Research Article

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Exact solutions of space–time fractional KdV–MKdV equation and Konopelchenko–Dubrovsky equation

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Abstract: In the present study, we deal with the space–time fractional KdV–MKdV equation and the space–time fractional Konopelchenko–Dubrovsky equation in the sense of the conformable fractional derivative. By means of the extend $\left(\frac{G'}{G}\right)$ -expansion method, many exact solutions are obtained, which include hyperbolic function solutions, trigonometric function solutions and rational solutions. The results show that the extend $\left(\frac{G'}{G}\right)$ -expansion method is an efficient technique for solving nonlinear fractional partial equations. We also provide some graphical representations to demonstrate the physical features of the obtained solutions.

Keywords: space–time fractional KdV–MKdV equation, space–time fractional Konopelchenko–Dubrovsky equation, extend $\left(\frac{G'}{G}\right)$ -expansion method, conformable fractional derivative

1 Introduction

In recent years, the study of nonlinear fractional partial differential equations (NFPDEs) has been one of the hot research topics, as a lot of important phenomena in scientific and engineering fields [1–11] can be well described by these equations. For example, in ref. [10], Kumar et al.

used a fractional homotopy analysis transform method to study the fractional vibration equation involving the Atangana–Baleanu fractional derivative; Singh et al. applied a homotopy analysis transform method to analyze the dynamical behavior of fractional fish farm model in ref. [11]. In order to help people understand those physical phenomena that are modeled by NFPDEs better, finding solutions of NFPDEs is essential. As we know, there are many methods [12–17] that have been used to obtain solutions of nonlinear partial differential equations. For example, in ref. [13], Yusuf et al. obtained many solutions of the Caudrey–Dodd–Gibbon equation by using Hirota’s bilinear method; in ref. [14], Sulaiman et al. used the Hirota bilinear method to construct solutions of the $(3 + 1)$ -dimensional soliton equation; in ref. [15], Abdel-Gawad et al. employed the unified method to construct rogue wave solutions for the complex coupled Kadomtsev–Petviashvili equation; in ref. [16], based on the logarithmic transformation, Yusuf et al. obtained many solutions of the Heisenberg ferromagnetic spin chain equation, which consist of rational solutions, interaction and combined multi-wave solutions; and Ghanbari et al. [17] employed the generalized exponential rational function method to study solutions of the Biswas–Milovic equation. But for most of the NFPDEs, it is difficult to find their solutions. Therefore, we should find an effective and practical method to study those equations. In the past few decades, many effective methods have been developed to study solutions of NFPDEs, such as homotopy analysis transform method [18,19], first integral method [20,21], homotopy perturbation method [22,23], Kudryashov method [24], variational iteration method [25,26], local discontinuous Galerkin method [27], sub-ordinary differential equation method [28,29], residual power series method [30,31] and so on.

More recently, a new definition of the fractional derivative has been proposed by Khail et al. [32]. They named it conformable fractional derivative. On the basis of Khail et al.’s work, many scholars [33–39,40] have carried out a lot of research in this issue and have achieved fruitful results.

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In this study, we make use of the extend $\left(\frac{G'}{G}\right)$ -expansion method to the space–time fractional KdV–MKdV and Konopelchenko–Dubrovsky equations as follows:

- The space–time fractional KdV–MKdV equation:

$$D_t^\beta u + (p + qu - ru^2)D_x^\beta u + D_x^{3\beta} u = 0.$$

- The space–time fractional Konopelchenko–Dubrovsky equation:

$$\begin{cases} D_t^\beta u - D_x^{3\beta} u - 6puD_x^\beta u + \frac{3}{2}q^2u^2D_x^\beta u \\ - 3D_y^\beta w + 3qwD_x^\beta u = 0, \\ D_x^\beta w = D_y^\beta u. \end{cases}$$

The rest of this article is arranged as follows: in Section 2, we describe the conformable fractional derivative and give the major steps of the extend $\left(\frac{G'}{G}\right)$ -expansion method for obtaining exact solutions of NFPDEs here. In Sections 3 and 4, we construct exact solutions of the space–time fractional KdV–MKdV equation and the Konopelchenko–Dubrovsky equation. In Section 5, we give some conclusions.

