Research Article

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An optimal system of group-invariant solutions and conserved quantities of a nonlinear fifth-order integrable equation

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Abstract: In this work, we perform Lie group analysis on a fifth-order integrable nonlinear partial differential equation, which was recently introduced in the literature and contains two dispersive terms. We determine a one-parameter group of transformations, an optimal system of group invariant solutions, and derive the corresponding analytic solutions. Topological kink, periodic and power series solutions are obtained. The existence of a variational principle for the underlying equation is proven using Helmholtz conditions and, thereafter, both local and nonlocal conserved quantities are obtained by utilising Noether's theorem and a homotopy integral approach.

Keywords: optimal system, one-parameter group, power series solutions, Noether's theorem, multiplier approach

1 Introduction

The importance of studying nonlinear partial differential equations (NLPDEs) cannot be overemphasised as they model behaviours and interrelations of physical quantities. A large number of scholars continue to research different aspects of NLPDEs, see, for example, ref. [1–49]. As the NLPDEs model physical phenomena of the real world,

determining their exact solutions is therefore a step further in understanding the world around us. Famous NLPDEs include the Korteweg–de Vries (KdV) equation [3], which replicates the propagation of waves in shallow water, the Benjamin–Bona–Mahony (BBM) equation [4], which models long surface gravity waves with a small amplitude, the Kadomtsev–Petviashvili (KP) equation [5], a model of water waves in cases where the ratio of water depth to wavelength is very small and the Camassa–Holm equation [6], which models shallow water waves, just to mention a few.

The most general form of a fifth-order KdV equation reads [7]

$$u_t + \alpha u^2 u_x + \nu u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0 \quad (1.1)$$

with arbitrary real constants α , ν and γ . Equation (1.1) models long waves propagating in shallow water and in the presence of gravity. Several famous fifth-order KdV equations emanate from equation (1.1), for instance, the fifth-order Sawada–Kotera equation [8]

$$u_t + 45u^2 u_x + 15u_x u_{xx} + 15u u_{xxx} + u_{xxxxx} = 0$$

occurs when $\alpha = 3\nu$, $\nu = \gamma$ and $\gamma = 15$. On the other hand, when $\alpha = 5/2\nu$, $\nu = 5/4\gamma$ and $\gamma = 20$, we have the nonlinear fifth-order Kaup–Kupershmidt equation [9]

$$u_t + 20u^2 u_x + 25u_x u_{xx} + 10u u_{xxx} + u_{xxxxx} = 0.$$

In this article, we study a newly formulated integrable fifth-order equation

$$F \equiv \beta u_{xxx} - 4u_t u_{xxx} - 12u_{xx} u_{tx} - 8u_x u_{txx} - u_{txxx} + u_{ttt} = 0, \quad \beta \neq 0. \quad (1.2)$$

Equation (1.2) was developed in ref. [10], where the researchers performed a Painlevé test to prove its integrability. The authors went further to find different types of soliton solutions of (1.2) using Hirota's method. In a sense, equation (1.2) is a second-generation hybrid between the fifth-order Kawahara equation [11]

$$u_t + \alpha u u_x + \nu u_{xxx} + \gamma u_{xxxxx} \quad (1.3)$$

and another fifth-order equation [12]

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$$u_{xxx} - 4u_t u_{xxx} - 12u_{xx} u_{tx} - 8u_x u_{txx} - u_{txxx} + u_{ttt} = 0. \quad (1.4)$$

In other words, equation (1.2) is a progression from equation (1.4) due to the addition of a second dispersive term (u_{xxx}), by lending credence from the Kawahara equation, which also has the two dispersive terms (u_{xxx}) and (u_{txxx}). Notwithstanding this backdrop, integrable models are of great interest to mathematicians and physicists for the plethora of information they possess. The integrability of equation (1.2), which has been proven in ref. [10], makes it a significant model and worth studying. In our work, we seek to obtain a more comprehensive set of solutions with the aid of an optimal system of one-dimensional subalgebras. Also, we derive a group of transformations corresponding to the symmetry algebra of (1.2).

Although the solutions of the authors [10] are extensive, it will be seen that our solutions subsume those in ref. [10] since we obtain hyperbolic, parabolic, elliptic and power series solutions. We will go a step further to determine the conserved vectors attributable to this equation.

The dawn of NLPDEs brought about a commensurate surge in methods for obtaining their analytical solutions. One of the most significant of these techniques emanates from the Lie group theory [13–18], courtesy of Marius Sophus Lie (1842–1899). Other methods include the multiple exponential function method [19], Hirota's bilinear approach [20], the (G'/G) -expansion technique [21,22], the homogeneous balance method [23], the simplest equation method [24], Kudryashov's method [25], the elliptic function technique [26] and the F -expansion method [27], to mention but a few.

Conserved quantities are a subject of keen interest in the fields such as theoretical and quantum mechanics [15,16,28–38]. In isolated systems, energy, mass, charge, linear and angular momentum are conserved. Conserved vectors may be used to check for integrability of differential equations (DEs) and the fidelity of numerical solution methods. In this article, we employ two different methods of determining conserved quantities, namely, Noether's approach and the multiplier method.

2 Lie group analysis of (1.2)

2.1 Infinitesimal generators

For a one-parameter group of transformations

$$\begin{aligned} \bar{t} &\rightarrow t + a\xi^1(t, x, u), \quad \bar{x} \rightarrow x + a\xi^2(t, x, u), \\ \bar{u} &\rightarrow u + a\eta(t, x, u) \end{aligned}$$

with a small parameter a , we have the corresponding infinitesimal generator

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u}. \quad (2.5)$$

The vector field (2.5) in conjunction with (1.2) satisfies the invariance condition

$$\begin{aligned} \text{pr}^{(5)}X(\beta u_{xxx} - 4u_t u_{xxx} - 12u_{xx} u_{tx} - 8u_x u_{txx} \\ - u_{txxx} + u_{ttt})|_{F=0} = 0. \end{aligned} \quad (2.6)$$

