and gamma gamma (GG) model.

The survival time and bladder cancer data sets are basically described in Table 9.

The two data sets mainly differ from their dispersion measures, with a high variance for the cancer patient data. They also possess different right-skewed and kurtosis natures.

Table 9: Preliminary description of the two data sets

Data set	n	Min	Max	Mean	Median	Var	Skew	Kur
Survival times	72	0.1	5.55	1.768	1.495	1.055	1.371	2.225
Cancer patient	128	0.080	79.050	9.366	6.395	109.562	3.326	16.154

Figure 7: Boxplot and TTT plot for the survival times data set.

Figure 8: Boxplot and TTT plot for the bladder cancer data set.

The boxplots and total time test (TTT) plots of the survival time and bladder cancer data sets are shown in Figures 7 and 8, respectively.

The boxplots in Figures 7 and 8 confirm that the data of the survival time and bladder cancer data sets are

right-skewed. Also, for the both data sets, the first part of the box (i.e., left of the median) is shorter than the other part, meaning that the smaller values are closer together than the greater ones. As immediate remark, the normal model is not adequate for these data.

Table 10: Estimates of the model parameters for the survival times data

Models	MLEs (and SEs)			
STLIL (α, β, λ)	1.086 (1.621)	1.147 (0.818)	4.653 (7.573)	
SIL (α, β)	8.105 (4.503)	0.223 (0.137)		
TLWL (θ, α, a, b)	0.062 (0.018)	39.763 (9.534)	0.971 (0.513)	12.220 (2.599)
TLL (α, a, b)	36.601 (3.844)	18.208 (1.127)	2.798 (0.562)	
WL (α, β, a, b)	20.737 (2.273)	2.198 (0.565)	0.255 (0.248)	1.412 (1.788)
KwL (α, β, a, b)	0.865 (3.592)	4.158 (8.508)	2.317 (0.583)	17.944 (2.790)
BL (α, β, a, b)	22.869 (5.174)	2.590 (0.337)	2.615 (0.578)	12.495 (8.510)
EL (θ, a, b)	26.889 (1.690)	26.371 (3.462)	2.868 (0.582)	
GzL (θ, α, a, b)	0.003 (0.0004)	0.167 (0.008)	0.234 (0.010)	10.869 (0.017)
MOE (α, β)	8.778 (3.555)	1.379 (0.193)		
BrXE (θ, β)	0.475 (0.060)	0.206 (0.012)		
KwE (a, b, β)	3.304 (1.106)	1.100 (0.764)	1.037 (0.614)	
BE (a, b, β)	0.807 (0.696)	3.461 (1.003)	1.331 (0.855)	
KwMOE (α, β, a, b)	0.373 (0.136)	3.478 (0.862)	3.306 (0.781)	0.299 (1.113)

Table 11: Estimates of the model parameters for the bladder cancer data

Models	MLEs (and SEs)				
STLIL (α, β, λ)	3.782 (6.451)	8.774 (6.459)	0.417 (0.644)		
SIL (α, β)	1.907 (0.362)	4.684 (1.271)			
SW (α, β)	0.992 (0.065)	0.061 (0.012)			
KwLL (α, β, a, b)	4.658 (13.163)	0.298 (0.167)	7.866 (4.493)	112.881 (243.364)	
TCWG ($\alpha, \beta, \lambda, \gamma$)	106.069 (124.800)	1.712 (0.099)	0.217 (0.610)	0.009 (0.007)	
KwEBXII (a, b, c, β, k)	2.780 (44.510)	67.636 (104.728)	0.338 (0.385)	3.083 (49.353)	0.839 (1.723)
BEBXII (a, b, c, β, k)	22.186 (21.956)	20.277 (17.296)	0.224 (0.144)	1.780 (1.076)	1.306 (1.079)
GIG (a, b, c, β, k)	2.327 (0.369)	0.0002 (0.0002)	17.931 (7.385)	0.543 (0.042)	0.001 (0.0003)
BFr (a, b, α, β)	12.526 (24.469)	33.342 (36.348)	27.753 (71.507)	0.169 (0.104)	
ETGR ($\alpha, \beta, \lambda, \delta$)	7.376 (5.389)	0.047 (0.004)	0.118 (0.260)	0.049 (0.036)	
TMW ($a, \alpha, \beta, \lambda$)	0.0002 (0.011)	0.121 (0.024)	0.896 (0.626)	0.407 (0.407)	
TAW ($a, b, \alpha, \beta, \lambda$)	0.00003 (0.006)	1.007 (0.035)	0.114 (0.032)	0.972 (0.125)	0.163 (0.280)
GTW ($a, b, \alpha, \beta, \lambda$)	2.797 (1.117)	0.013 (7.214)	0.299 (0.151)	0.654 (0.121)	0.002 (1.769)
BEP (a, b, λ, σ)	2.521 (0.511)	14.218 (0.592)	5.939 (2.182)	23.876 (12.403)	
EE (λ, σ)	1.221 (0.149)	0.121 (0.014)			
KwW (a, b, λ, σ)	8.18 (2.584)	8.805 (5.268)	0.274 (0.064)	2.348 (1.743)	
BW (a, b, λ, σ)	0.685 (0.149)	2.913 (1.918)	2.631 (1.164)	0.8 (0.499)	
GW (a, b, σ)	3.508 (0.776)	0.538 (0.063)	0.743 (0.509)		
KwG (a, b, λ, σ)	25.045 (0.535)	0.618 (0.084)	0.058 (0.008)	7.113 (0.581)	
BG (a, b, λ, σ)	0.065 (0.011)	4.236 (0.011)	19.655 (0.755)	0.384 (0.039)	
GG (a, b, σ)	1.09 (0.718)	1.078 (0.71)	8.004 (1.107)		

Let us now comment on the TTT plots. In Figure 7, since the colored TTT line is concave, the corresponding empirical HRF is increasing for the survival time data set. In Figure 8, since the colored TTT line is first concave then convex, the corresponding empirical HRF is unimodal for the bladder cancer data set. For details on the possible interpretation of the TTT plots see ref. [1].