2 Conformable fractional derivative and extend $\left(\frac{G'}{G}\right)$ -expansion method

At first, we give the definition and important properties of the conformable fractional derivative.

Definition 2.1. The conformable fractional derivative of g of order α is defined by ref. [32]

$$D_\beta(g)(t) = \lim_{\tau \rightarrow 0} \frac{g(t + \tau t^{1-\beta}) - g(t)}{\tau}, \quad g: [0, \infty) \rightarrow \mathbb{R}$$

for all $t > 0$, $0 < \beta < 1$.

Some important properties of the conformable fractional derivative are listed in ref. [33–39] as follows:

$$D_\beta(t^\delta) = \delta t^{\delta-\beta}, \quad \delta \in \mathbb{R}, \quad (1)$$

$$D_\beta(af + bg) = aD_\beta(f) + bD_\beta(g), \quad a, b \in \mathbb{R}, \quad (2)$$

$$D_\beta(fg) = fD_\beta(g) + gD_\beta(f), \quad (3)$$

$$D_\beta\left(\frac{f}{g}\right) = \frac{gD_\beta(f) - fD_\beta(g)}{g^2}, \quad (4)$$

$$D_\beta(f \circ g)(t) = t^{1-\beta}g'(t)f'(g(t)). \quad (5)$$

Next, we give the concrete steps of the extend $\left(\frac{G'}{G}\right)$ -expansion method as follows:

Step 1. Let us assume that an NFPDE

$$P(v, D_t^{\beta_1}v, D_x^{\beta_2}v, \dots) = 0, \quad 0 < \beta_1, \beta_2 \leq 1, \quad (6)$$

where $D_t^{\beta_1}v$ and $D_x^{\beta_2}v$ are the conformable fractional derivative of v , can be translated into the following ordinary differential equation:

$$P(v, cD_\xi v, kD_\theta v, \dots) = 0, \quad (7)$$

by using

$$\theta = k \frac{x^{\beta_2}}{\beta_2} + c \frac{t^{\beta_1}}{\beta_1} + \theta_0. \quad (8)$$

Step 2. Assuming that equation (7) has the following solution:

$$v(\theta) = \sum_{j=-m}^m b_j \left(\frac{G'}{G}\right)^j, \quad (9)$$

where b_j ($j = -m, -m+1, \dots, m-1, m$) are under-determined constants, and $G(\theta)$ satisfies the following second-order Riccati equation:

$$G''(\theta) + \eta G'(\theta) + \omega G(\theta) = 0, \quad (10)$$

where η and ω are constants. As is known to all, equation (10) has the following exact solutions:

$$\left(\frac{G'}{G}\right) = \begin{cases} -\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{\left(A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)\right)}{\left(A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)\right)}, & \eta^2 - 4\omega > 0, \\ -\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \frac{\left(-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)\right)}{\left(A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)\right)}, & \eta^2 - 4\omega < 0, \\ \frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2}, & \eta^2 - 4\omega = 0, \end{cases} \quad (11)$$

where A_1 and A_2 are constants.

Step 3. Substituting equations (9) and (10) into equation (7) and getting all terms with the same order of $\left(\frac{G'}{G}\right)^j$, then setting each coefficient of $\left(\frac{G'}{G}\right)^j$ to zero, we could get the set of under-determined algebraic equations for k , c and b_j .

Step 4. Substituting equation (9) given the values of k , c and b_j determined in Step 3 into equation (7), we can finally get the exact solutions of equation (6).