Here, $\text{pr}^{(5)}X$ is the fifth-order extension of (2.5) given by [15]

$$\begin{aligned} \text{pr}^{(5)}X = X + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} \\ + \zeta_{ttt} \frac{\partial}{\partial u_{ttt}} + \zeta_{txx} \frac{\partial}{\partial u_{txx}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}} + \zeta_{txxx} \frac{\partial}{\partial u_{txxx}} \end{aligned}$$

and $\zeta_t, \zeta_x, \zeta_{tx}, \zeta_{xx}, \zeta_{ttt}, \zeta_{txx}, \zeta_{xxx}, \zeta_{txxx}$ are prolongation coefficient functions which can be obtained from the following general formulae:

$$\begin{aligned} \zeta_i &= D_i(W) + \xi^j u_{ij}, \\ \zeta_{i_1 \dots i_s} &= D_{i_1} \dots D_{i_s}(W) + \xi^j u_{ji_1 \dots i_s}, \quad s > 1, \end{aligned} \quad (2.7)$$

where W is the Lie characteristic function given by

$$W = \eta - \xi^i u_i \quad (2.8)$$

and the total derivatives D_t and D_x are defined as

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xt} \frac{\partial}{\partial u_t} + \dots. \end{aligned} \quad (2.9)$$

From (2.6), we obtain the following system of 12 linear homogeneous PDEs:

$$\begin{aligned} \xi_x^1 &= 0, \xi_u^1 = 0, \xi_{tt}^1 = 0, \xi_{tx}^1 = 0, \xi_{tu}^1 = 0, \xi_t^2 = 0, \xi_u^2 = 0, \\ &= 0, \\ 2\xi_x^2 - \xi_{tt}^1 &= 0, \eta_{ttt} = 0, \eta_x = 0, \xi_t^1 + 2\eta_u = 0, \\ 8\eta_t - 3\beta\xi_t^1 &= 0. \end{aligned} \quad (2.10)$$

Solving system (2.10) yields the generator coefficients

$$\begin{aligned} \xi^1 &= \frac{8C_2}{3\beta}t + C_1, \xi^2 = \frac{4C_2}{3\beta}x + C_4, \eta = -\frac{4C_2}{3\beta}u + C_2t \\ &+ C_3, \end{aligned} \quad (2.11)$$

and ultimately we obtain a four-dimensional Lie algebra L_4 , spanned by the generators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 8t \frac{\partial}{\partial t} + 4x \frac{\partial}{\partial x} + (3\beta t - 4u) \frac{\partial}{\partial u}, \\ X_4 &= \frac{\partial}{\partial u}. \end{aligned} \quad (2.12)$$

2.2 Group transformations of known solutions

The one-parameter groups G_i of transformations generated by the four-dimensional Lie algebra (2.12) can be obtained by solving the Lie equations [18] and these are

$$\begin{aligned} G_1 : (t, x, u) &\rightarrow (t + a, x, u), \\ G_2 : (t, x, u) &\rightarrow (t, x + a, u), \\ G_3 : (t, x, u) &\rightarrow \left(te^{8a}, xe^{4a}, ue^{-4a} + \frac{1}{4}\beta te^{-4a}e^{12a} - 1 \right), \\ G_4 : (t, x, u) &\rightarrow (t, x, u + a). \end{aligned}$$

Thus, we have the following theorem.

Theorem. *If $u = m(t, x)$ is a solution of the PDE (1.2), then so are the functions*

$$\begin{aligned} G_1 : u^{(1)} &= m(t - a, x), \\ G_2 : u^{(2)} &= m(t, x - a), \\ G_3 : u^{(3)} &= e^{-4a}m(te^{-8a}, xe^{-4a}) + \frac{1}{4}\beta t(1 - e^{-12a}), \\ G_4 : u^{(4)} &= m(t, x) + a. \end{aligned}$$

This means that from any known solution of (1.2) one can obtain infinitely many new exact solutions of (1.2) by the repeated use of the above family of solutions.

2.3 Optimal system of one-parameter group-invariant solutions

We now utilise the method given in ref. [15] to find an optimal system of one-dimensional subalgebras corresponding to the Lie algebra (2.12). Accordingly, we begin by determining the commutation relations, $[X_i, X_j] = X_i(X_j) - X_j(X_i)$, between vector fields (2.12). This leads to the commutator table given in Table 1.

The Lie series

$$\text{Ad}(\exp(\varepsilon X_i))X_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{Ad} X_i)^n(X_j)$$

Table 1: Commutation relations of four-dimensional Lie algebra (2.12)

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	$8X_1 + 3\beta X_4$	0
X_2	0	0	$4X_2$	0
X_3	$-8X_1 - 3\beta X_4$	$-4X_2$	0	$4X_4$
X_4	0	0	$-4X_4$	0

along with the results of Table 1 yields the adjoint representations, which are presented in Table 2.

Using Tables 1 and 2, and the procedure given in ref. [15], we obtain an optimal system of one-dimensional subalgebras spanned by $X_1, X_2, X_3, X_1 \pm X_2, X_2 \pm X_4$. However, we note that the vector field $X_1 - X_2$ can be mapped to the vector field $X_1 + X_2$ by the discrete symmetry $(t, x, u) \mapsto (t, -x, u)$. This leaves us with the following optimal system of one-dimensional subalgebras of (1.2) that is spanned by

$$X_1, X_2, X_3, X_1 + X_2, X_2 \pm X_4. \quad (2.13)$$

We now determine an optimal system of group-invariant solutions corresponding to (2.13). This will give us closed-form solutions of (1.2).