The ML estimates of the model parameters, and their standard errors (SEs), are provided in Tables 10 and 11

Table 12: Goodness-of-fits of the considered models for the survival time data set

Models	AIC	CAIC	HQIC	A*	W*
STLIL	198.861	199.214	201.58	0.412	0.060
SIL	214.845	215.019	216.658	1.039	0.096
TLWL	209.736	210.094	213.361	0.425	0.067
TLL	212.164	212.340	214.883	0.716	0.103
WL	213.566	213.924	217.191	0.733	0.112
KwL	212.850	213.208	216.476	0.650	0.097
BL	213.815	214.173	217.440	0.715	0.105
EL	212.266	212.442	214.985	0.717	0.102
GzL	217.630	217.988	221.255	1.098	0.185
MOE	210.360	210.530	212.161	1.182	0.173
BrXE	235.303	235.502	237.104	2.900	0.522
KwE	209.422	209.773	212.120	0.743	0.112
BE	207.384	207.732	210.081	0.983	0.149
KwMOE	207.818	208.422	211.419	211.419	0.109

for the survival time and bladder cancer data sets, respectively.

Table 13: Goodness-of-fits of the considered models for the bladder cancer data set

Models	AIC	CAIC	HQIC	A*	W*
STLIL	825.934	826.127	829.41	0.178	0.027
SIL	829.000	829.096	831.318	0.374	0.044
SW	832.650	832.746	834.968	0.719	0.116
KwLL	829.531	829.857	834.167	0.317	0.049
TCWG	829.995	830.320	834.63	0.306	0.043
KwEBXII	831.651	832.143	837.445	0.320	0.048
BEBXII	841.268	841.760	855.528	0.900	0.134
GIG	839.824	840.316	854.085	2.618	0.410
BFr	842.965	843.290	854.373	1.121	0.168
ETGR	866.350	866.675	877.758	2.361	0.398
TMW	836.450	836.775	847.858	3.125	0.760
TAW	838.478	838.970	852.739	3.113	0.703
GTW	831.347	831.839	837.141	0.306	0.047
TLEP	829.36	829.789	830.05	0.3044	0.0467
BGP	835.569	846.977	835.894	0.659	0.096
EE	830.156	835.86	830.252	0.673	0.112
KwW	829.29	840.698	829.615	0.281	0.043
BW	829.398	840.806	829.723	0.289	0.044
GW	827.719	836.275	827.913	0.317	0.049
KwG	830.371	841.779	830.696	0.375	0.060
BG	830.275	841.683	830.601	0.358	0.056
GG	832.719	841.275	832.912	0.717	0.120

Figure 9: Fits of the ECDF, EPDF, PP plot and esf for the survival times data set.

From Table 10, for the survival times data, we see that the parameters α , β and λ of the STLIL model are estimated by $\hat{\alpha} = 1.086$, $\hat{\beta} = 1.147$ and $\hat{\lambda} = 4.653$, respectively. In particular, one can remark that λ is estimated far to 1, making the difference with the SIL model originally defined with $\lambda = 1$ (having completely different estimates for α and β). Also, from Table 11, for the bladder cancer data, we see that the parameters α , β and λ of the STLIL model are estimated by $\hat{\alpha} = 3.782$, $\hat{\beta} = 8.774$ and $\hat{\lambda} = 0.417$, respectively. The remark on the importance of λ still holds.

The goodness-of-fits of the considered models are presented in Tables 12 and 13 for the survival time and bladder cancer data sets, respectively.

From Tables 12 and 13, we note that the STLIL model possesses the smallest values of the goodness-of-fits as compared to the competitors. Therefore, the STLIL model provides the best fits for both data sets. This provides evidence of the statistical relevance of the STL-G family in a concrete data analysis scenario.

Figures 9 and 10 illustrate graphically the fit behavior of the STLIL model by plotting various estimated objects on

Figure 10: Fits of the ECDF, EPDF, PP plot and esf for the bladder cancer data set.

diverse approximate representations of the “true” distribution of the data. More precisely, for the two data sets, we plot the estimated CDF (ECDF) on the empirical CDF, the estimated PDF (EPDF) on the histogram, the probability–probability (PP) plot and the estimated survival function (ESF) on the empirical survival function.

From Figures 9 and 10, for the two data sets, we see that the colored estimated objects fit well the corresponding empirical black object, as expected.

6 Conclusion

In this paper, a new facet of the sine-G family is explored, through the use of the Topp-Leone family as parent. The resulting family of distributions refers to the STL-G family. With the use of different approaches, we demonstrated that the STL-G family possesses nice mathematical and practical qualities, making it quite applicable for data analysis. In particular, we show that the maximum likelihood methodology can be applied in a straightforward manner to the STL-G models. Also, with the consideration of the inverse Lomax distribution as parent and two important data sets, the corresponding model reveals to be superior to more than 20 existing models including the former sine-G models, with the use of standard goodness-of-fits criteria. Hence, the STL-G family has the required qualities to belong to the top modern families, with possible nice applications beyond those investigated in this study.

For example, the STL-G provides an arsenal of pliant trigonometric functions which can be used for the construction of various mathematical objects, such as diverse “kernels functions” or can be involved in the definition of various differential equations, with modern applications in physics, among others. In these regards, we may refer to refs. [7], [8] and [9].

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