3 Space–time fractional KdV–MKdV equation

Let us consider the space–time fractional KdV–MKdV equation:

$$D_t^\beta u + (p + qu - ru^2)D_x^\beta u + D_x^3 u = 0. \quad (12)$$

In order to get the exact solutions of equation (12), we use the following transformation:

$$u = u(\theta), \quad \theta = k \frac{x^\beta}{\beta} + c \frac{t^\beta}{\beta} + \theta_0, \quad (13)$$

then equation (12) can be translated into

$$cD_\theta u + k(p + qu - ru^2)D_\theta u + k^3 D_\theta^3 u = 0. \quad (14)$$

With the help of the homogeneous balance method, we assume that the solution of equation (14) can be represented as:

$$u = b_0 + b_1 \left(\frac{G'}{G}\right) + b_{-1} \left(\frac{G'}{G}\right)^{-1}. \quad (15)$$

Substituting equations (15) and (10) into equation (14) and setting each coefficient of $\left(\frac{G'}{G}\right)^j$ to zero, we get a system of algebraic equations about b_0 , b_1 , b_{-1} , k , c . Solving this system, we can get:

Case 1.

$$b_0 = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r}, \quad b_1 = \pm \frac{\sqrt{6}k}{\sqrt{r}}, \quad b_{-1} = 0, \\ c = k \left[\frac{-q^2 + 2k^2r(\eta^2 - 4\omega)}{4r} - p \right].$$

Case 2.

$$b_0 = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r}, \quad b_1 = 0, \quad b_{-1} = \pm \frac{\sqrt{6}\omega k}{\sqrt{r}}, \\ c = k \left[\frac{-q^2 + 2k^2r(\eta^2 - 4\omega)}{4r} - p \right].$$

Case 3.

$$b_0 = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r}, \quad b_1 = \pm \frac{\sqrt{6}k}{\sqrt{r}}, \quad b_{-1} = \pm \frac{\sqrt{6}\omega k}{\sqrt{r}}, \\ c = k \left[\frac{-q^2 + 2k^2r(\eta^2 + 8\omega)}{4r} - p \right].$$

When $\eta^2 - 4\omega > 0$, we can get the following hyperbolic function solutions of equation (12) by using Cases 1–3 and the solutions of equation (10):

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r(\eta^2 - 4\omega)}k}{2r} \\ \times \frac{\left(A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)}{\left(A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)}, \quad (16)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2r(\eta^2 - 4\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r} \pm \frac{\sqrt{6}\omega k}{\sqrt{r}} \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{\left(A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)}{\left(A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)} \right]^{-1}, \quad (17)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2r(\eta^2 - 4\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r(\eta^2 - 4\omega)}k}{2r} \times \frac{\left(A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)}{\left(A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)} \pm \frac{\sqrt{6}\omega k}{\sqrt{r}} \\ \times \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{\left(A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)}{\left(A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) \right)} \right]^{-1}, \quad (18)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2 r(\eta^2 + 8\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

When $\eta^2 - 4\omega < 0$, we can obtain trigonometric function solutions of equation (12) as follows:

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r(4\omega - \eta^2)}k}{2r} \times \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right), \quad (19)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2 r(\eta^2 - 4\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r} \pm \frac{\sqrt{6}\omega k}{\sqrt{r}} \times \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \quad (20)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2 r(\eta^2 - 4\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r(4\omega - \eta^2)}k}{2r} \times \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \pm \frac{\sqrt{6}\omega k}{\sqrt{r}} \times \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \quad (21)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2 r(\eta^2 + 8\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

When $\eta^2 - 4\omega = 0$, we obtain rational solutions of equation (12) from Cases 1 to 3:

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r}kA_2}{r(A_1 + A_2\theta)}, \quad (22)$$

where $\theta = k \frac{x^\beta}{\beta} - k \left(\frac{q^2}{4r} + p \right) \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q \pm \sqrt{6}k\eta\sqrt{r}}{2r} \pm \frac{\sqrt{6}\omega k}{\sqrt{r}} \left[\frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2} \right]^{-1}, \quad (23)$$

(a)

(b)

(c)

(d)

Figure 1: Solution (16) with different values of β . (a) $\beta = 0.25$, (b) $\beta = 0.5$, (c) $\beta = 0.75$ and (d) $\beta = 1$, respectively, with $k = r = 1$, $\eta = 3$, $\omega = 2$, $q = -2$, $A_2 = \theta_0 = 0$.