2.3.1 Cases X_1, X_2 and $X_2 \pm X_4$

From the generators X_1, X_2 and $X_2 \pm X_4$, we obtain the group-invariant solutions:

$$u(t, x) = \phi_1(\xi), \quad \xi = x, \quad (2.14)$$

$$u(t, x) = \phi_2(\xi), \quad \xi = t, \quad (2.15)$$

$$u(t, x) = \phi_3(\xi) \pm x, \quad \xi = t, \quad (2.16)$$

respectively. Each of the group-invariant solutions in (2.14)–(2.16) transform equation (1.2) into the ODE $\phi_j'''(\xi) = 0$, $j = 1, 2, 3$. This is easily solved to give the solution

$$\phi_j(\xi) = \frac{1}{2}C_1\xi^2 + C_2\xi + C_3, \quad j = 1, 2, 3, \quad (2.17)$$

with C_1, C_2 and C_3 integration constants. Thus,

$$u_1 = \frac{1}{2}C_1x^2 + C_2x + C_3, \quad (2.18)$$

$$u_2 = \frac{1}{2}C_1t^2 + C_2t + C_3, \quad (2.19)$$

$$u_3 = \frac{1}{2}C_1t^2 + C_2t + C_3 \pm x \quad (2.20)$$

Table 2: Adjoint table of Lie algebra (2.12)

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	$-8\epsilon X_1 + X_3 - 3\beta\epsilon X_4$	X_4
X_2	X_1	X_2	$-4\epsilon X_2 + X_3$	X_4
X_3	$e^{8\epsilon}X_1 + \frac{1}{4}\beta e^{-4\epsilon}(e^{12\epsilon} - 1)$	$e^{4\epsilon}X_2$	X_3	$e^{-4\epsilon}X_4$
X_4	X_1	X_2	$X_3 + 4\epsilon X_4$	X_4

are the group-invariant solutions of (1.2) under the optimal system elements X_1 , X_2 and $X_2 \pm X_4$, respectively.

2.3.2 Case $X_1 + X_2$

We now focus on the optimal system element $X_1 + X_2$. Using the usual Lie theory, we obtain the group-invariant solution $u(t, x) = \phi(\xi)$, $\xi = x - t$, which transforms equation (1.2) into the nonlinear ordinary differential equation (NLODE)

$$\phi'''' + (\beta - 1)\phi''' + 12(\phi'''\phi' + \phi'^2) = 0. \quad (2.21)$$

2.3.2.1 Solutions of (1.2) by direct integration of (2.21)

Integrating (2.21) twice with respect to ξ gives

$$\phi''' + 6\phi'^2 + (\beta - 1)\phi' + C_1\xi + C_2 = 0 \quad (2.22)$$

with integration constants C_1 and C_2 . Equation (2.22) cannot be solved to produce solutions in terms of classical functions, see for example ref. [39–42]. At this point, we are compelled to explore two interesting cases which lead to elliptic and hyperbolic solutions.

(i) The Jacobi elliptic function solutions of (1.2)

Consider the NLODE (2.22) with $C_1 = 0$ and let $\phi'(\xi) = (1/2)\Theta(\xi)$. Then equation (2.22) becomes

$$\Theta'' + 3\Theta^2 - \rho\Theta + C_4 = 0, \quad (2.23)$$

where $\rho = 1 - \beta$ and $C_4 = 2C_2$. The NLODE (2.23) is a principal equation and arises from many famous NLPDEs. See for example [39]. Multiplying (2.23) by Θ' and integrating with respect to ξ yield

$$\Theta'^2 + 2\Theta^3 - \rho\Theta^2 + 2C_4\Theta + 2C_5 = 0. \quad (2.24)$$

Equation (2.24) leads us to the well-known Jacobi elliptic cosine function solution of (2.23), namely [43–45]

$$\Theta(\xi) = \mu_2 + (\mu_1 - \mu_2)\operatorname{cn}^2\left\{\sqrt{\frac{\mu_1 - \mu_3}{2}}\xi, M^2\right\}, \quad (2.25)$$

$$M^2 = \frac{\mu_1 - \mu_2}{\mu_1 - \mu_3}.$$

Here μ_1 , μ_2 and μ_3 (with $\mu_1 \geq \mu_2 \geq \mu_3$, $0 \leq M^2 \leq 1$) are roots of the algebraic equation

$$\Theta^3 - \frac{1}{2}\rho\Theta^2 + C_4\Theta + C_5 = 0$$

and thus satisfy the DE

$$\Theta'^2 = -2(\Theta - \mu_1)(\Theta - \mu_2)(\Theta - \mu_3).$$

Moreover, $C_4 = 2C_2 = \mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3$ and $C_5 = \mu_1\mu_2\mu_3$. Finally, we recuperate the variable u to obtain the solution of (1.2) as

$$u(t, x) = \frac{(\mu_1 - \mu_2)\operatorname{sn}(A\xi|M^2)\operatorname{COS}^{-1}\{\operatorname{dn}(A\xi|M^2)\}}{A\sqrt{1 - \operatorname{dn}(A\xi|M^2)^2}} + \mu_2\xi, \quad (2.26)$$

where $A = \sqrt{\mu_1 - \mu_3}/\sqrt{2}$, $\beta = 1 - 2(\mu_1 + \mu_2 + \mu_3)$ and $\xi = x - t$.

The analytic solution (2.26) is periodic in nature, and its profile is sketched in Figure 1.

(i) The hyperbolic solutions of (1.2)

By letting the integration constants C_1 and C_2 equal to zero and $\phi'(\xi) = \psi(\xi)$ in (2.22), equation (2.22) becomes

$$\psi'' + (\beta - 1)\psi + 6\psi^2 = 0. \quad (2.27)$$

Multiplying by ψ' and integrating with respect to ξ lead us to

$$\psi'^2 + 4\psi^3 + (\beta - 1)\psi^2 = 0, \quad (2.28)$$

where once again, the integration constant is taken as zero. The solution of equation (2.28) is

$$\psi(\xi) = -\frac{1}{4}(\beta - 1) \times \left[1 - \tanh^2\left\{\frac{1}{2}(\pm\sqrt{1-\beta}C_3 - \sqrt{1-\beta}\xi)\right\}\right], \quad (2.29)$$

Figure 1: Profile of periodic solution (1.2)

where C_3 is the integration constant. In order to recover the variable u , we integrate (2.29) with respect to ξ to obtain the topological kink soliton solution

$$u(t, x) = \frac{1 - \beta}{2\sqrt{1 - \beta}} \tanh\left(\frac{1}{2}\sqrt{1 - \beta}(\xi \pm C_3)\right) \quad (2.30)$$

with its profile shown in Figure 2.

Remark. Travelling wave solutions of (1.2) can be obtained by taking the linear combination $X_1 + vX_2$, where v is the wave velocity. The corresponding group-invariant solution for this combination of vector fields is $u(t, x) = \phi(\xi)$, $\xi = x - vt$, which consequently yields the travelling wave solution of (1.2) as

$$u(t, x) = \frac{(\mu_1 - \mu_2) \operatorname{sn}(A\xi|M^2) \cos^{-1}\{\operatorname{dn}(A\xi|M^2)\}}{A\sqrt{1 - \operatorname{dn}(A\xi|M^2)^2}} + \mu_2\xi, \quad (2.31)$$

where $A = \sqrt{\mu_1 - \mu_3}/\sqrt{2}$, $\beta = v\{v^2 - 2(\mu_1 + \mu_2 + \mu_3)\}$ and $\xi = x - vt$.

2.3.2.2 Solutions of (1.2) via the (G'/G) -expansion method

Using the (G'/G) -expansion technique [21,22], we now solve the NLODE (2.21). Let equation (2.21) have the formal solution

$$\phi(\xi) = \sum_{i=0}^m A_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i. \quad (2.32)$$

We aim to find the values of the coefficients $A_0, A_1, \dots, A_m$. In the case of the NLODE (2.21), we obtain $m = 1$. See, for example [21]. Thus, from (2.32) we have

$$\phi(\xi) = A_0 + A_1 \left(\frac{G'(\xi)}{G(\xi)} \right). \quad (2.33)$$

Figure 2: Profile of topological kink soliton (2.30).

Figure 3: Topological kink soliton profile of (2.35).

Substituting (2.33) into (2.21) and making use of the linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (2.34)$$

where λ and μ take arbitrary real values, we obtain an algebraic equation in A_0 and A_1 , whose solution is

$$A_0 = A_0, \quad A_1 = 1, \quad \beta = 1 - (\lambda^2 - 4\mu).$$

Thus, we have the following three types of solutions for (1.2):

(i) When $1 - \beta > 0$, we have

$$u(t, x) = A_0 + \Omega_1 \left(\frac{B_1 \cosh(\Omega_1 \xi) + B_2 \sinh(\Omega_1 \xi)}{B_1 \sinh(\Omega_1 \xi) + B_2 \cosh(\Omega_1 \xi)} \right) - \frac{\lambda}{2}, \quad (2.35)$$

where $\Omega_1 = \sqrt{1 - \beta}/2$, $\xi = x - t$ and A_0 , B_1 and B_2 are arbitrary constants. The solution profile of (2.35) is presented in Figure 3.

(ii) When $1 - \beta < 0$, we get

$$u(t, x) = A_0 + \Omega_2 \left(\frac{B_1 \sin(\Omega_2 \xi) - B_2 \cos(\Omega_2 \xi)}{B_1 \cos(\Omega_2 \xi) + B_2 \sin(\Omega_2 \xi)} \right) - \frac{\lambda}{2}, \quad (2.36)$$

where $\Omega_2 = \sqrt{\beta - 1}/2$, $\xi = x - t$ and A_0 , B_1 and B_2 are arbitrary constants. The solution profile of (2.36) is sketched in Figure 4.

(iii) When $\beta = 1$, we obtain

$$u(t, x) = A_0 + \frac{B_2}{B_2 \xi + B_1} - \frac{\lambda}{2}, \quad (2.37)$$

where $\xi = x - t$ and A_0 , B_1 and B_2 are arbitrary constants. The profile of solution (2.37) is given in Figure 5.

2.3.3 Case X_3

Finally, we consider the optimal system element X_3 , which culminates in the group-invariant solution $u(t, x) = (1/\sqrt{t})\phi(\xi) + 1/4\beta t$, $\xi = x/\sqrt{t}$. Thus, equation (1.2) is transformed into the NLODE

$$4\xi\phi^{(5)} + 20\phi^{(4)} + 48\xi\phi'\phi''' - \xi^3\phi''' + 96\xi\phi'^2 + 96\xi\phi\phi'' + 16\phi\phi''' + 192\phi'\phi'' - 33\xi\phi' - 12\xi^2\phi'' - 15\phi = 0. \quad (2.38)$$

The power series solution method is ideal for solving such complicated nonlinear differential equation (2.38). See, for example ref. [46–49]. To begin, let the solution of (2.38) take the form

Figure 4: Periodic behaviour with singularities for (2.36).

$$\phi(\xi) = \sum_{z=0}^{\infty} g_z \xi^z \quad (2.39)$$

with constants g_z , $z = 0, 1, 2, \dots$, to be determined. The various derivatives of (2.39) are thus

$$\begin{aligned} \phi'(\xi) &= \sum_{z=0}^{\infty} (z+1) g_{z+1} \xi^z, \\ \phi''(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2) g_{z+2} \xi^z, \\ \phi'''(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2)(z+3) g_{z+3} \xi^z, \\ \phi^{(4)}(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2)(z+3)(z+4) g_{z+4} \xi^z, \\ \phi^{(5)}(\xi) &= \sum_{z=0}^{\infty} (z+1)(z+2)(z+3)(z+4)(z+5) g_{z+5} \xi^z. \end{aligned} \quad (2.40)$$

Substituting the results of (2.40) into (2.38) gives

$$\begin{aligned} &4 \sum_{z=1}^{\infty} z(z+1)(z+2)(z+3)(z+4) g_{z+4} \xi^z \\ &+ 20 \sum_{z=0}^{\infty} (z+1)(z+2)(z+3)(z+4) g_{z+4} \xi^z \\ &+ 48 \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} (j+1)(z-j)(z+1-j)(z+2-j) \\ &\times g_{j+1} g_{z+2-j} \xi^z - \sum_{z=3}^{\infty} (z-2)(z-1) z g_z \xi^z \\ &+ 96 \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} (j+1)(z-j) g_{j+1} g_{z-j} \xi^z \\ &+ 96 \sum_{z=1}^{\infty} \sum_{j=0}^{z-1} (z-j)(z+1-j) g_j g_{z+1-j} \xi^z \\ &+ 16 \sum_{z=0}^{\infty} \sum_{j=0}^z (z+1-j)(z+2-j) \\ &\times (z+3-j) g_j g_{z+3-j} \xi^z \\ &+ 192 \sum_{z=0}^{\infty} \sum_{j=0}^z (j+1)(z+1-j) \\ &\times (z+2-j) g_{j+1} g_{z+2-j} \xi^z \\ &- 33 \sum_{z=1}^{\infty} z g_z \xi^z - 12 \sum_{z=2}^{\infty} (z-1) z g_z \xi^z \\ &- 15 \sum_{z=0}^{\infty} g_z \xi^z = 0. \end{aligned} \quad (2.41)$$