where $\theta = k \frac{x^\beta}{\beta} - k \left(\frac{q^2}{4r} + p \right) \frac{t^\beta}{\beta} + \theta_0$.

$$u = \frac{q}{2r} \pm \frac{\sqrt{6r} k A_2}{r(A_1 + A_2 \theta)} \pm \frac{\sqrt{6\omega} k}{\sqrt{r}} \left[\frac{A_2}{A_1 + A_2 \theta} - \frac{\eta}{2} \right]^{-1}, \quad (24)$$

where $\theta = k \frac{x^\beta}{\beta} + k \left[\frac{-q^2 + 2k^2 r(\eta^2 + 8\omega)}{4r} - p \right] \frac{t^\beta}{\beta} + \theta_0$.

We show some properties of the solution $u = \frac{q}{2r} +$

$$\frac{\sqrt{6r(\eta^2 - 4\omega)} k}{2r} \left(\frac{A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)}{A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)} \right) \text{ in equation (16)}$$

for the space–time fractional KdV–MKdV equation in Figure 1 for different values of β . It could be observed that the width of the wave is increasing when we increase the fractional order β .

4 Space–time fractional Konopelchenko–Dubrovsky equation

In this section, we consider the following space–time fractional Konopelchenko–Dubrovsky equation:

$$\begin{cases} D_t^\beta u - D_x^{3\beta} u - 6pu D_x^\beta u + \frac{3}{2} q^2 u^2 D_x^\beta u - 3D_y^\beta w \\ \quad + 3qw D_x^\beta u = 0, \\ D_x^\beta w = D_y^\beta u. \end{cases} \quad (25)$$

Similarly, we assume that equation (25) has the following solution by using the method mentioned in Section 2:

$$\begin{cases} u(\theta) = b_0 + b_1 \left(\frac{G'}{G} \right) + b_{-1} \left(\frac{G'}{G} \right)^{-1}, \\ w(\theta) = d_0 + d_1 \left(\frac{G'}{G} \right) + d_{-1} \left(\frac{G'}{G} \right)^{-1}, \end{cases} \quad (26)$$

where $\theta = k \frac{x^\beta}{\beta} + l \frac{y^\beta}{\beta} + c \frac{t^\beta}{\beta} + \theta_0$.

By substituting equations (26) and (10) into equation

(25) and setting each coefficient of $\left(\frac{G'}{G} \right)^i$ to zero, we get a set of algebraic equations about $b_0, b_1, b_{-1}, d_0, d_1, d_{-1}, k, l, c$. Solving these equations, we get:

Case 1.

$$b_0 = \frac{-lq + 2kp \pm k^2 q \eta}{kq^2}, \quad b_1 = \pm \frac{2k}{q}, \quad b_{-1} = 0, \quad d_1 = \pm \frac{2l}{q},$$

$$d_{-1} = 0, \quad c = \frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right)$$

$$- \frac{k^3}{2} (\eta^2 - 4\omega).$$

Case 2.

$$b_0 = \frac{-lq + 2kp \pm k^2 q \eta}{kq^2}, \quad b_1 = 0, \quad b_{-1} = \pm \frac{2k\omega}{q}, \quad d_1 = 0,$$

$$d_{-1} = \pm \frac{2l\omega}{q}, \quad c = \frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right)$$

$$- \frac{k^3}{2} (\eta^2 - 4\omega).$$

Case 3.

$$b_0 = \frac{-lq + 2kp \pm k^2 q \eta}{kq^2}, \quad b_1 = \pm \frac{2k}{q},$$

$$b_{-1} = \pm \frac{2k\omega}{q}, \quad d_1 = \pm \frac{2l}{q},$$

$$d_{-1} = \pm \frac{2l\omega}{q}, \quad c = \frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right)$$

$$- \frac{k^3}{2} (\eta^2 + 8\omega).$$

Similarly, using Cases 1–3, and the solutions of equation (10), we can obtain the exact solutions of equation (25) as follows:

When $\eta^2 - 4\omega > 0$,

$$\begin{cases} u = \frac{2kp - lq}{kq^2} \pm \frac{\sqrt{(\eta^2 - 4\omega)} k}{q} \left(\frac{A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)}{A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)} \right), \\ w = \frac{d_0 q \pm l\eta}{q} \pm \frac{\sqrt{(\eta^2 - 4\omega)} l}{q} \left(\frac{A_1 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)}{A_1 \cosh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right) + A_2 \sinh \left(\frac{\sqrt{\eta^2 - 4\omega}}{2} \theta \right)} \right), \end{cases} \quad (27)$$

where $\theta = k \frac{\chi^\beta}{\beta} + l \frac{\nu^\beta}{\beta} + \left[\frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right) - \frac{k^3}{2}(\eta^2 - 4\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

$$\left\{ \begin{aligned} u &= \frac{-lq + 2kp \pm k^2q\eta}{kq^2} \pm \frac{2k\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \right]^{-1}, \\ w &= d_0 \pm \frac{2l\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \right]^{-1}, \end{aligned} \right. \quad (28)$$

where $\theta = k \frac{\chi^\beta}{\beta} + l \frac{\nu^\beta}{\beta} + \left[\frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right) - \frac{k^3}{2}(\eta^2 - 4\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

$$\left\{ \begin{aligned} u &= \frac{2kp - lq}{kq^2} \pm \frac{\sqrt{(\eta^2 - 4\omega)}k}{q} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \\ &\quad \pm \frac{2k\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \right]^{-1}, \\ w &= \frac{d_0q \pm l\eta}{q} \pm \frac{\sqrt{(\eta^2 - 4\omega)}l}{q} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \\ &\quad \pm \frac{2l\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{\eta^2 - 4\omega}}{2} \frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}}{2}\theta\right)} \right]^{-1}, \end{aligned} \right. \quad (29)$$

where $\theta = k \frac{\chi^\beta}{\beta} + l \frac{\nu^\beta}{\beta} + \left[\frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right) - \frac{k^3}{2}(\eta^2 + 8\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

When $\eta^2 - 4\omega < 0$,

$$\left\{ \begin{aligned} u &= \frac{2kp - lq}{kq^2} \pm \frac{\sqrt{(4\omega - \eta^2)}k}{q} \frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}, \\ w &= \frac{d_0q \pm l\eta}{q} \pm \frac{\sqrt{(4\omega - \eta^2)}l}{q} \frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}, \end{aligned} \right. \quad (30)$$

where $\theta = k \frac{\chi^\beta}{\beta} + l \frac{\nu^\beta}{\beta} + \left[\frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right) - \frac{k^3}{2}(\eta^2 - 4\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

$$\left\{ \begin{aligned} u &= \frac{-lq + 2kp \pm k^2q\eta}{kq^2} \pm \frac{2k\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \\ w &= d_0 \pm \frac{2l\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \end{aligned} \right. \quad (31)$$

where $\theta = k\frac{x^\beta}{\beta} + l\frac{y^\beta}{\beta} + \left[\frac{3l^2}{k} + k\left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0\right) - \frac{k^3}{2}(\eta^2 - 4\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

$$\left\{ \begin{aligned} u &= \frac{2kp - lq}{kq^2} \pm \frac{\sqrt{(4\omega - \eta^2)}k}{q} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \\ &\quad \pm \frac{2k\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \\ w &= \frac{d_0q \pm l\eta}{q} \pm \frac{\sqrt{(4\omega - \eta^2)}l}{q} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \\ &\quad \pm \frac{2l\omega}{q} \left[-\frac{\eta}{2} + \frac{\sqrt{4\omega - \eta^2}}{2} \left(\frac{-A_1 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)}{A_1 \cos\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right) + A_2 \sin\left(\frac{\sqrt{4\omega - \eta^2}}{2}\theta\right)} \right) \right]^{-1}, \end{aligned} \right. \quad (32)$$

where $\theta = k\frac{x^\beta}{\beta} + l\frac{y^\beta}{\beta} + \left[\frac{3l^2}{k} + k\left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0\right) - \frac{k^3}{2}(\eta^2 + 8\omega) \right] \frac{t^\beta}{\beta} + \theta_0$.