Equation (2.41) leads to

$$\begin{aligned} &480 g_5 \xi + 2880 g_6 \xi^2 + 4 \sum_{z=3}^{\infty} z(z+1)(z+2) \\ &\times (z+3)(z+4) g_{z+4} \xi^z + 480 g_4 + 2400 g_5 \xi \\ &+ 7200 g_6 \xi^2 + 20 \sum_{z=3}^{\infty} (z+1)(z+2)(z+3) \\ &\times (z+4) g_{z+4} \xi^z + 288 g_1 g_3 \xi + 1152 g_1 g_4 \xi^2 \\ &+ 48 \sum_{z=3}^{\infty} \sum_{j=0}^{z-1} (j+1)(z-j)(z+1-j) \\ &\times (z+2-j) g_{j+1} g_{z+2-j} \xi^z - \sum_{z=3}^{\infty} (z-2)(z-1) z g_z \xi^z \\ &+ 96 g_1^2 \xi + 192 g_1 g_2 \xi^2 + 96 \sum_{z=3}^{\infty} \sum_{j=0}^{z-1} (j+1)(z-j) \\ &\times g_{j+1} g_{z-j} \xi^z + 192 g_0 g_2 \xi + 576 g_0 g_3 \xi^2 \\ &+ 96 \sum_{z=3}^{\infty} \sum_{j=0}^{z-1} (z-j)(z+1-j) g_j g_{z+1-j} \xi^z + 96 g_0 g_3 \\ &+ 384 g_0 g_4 \xi + 960 g_0 g_5 \xi^2 + 16 \sum_{z=3}^{\infty} \sum_{j=0}^z (z+1-j) \\ &\times (z+2-j)(z+3-j) g_j g_{z+3-j} \xi^z + 384 g_1 g_2 \\ &+ 1152 g_1 g_3 \xi + 2304 g_1 g_4 \xi^2 + 192 \sum_{z=3}^{\infty} \sum_{j=0}^z (j+1) \\ &\times (z+1-j)(z+2-j) g_{j+1} g_{z+2-j} \xi^z - 33 g_1 \xi \\ &- 66 g_2 \xi^2 - 33 \sum_{z=3}^{\infty} z g_z \xi^z - 12 g_2 \xi^2 - 12 \sum_{z=3}^{\infty} (z-1) \\ &\times z g_z \xi^z - 15 g_0 - 15 g_1 \xi - 15 g_2 \xi^2 - 15 \sum_{z=3}^{\infty} g_z \xi^z = 0. \end{aligned} \quad (2.42)$$

Now comparing coefficients of ξ in equation (2.42) we have, for $z = 0, 1, 2$:

$$96 g_0 g_3 + 384 g_1 g_2 - 15 g_0 + 480 g_4 = 0, \quad (2.43)$$

$$192 g_0 g_2 + 384 g_0 g_4 + 96 g_1^2 + 1440 g_1 g_3 - 48 g_1 + 2880 g_5 = 0, \quad (2.44)$$

$$576 g_0 g_3 + 960 g_0 g_5 + 192 g_1 g_2 + 3456 g_1 g_4 - 93 g_2 + 10080 g_6 = 0 \quad (2.45)$$

and generally for $z \geq 3$ we have the recurrence relation

$$\begin{aligned}
g_{z+4} = & -\frac{1}{(4z+20)(z+1)(z+2)(z+3)(z+4)} \\
& \times \left[48 \sum_{j=0}^{z-1} (j+1)(z-j)(z+1-j)(z+2-j) \right. \\
& \times g_{j+1}g_{z+2-j} - 33zg_z - 12(z-1)zg_z - 15g_m \\
& + 96 \sum_{j=0}^{z-1} (j+1)(z-j)g_{j+1}g_{z-j} \\
& + 96 \sum_{j=0}^{z-1} (z-j)(z+1-j)g_jg_{z+1-j} \\
& + 16 \sum_{j=0}^m (z+1-j)(z+2-j)(z+3-j) \\
& \times g_jg_{z+3-j} - (z-2)(z-1)zg_z + 192 \sum_{j=0}^m (j+1) \\
& \left. \times (z+1-j)(z+2-j)g_{j+1}g_{z+2-j} \right]. \quad (2.46)
\end{aligned}$$

From equations (2.43)–(2.45) and for arbitrary $g_0, \dots, g_3$, we have the following:

$$\begin{aligned}
g_4 &= \frac{1}{32}g_0 - \frac{1}{5}g_0g_3 - \frac{4}{5}g_1g_2, \\
g_5 &= \frac{2}{75}g_0^2g_3 - \frac{1}{15}g_0g_2 + \frac{8}{75}g_0g_1g_2 - \frac{1}{240}g_0^2 \\
&\quad - \frac{1}{30}g_1^2 - \frac{1}{2}g_1g_3 + \frac{1}{60}g_1, \\
g_6 &= \frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 \\
&\quad + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0 \\
&\quad + \frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 \\
&\quad - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2. \quad (2.47)
\end{aligned}$$

Using the recursion formula (2.46), successive terms g_m , $m = 7, 8, \dots, \infty$ can be determined uniquely. Thus, the power series solution of (2.38) can be written as

$$\begin{aligned}
\phi(\xi) &= g_0 + g_1\xi + g_2\xi^2 + g_3\xi^3 \\
&\quad + \left(\frac{1}{32}g_0 - \frac{1}{5}g_0g_3 - \frac{4}{5}g_1g_2 \right) \xi^4 \\
&\quad + \left(\frac{2}{75}g_0^2g_3 - \frac{1}{15}g_0g_2 + \frac{8}{75}g_0g_1g_2 \right. \\
&\quad \left. - \frac{1}{240}g_0^2 - \frac{1}{30}g_1^2 - \frac{1}{2}g_1g_3 + \frac{1}{60}g_1 \right) \xi^5 \\
&\quad + \left(\frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 \right. \\
&\quad + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0 \\
&\quad + \frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 \\
&\quad \left. - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2 \right) \xi^6 + \sum_{z=3}^{\infty} g_z + g\xi^{z+4}. \quad (2.48)
\end{aligned}$$