When $\eta^2 - 4\omega = 0$,

$$\left\{ \begin{aligned} u &= \frac{2kp - lq}{kq^2} \pm \frac{2kA_2}{q(A_1 + A_2\theta)}, \\ w &= \frac{d_0q \pm l\eta}{q} \pm \frac{2lA_2}{q(A_1 + A_2\theta)}, \end{aligned} \right. \quad (33)$$

where $\theta = k\frac{x^\beta}{\beta} + l\frac{y^\beta}{\beta} + \left[\frac{3l^2}{k} + k\left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0\right) \right] \frac{t^\beta}{\beta} + \theta_0$.

$$\left\{ \begin{aligned} u &= \frac{-lq + 2kp \pm k^2q\eta}{kq^2} \pm \frac{2k\omega}{q} \left[\frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2} \right]^{-1}, \\ w &= d_0 \pm \frac{2l\omega}{q} \left[\frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2} \right]^{-1}, \end{aligned} \right. \quad (34)$$

where $\theta = k\frac{x^\beta}{\beta} + l\frac{y^\beta}{\beta} + \left[\frac{3l^2}{k} + k\left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0\right) \right] \frac{t^\beta}{\beta} + \theta_0$.

Figure 2: Solution u in equation (27) with different values of β . (a) $\beta = 0.25$, (b) $\beta = 0.5$, (c) $\beta = 0.75$ and (d) $\beta = 1$, respectively, with $k = l = q = 1$, $\omega = p = 2$, $\eta = 3$, $d_0 = 5$, $x = 0$, $A_2 = \theta_0 = 0$.

$$\begin{cases} u = \frac{2kp - lq}{kq^2} \pm \frac{2kA_2}{q(A_1 + A_2\theta)} \\ \quad \pm \frac{2k\omega}{q} \left[\frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2} \right]^{-1}, \\ w = \frac{d_0q \pm l\eta}{q} \pm \frac{2lA_2}{q(A_1 + A_2\theta)} \\ \quad \pm \frac{2l\omega}{q} \left[\frac{A_2}{A_1 + A_2\theta} - \frac{\eta}{2} \right]^{-1}, \end{cases} \quad (35)$$

$$\text{where } \theta = k \frac{x^\beta}{\beta} + l \frac{y^\beta}{\beta} + \left[\frac{3l^2}{k} + k \left(\frac{6p^2}{q^2} \pm 3l\eta - 3qd_0 \right) - \frac{k^3}{2}(\eta^2 + 8\omega) \right] \frac{t^\beta}{\beta} + \theta_0$$

Taking the solution $u = \frac{2kp - lq}{kq^2} + \frac{\sqrt{(\eta^2 - 4\omega)k}}{q}$

$$\left(\frac{A_1 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}\theta}{2}\right) + A_2 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}\theta}{2}\right)}{A_1 \cosh\left(\frac{\sqrt{\eta^2 - 4\omega}\theta}{2}\right) + A_2 \sinh\left(\frac{\sqrt{\eta^2 - 4\omega}\theta}{2}\right)} \right) \text{ in equation (27), for ex-}$$

ample, we show the effect of fractional order for the space-time fractional Konopelchenko–Dubrovsky equation in Figure 2 for different values of β . It is shown that the width of the wave is increasing when we increase the fractional order β .

Remark 1. From Figures 1 and 2, we can see the effect of fractional derivative on solutions (16) and (27). For other solutions of equations (12) and (25), we can also find the effect of fractional order β , but we omit them for simplicity.

Remark 2. To the best of our knowledge, the solutions to the space-time fractional KdV–MKdV equation and the space-time fractional Konopelchenko–Dubrovsky equation in the sense of the conformable fractional derivative have never been documented before in the literature, so these solutions are new. The reported solutions in this study may give insights into many considerable physical processes.