Finally, the solution of (1.2) is

$$\begin{aligned}
u(t, x) &= g_0 + g_1 \frac{x}{\sqrt{t}} + g_2 \frac{x^2}{t} + g_3 \frac{x^3}{t^{3/2}} \\
&\quad + \left(\frac{1}{32}g_0 - \frac{1}{5}g_0g_3 - \frac{4}{5}g_1g_2 \right) \frac{x^4}{t^2} \\
&\quad + \left(\frac{2}{75}g_0^2g_3 - \frac{1}{15}g_0g_2 + \frac{8}{75}g_0g_1g_2 \right. \\
&\quad \left. - \frac{1}{240}g_0^2 - \frac{1}{30}g_1^2 - \frac{1}{2}g_1g_3 + \frac{1}{60}g_1 \right) \frac{x^5}{t^{5/2}} \\
&\quad + \left(\frac{1}{2520}g_0^3 - \frac{4}{1575}g_3g_0^3 - \frac{16}{1575}g_1g_2g_0^2 \right. \\
&\quad + \frac{2}{315}g_2g_0^2 + \frac{1}{315}g_1^2g_0 - \frac{31}{2520}g_1g_0 \\
&\quad + \frac{61}{525}g_1g_3g_0 - \frac{2}{35}g_3g_0 + \frac{48}{175}g_1^2g_2 \\
&\quad \left. - \frac{2}{105}g_1g_2 + \frac{31}{3360}g_2 \right) \frac{x^6}{t^3} \\
&\quad - \sum_{z=3}^{\infty} \frac{(z+4)!}{(4z+20)z!} \left(48 \sum_{j=0}^{z-1} (j+1)(z-j) \right. \\
&\quad \times (z+1-j)(z+2-j)g_{j+1}g_{z+2-j} = 33zg_z \\
&\quad - 12(z-1)zg_z - 15g_z + 96 \sum_{j=0}^{z-1} (j+1) \\
&\quad \times (z-j)g_{j+1}g_{z-j} + 96 \sum_{j=0}^{z-1} (z-j) \\
&\quad \times (z+1-j)g_jg_{z+1-j} + 16 \sum_{j=0}^z (z+1-j) \\
&\quad \times (z+2-j)(z+3-j)g_jg_{z+3-j} \\
&\quad - (z-2)(z-1)zg_z + 192 \sum_{j=0}^z (j+1) \\
&\quad \times (z+1-j)(z+2-j)g_{j+1}g_{z+2-j} \left. \right) \left(\frac{x}{\sqrt{t}} \right)^{z+4}. \quad (2.49)
\end{aligned}$$

The graphical depiction of the behavioural pattern of solution (2.49) is given in Figure 6.

3 Conserved quantities of (1.2)

3.1 Noether's approach

We begin by applying the classical Noether approach [28] to determine conserved quantities of (1.2). The Helmholtz conditions dictate that for an NLPDE to have a variational principle, it must, among other things, have an even-order [29]. Equation (1.2) is of order five and thus cannot have a variational principle. To remedy this we introduce $u = v_x$ to transform (1.2) into the sixth-order NLPDE

Figure 5: Bright and dark solitons (2.37) with a singularity.

$$F^* \equiv \beta v_{xxxx} - 4v_{tx} v_{xxxx} - 12v_{xxx} v_{bxx} - 8v_{xx} v_{bxxx} - v_{bxxxx} + v_{tttx} = 0. \quad (3.50)$$

For the transformed equation (3.50) to have a variational principle, it must indeed satisfy the Helmholtz conditions

$$\frac{\partial F^*}{\partial(\partial^q v)} = (-1)^q (E_v^q(F^*))^t. \quad (3.51)$$

Here $q = 1, \dots, Q$, where Q is the order of equation (3.50). Due to the presence of the transpose t , generally the number of field variables v should equal the number of components in system F^* , else the Helmholtz conditions do not hold and extension F^* would not possess a

variational principle. Also, E_v^q represents higher Euler operators, which are given by

$$E_v^{(r)}(F^*) = \frac{\partial F^*}{\partial(\partial^r v)} - \binom{r+1}{r} D \cdot \frac{\partial F^*}{\partial(\partial^{r+1} v)} + \dots + \binom{q}{r} (-D)^{q-r} \cdot \frac{\partial F^*}{\partial(\partial^r v)} \quad (3.52)$$

with $r = 1 \dots q$ and D represents the total derivatives $D = (D_t, D_x)$. It can be shown with relative ease that a reckoning of (3.51) will confirm that equation (3.50) indeed has a variational principle. A Lagrangian can thus be recovered for (3.50) as

Figure 6: Depiction of cumulative partial sums up to T_6 for solution (2.49).

$$\begin{aligned}\mathcal{L} = & \frac{8}{3}v_{tx}(v_{xxx} + v_x v_{xx}) + \frac{4}{3}v_{xxx}(v_{tx} + v_x v_{tx}) \\ & + \frac{1}{2}\beta v_{xx}^2 + \frac{1}{2}v_{xxx}v_{tx} + \frac{1}{2}v_{tt}v_{tx} - 4v_{xxx}v_{tx}. \quad (3.53)\end{aligned}$$

The Lagrangian (3.53) conforms with condition $\delta\mathcal{L}/\delta v = F^* = 0$, as expected.

In order to obtain the variational symmetries $Y = \xi^1\partial/\partial t + \xi^2\partial/\partial x + \eta\partial/\partial v$ corresponding to Lagrangian (3.53), we shall use the invariance condition

$$\text{pr}^{(3)}Y(\mathcal{L}) + \mathcal{L}D_t\xi^1 + D_x\xi^2 = D_tB^1 + D_xB^2, \quad (3.54)$$

where $\text{pr}^{(3)}Y$ is the third prolongation of Y obtainable from

$$\begin{aligned}\text{pr}^{(3)}Y = & Y + \zeta_x \frac{\partial}{\partial v_x} + \zeta_{tt} \frac{\partial}{\partial v_{tt}} + \zeta_{tx} \frac{\partial}{\partial v_{tx}} \\ & + \zeta_{xx} \frac{\partial}{\partial v_{xx}} + \zeta_{tx} \frac{\partial}{\partial v_{tx}} + \zeta_{xxx} \frac{\partial}{\partial v_{xxx}}.\end{aligned}$$

Note that in (3.54), ξ^1, ξ^2, η, B^1 and B^2 are functions of (t, x, v) . The invariance condition (3.54) yields the following system of 12 linear PDEs:

$$\begin{aligned}\xi_t^1 = 0, \xi_x^1 = 0, \xi_v^1 = 0, \xi_t^2 = 0, \xi_x^2 = 0, \xi_v^2 = 0, \\ \eta_{tt} = 0, \eta_x = 0, \eta_v = 0, B_v^1 = 0, B_v^2 = 0, B_t^1 + B_x^2(3.55) \\ = 0.\end{aligned}$$

From system (3.55), we can readily infer without any tedious calculations that

$$\begin{aligned}\xi^1 = C_1, \xi^2 = C_2, \eta = C_3t + C_4, B^1 = F(t, x), \\ B^2 = H(t, x) + P(t)\end{aligned}$$

with $H(t, x) = -\int F_t(t, x)dx$. The above generator coefficients lead us to the variational symmetries

$$\begin{aligned}Y_1 = \frac{\partial}{\partial t}, B^1 = B^2 = 0, \\ Y_2 = \frac{\partial}{\partial x}, B^1 = B^2 = 0, \\ Y_3 = t\frac{\partial}{\partial v}, B^1 = B^2 = 0, \\ Y_4 = \frac{\partial}{\partial v}, B^1 = B^2 = 0.\end{aligned} \quad (3.56)$$

For the Noether symmetry generators (3.56) associated with Lagrangian (3.53), we derive conserved vectors [18]

$$T^i = N^i(\mathcal{L}) - B^i, i = 1, \dots, n \quad (3.57)$$

such that $D_tT^t + D_xT^x = 0$ for all solutions of (3.50). Here N^i is the Noether operator given by

$$\begin{aligned}N^i = \xi^i + W^\alpha \frac{\delta}{\delta v_i^\alpha} + \sum_{s \geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \delta \delta v_{i_1 i_2 \dots i_s}^\alpha, \quad (3.58) \\ i = 1, \dots, n,\end{aligned}$$

where the Euler operators with respect to derivatives of v^α are obtained from

$$\begin{aligned}\frac{\delta}{\delta v^\alpha} = \frac{\partial}{\partial v^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v_{i_1 i_2 \dots i_s}^\alpha}, \quad (3.59) \\ \alpha = 1, \dots, m,\end{aligned}$$

by replacing v^α with the corresponding derivatives. For example,

$$\begin{aligned}\frac{\delta}{\delta v_i^\alpha} = \frac{\partial}{\partial v_i^\alpha} + \sum_{s \geq 1} (-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial v_{i j_1 j_2 \dots j_s}^\alpha}, \quad (3.60) \\ i = 1, \dots, n, \alpha = 1, \dots, m.\end{aligned}$$

By utilising (3.53), (3.56) and (3.57), we obtain the following nonlocal conserved quantities of equation (1.2):

$$\begin{aligned}T_1^t = \frac{1}{36} \{ 2(16u_t + 9\beta)u_x^2 + 8 \left(8uu_{tx} - 9u_{xx} \left(\int u_t dx \right) \right) u_x \\ - 8 \left(u_{xxx} \int u_t dx - 7u_t u_{xx} \right) u + 3(4u_{xx} u_{tx} \\ + \left(2u_{xxx} - 3 \int u_{tt} dx \right) u_t + (9u_{tt} - 2u_{xxx}) \int u_t dx \} \},\end{aligned}$$

$$\begin{aligned}T_1^x = \frac{1}{36} \left\{ 4 \left((9\beta - 34u_t)u_{xx} \int u_t dx + 2u \int u_{tt} dx - 3u_{tt} \right) \right. \\ + 32u_x^2 \int u_{tt} dx - 18u_{tx}^2 \\ - 4 \left(9\beta u_t - 20u_t^2 + 56u_{tx} \int u_t dx + 16uu_{tt} \right) u_x \\ - 64uu_t u_{tx} + 8uu_{tx} \int u_t dx + 24u_t u_{tx} \\ - 30u_{txx} \int u_t dx + 6u_{xxx} \int u_{tt} dx - 9 \left(\int u_{tt} dx \right)^2 \\ \left. + 9 \left(\int u_t dx \right) \left(\int u_{ttt} dx \right) \right\};\end{aligned}$$

$$\begin{aligned}T_2^t = \frac{1}{36} \left\{ \left(6u_{xxx} - 9 \int u_{tt} dx - 96uu_{xx} \right) u_x - 18u_t^2 \right. \\ + 32u_x^3 - 6u_{xx}^2 + (27u_{tt} - 8uu_{xxx} - 6u_{xxx})u \},\end{aligned}$$

$$\begin{aligned}T_2^x = \frac{1}{36} \left\{ \left(4u_{xx}(9\beta - 32u_t) - 30u_{txx} + 9 \int u_{ttt} dx \right) u \right. \\ + 2(56u_t - 9\beta)u_x^2 + 8u^2 u_{tx} - 12u_{xx} u_{tx} \\ + 8u_x(3u_{tx} - 26uu_{tx}) + 3u_t \left(3 \int u_{tt} dx + 2u_{xxx} \right) \} \};\end{aligned}$$

$$T_3^t = \frac{1}{2}u_t + \frac{1}{36}t\{72u_x u_{xx} - 27u_{tt} + 8uu_{xxx} + 6u_{xxxx}\},$$

$$T_3^x = \frac{1}{36}\left\{224tu_x u_{tx} - 4(2u + 9\beta t - 34tu_t)u_{xx} - 8tuu_{txx} + 30tu_{txxx} + 9\int u_{tt} dx - 9t\int u_{ttt} dx - 32u_x^2 - 6u_{xxx}\right\};$$

$$T_4^t = 2u_x u_{xx} - \frac{3}{4}u_{tt} + \frac{2}{9}uu_{xxx} + \frac{1}{6}u_{xxxx},$$

$$T_4^x = \frac{1}{36}\left\{4(34u_t - 9\beta)u_{xx} + 224u_x u_{tx} - 8uu_{txx} + 30u_{txxx} - 9\int u_{ttt} dx\right\}.$$

3.2 Multiplier approach

One advantage of the multiplier approach is that it does not require the existence of a variational principle in order to obtain conserved quantities of DEs [15].