5 Conclusion

In this essay, we have successfully applied the extend $\left(\frac{G'}{G}\right)$ -expansion method to gain exact solutions of the space-time fractional KdV–MKdV equation and the space-time fractional Konopelchenko–Dubrovsky

equation. Consequently, a number of solutions in terms of hyperbolic function solutions, trigonometric function solutions and rational solutions are obtained. Furthermore, we depicted some of these solutions in Figures 1 and 2 for different fractional orders β in order to explain the physical property. The extend $\left(\frac{G'}{G}\right)$ -expansion method can also be used to solve other NFPDEs. As we all know, when using the direct methods to study the exact solutions, how to choose an approach ansatz is very important. In addition, how to develop the extend $\left(\frac{G'}{G}\right)$ -expansion method by using a more generalized ansatz is one of our future studies. And how to extend other methods such as Hirota bilinear method and exponential rational function method to deal with NFPDEs is also a very important and worthy research task. We will study it in our future work.

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References

- [1] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. New York: Wiley; 1993.
- [2] Baleanu D, Deftferli O, Agrawal OP. A central difference numerical scheme for fractional optimal control problems. *Vib J Control*. 2009;15:583–97.
- [3] Singh J, Kumar D, Swroop R, Kumar S. A new fuzzy logic-based query expansion model for efficient information retrieval using relevance feedback approach. *Neural Comput Appl*. 2017;28:2557–80.
- [4] Hilfer R. Applications of fractional calculus in physics. New Jersey: World Scientific; 2000.
- [5] West BJ, Bolognab M, Grigolini P. Physics of fractal operators. New York: Springer; 2003.
- [6] Mohammadi F, Moradi L, Baleanu D, Jajarmi A. A hybrid functions numerical scheme for fractional optimal control problems: application to nonanalytic dynamic systems. *Vib J Control*. 2018;24(21):5030.
- [7] Kumar S, Kumar R, Cattani C, Samet B. Chaotic behaviour of fractional predator-prey dynamical system. *Chaos Solitons Fractals*. 2020;135:109811.
- [8] Kumar D, Singh J, Tanwar K, Baleanu D. A new fractional exothermic reactions model having constant heat source in porous media with power, exponential and Mittag-Leffler laws. *Int J Heat Mass Transf*. 2019;138:1222–7.
- [9] Podlubny I. Fractional differential equations. San Diego: Academic Press; 1999.
- [10] Kumar D, Singh J, Baleanu D. On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law. *Math Methods Appl Sci*. 2020;43(1):443–57.
- [11] Singh J, Kumar D, Baleanu D. A new analysis of fractional fish farm model associated with Mittag-Leffler-type kernel. *Int J Biomath*. 2020;13(2):2050010.
- [12] Tang B, Fan Y, Wang J, Chen S. Exact solutions for N-coupled nonlinear Schrödinger equations with variable coefficients. *Z Naturforsch*. 2016;71(7a):665–72.
- [13] Yusuf A, Sulaiman TA, Inc M, Bayram M. Breather wave, lump-periodic solutions and some other interaction phenomena to the Caudrey–Dodd–Gibbon equation. *Eur Phys J Plus*. 2020;135:563.
- [14] Sulaiman TA, Yusuf A, Atangana A. New lump, lump-kink, breather waves and other interaction solutions to the (3+1)-dimensional soliton equation. *Commun Theor Phys*. 2020;72:085004.
- [15] Abdel-Gawad HI, Tantawy M, Inc M, Yusuf A. Construction of rogue waves and conservation laws of the complex coupled Kadomtsev–Petviashvili equation. *Int J Mod Phys B*. 2020;34(12):2050115.
- [16] Yusuf A, Tchier F, Inc M. New interaction and combined multi-wave solutions for the Heisenberg ferromagnetic spin chain equation. *Eur Phys J Plus*. 2020;135:416.
- [17] Ghanbari B, Inc M, Yusuf A, Baleanu D, Bayram M. Families of exact solutions of Biswas–Milovic equation by an exponential rational function method. *Tbil Math J*. 2020;13(2):39–65.
- [18] Kumar S, Kumar A, Baleanu D. Two analytical methods for time-fractional nonlinear coupled Boussinesq–Burger’s equations arise in propagation of shallow water waves. *Nonlinear Dynam*. 2016;85(2):699–715.
- [19] Kumar S, Kumar A, Odibat Z, Aldhaifallah M, Nisar KS. A comparison study of two modified analytical approach for the solution of nonlinear fractional shallow water equations in fluid flow. *AIMS Math*. 2020;5(4):3035–55.
- [20] Javeed S, Saif S, Waheed A, Baleanu D. Exact solutions of fractional mBBM equation and coupled system of fractional Boussinesq–Burgers. *Res Phys*. 2018;9:1275–81.
- [21] Lu B. The first integral method for some time fractional differential equations. *J Math Anal Appl*. 2012;395(2):684–93.
- [22] Kumar D, Singh J, Kumar S. Numerical computation of nonlinear fractional Zakharov–Kuznetsov equation arising in ion-acoustic waves. *J Egypt Math Soc*. 2014;22(3):373–8.
- [23] Kumar D, Singh J, Sushila. Application of homotopy analysis-transform method to fractional biological population model. *Rom Rep Phys*. 2013;65(1):63.
- [24] Hosseini K, Ayati Z. Exact solutions of space-time fractional EW and modified EW equations using Kudryashov method. *Nonlinear Sci Lett A*. 2016;7:58.
- [25] Tang B, Wang X, Wei L, Zhang X. Exact solutions of fractional heat-like and wave-like equations with variable coefficients. *Int J Numer Methods Heat Fluid Flow*. 2014;24:455–67.

- [26] Tang B, Fan Y, Zhao J, Wang X. Solitary and compacton solutions of fractional KdV-like equations. *Open Phys.* 2016;1:328–36.
- [27] Wei L, He Y, Tang B. Analysis of the fractional Kawahara equation using an implicit fully discrete local discontinuous Galerkin method. *Numer Methods Partial Differ Equ.* 2013;29(5):1441–58.
- [28] Tang B, He Y, Wei L, Zhang X. A generalized fractional sub-equation method for fractional differential equations with variable coefficients. *Phys Lett A.* 2012;376:2588–90.
- [29] Zhao JP, Tang B, Kumar S, Hou YR. A fuzzy approach to robust control of stochastic nonaffine nonlinear systems. *Math Probl Eng.* 2012;2012:367802–17.
- [30] Zhang Y, Kumar A, Kumar S, Baleanu D. Residual power series method for time-fractional Schrödinger equations. *J Nonlinear Sci Appl.* 2016;9(11):5821–9.
- [31] Dubey VP, Kumar R, Kumar D, Khan I, Singh J. An efficient computational scheme for nonlinear time fractional systems of partial differential equations arising in physical sciences. *Adv Differ Equ.* 2020;2020:46.
- [32] Khalil R, Horani MA, Yousef A, Sababhehb M. A new definition of fractional derivative. *J Comput Appl Math.* 2014;264:65–70.
- [33] Abdeljawad T. On conformable fractional calculus. *J Comput Appl Math.* 2015;279:57–66.
- [34] Abdeljawad T, AL Horani M, Khalil R. Conformable fractional semigroups of operators. *J Semigroup Theory Appl.* 2015;2015:7.
- [35] Chen C, Jiang YL. Simplest equation method for some time-fractional partial differential equations with conformable derivative. *Comput Math Appl.* 2018;75(8):2978–88.
- [36] Eslami M. Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations. *Appl Math Comput.* 2016;285:141.
- [37] Chung WS. Fractional Newton mechanics with conformable fractional derivative. *J Comput Appl Math.* 2015;290:150–8.
- [38] Abdeljawad T, Alzabut J, Jarad F. A generalized Lyapunov-type inequality in the frame of conformable derivatives. *Adv Differ Equ.* 2017;2017(1):321.
- [39] Baleanu D, Jarad F, Ugurlu E. Singular conformable sequential differential equations with distributional potentials. *Quaest Math.* 2019;42(3):277–87.