3.2.1 Computation of conserved quantities of (1.2)

The determining condition for a multiplier, namely,

$$\frac{\delta}{\delta u}(\Lambda F) = 0 \quad (3.61)$$

will give us multipliers $\Lambda(t, x, u)$ with F given by equation (1.2). The Euler operator $\delta/\delta u$, in our case, is given by

$$\begin{aligned} \frac{\delta}{\delta u} = & \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_t D_x \frac{\partial}{\partial u_{tx}} + D_x^2 \frac{\partial}{\partial u_{xx}} \\ & - D_t^3 \frac{\partial}{\partial u_{ttt}} - D_t D_x^2 \frac{\partial}{\partial u_{txx}} - D_x^3 \frac{\partial}{\partial u_{xxx}} - D_t D_x^4 \frac{\partial}{\partial u_{txxxx}} \end{aligned}$$

and D_t, D_x are given by (2.9). From equation (3.61) and by following the procedure similar to the usual Lie symmetry algorithm, six multiplier determining equations are realised, which are

$$\Lambda_{uu} = 0, \Lambda_{xu} = 0, \Lambda_{xx} = 0, \Lambda_{tu} = 0, \Lambda_{tx} = 0, \Lambda_{ttt} = 0. \quad (3.62)$$

Solving equation (3.62) one obtains the multipliers corresponding to equation (1.2) as

$$\Lambda = C_1 u + C_2 x + \frac{1}{2}C_3 t^2 + C_4 t + C_5, \quad (3.63)$$

where $C_1, \dots, C_5$ are constants. We now employ the first homotopy integral formula [29]

$$\Phi = \int_0^1 \sum_{j=1}^k \partial_\lambda \partial^{j-1} u_{(\lambda)} \left(\sum_{l=j}^k (-D)^{l-j} \cdot \left(\frac{\partial F \Lambda}{\partial (\partial^l u)} \right) \right) \bigg|_{u=\lambda u} d\lambda, \quad (3.64)$$

where k is the highest-order derivative of F with equation (1.2) along with multipliers (3.63). We obtain

$$\begin{aligned} T = & C_2 uu_{tt} - 2C_1 uu_{xx} - 4C_2 uu_x u_{xx} + C_3 u \\ & - \frac{1}{2}C_2 uu_{xxxx} - \frac{1}{2}C_2 u_t^2 - C_3 tu_t - u_t C_4 \\ & + C_1 xu_{tt} + \frac{1}{2}C_3 t^2 u_{tt} + C_4 tu_{tt} + C_5 u_{tt} \end{aligned}$$

and

$$\begin{aligned} X = & C_5 \beta u_{xx} - C_1 xu_{txx} - C_4 tu_{txx} - C_1 \beta u_x \\ & - \frac{1}{2}C_3 t^2 u_{txxx} + C_1 \beta xu_{xx} + \frac{1}{2}C_3 \beta t^2 u_{xx} \\ & + C_4 \beta tu_{xx} - \frac{1}{2}C_2 uu_{txxx} - 8C_5 u_x u_{tx} \\ & + 6C_1 u_t u_x - \frac{1}{2}C_2 u_{tx} u_{xx} - 4C_5 u_t u_{xx} \\ & + \frac{1}{2}u_{txx} C_2 u_x + 2uu_{tx} C_1 + 4u_t C_2 u_x^2 \\ & + \frac{1}{2}C_2 u_t u_{xxx} - \frac{1}{2}C_2 \beta u_x^2 - C_5 u_{txxx} + C_1 u_{txx} \\ & - 4C_2 uu_{xx} u_t - 4C_1 xu_t u_{xx} - 2C_3 t^2 u_t u_{xx} \\ & - 4C_4 tu_t u_{xx} - 8C_1 xu_x u_{tx} - 4C_3 t^2 u_x u_{tx} \\ & + C_2 \beta uu_{xx} - 8C_4 tu_x u_{tx} - 4C_2 uu_{tx} u_x \end{aligned}$$

and consequently, we have the following five conserved quantities of (1.2):

$$\begin{aligned} T_1^t = & xu_{tt} - 2uu_{xx}, \\ T_1^x = & u_{txx} + \beta xu_{xx} - 8xu_x u_{tx} - 4xu_t u_{xx} \\ & + 2uu_{tx} - \beta u_x + 6u_t u_x - xu_{txxx}; \end{aligned}$$

$$\begin{aligned} T_2^t = & uu_{tt} - \frac{1}{2}uu_{xxxx} - 4uu_x u_{xx} - \frac{1}{2}u_t^2, \\ T_2^x = & 4u_t u_x^2 - \frac{1}{2}uu_{txxx} + \frac{1}{2}u_x u_{txx} - \frac{1}{2}u_{xx} u_{tx} + \frac{1}{2}u_t u_{xxx} \\ & - 4uu_x u_{tx} + \beta uu_{xx} - 4uu_t u_{xx} - \frac{1}{2}\beta u_x^2; \end{aligned}$$

$$\begin{aligned} T_3^t = & \frac{1}{2}t^2 u_{tt} + u - tu_t, \\ T_3^x = & \frac{1}{2}\beta t^2 u_{xx} - \frac{1}{2}t^2 u_{txxx} - 2t^2 u_t u_{xx} - 4t^2 u_x u_{tx}; \end{aligned}$$

$$\begin{aligned} T_4^t = & tu_{tt} - u_t, \\ T_4^x = & \beta tu_{xx} - 8tu_x u_{tx} - 4tu_t u_{xx} - tu_{txxx}; \\ T_5^t = & u_{tt}, \\ T_5^x = & \beta u_{xx} - 4u_t u_{xx} - 8u_x u_{tx} - u_{txxx}. \end{aligned}$$

Remarks. As can be seen, the conserved vectors obtained here are first integrals of equation (1.2). Again, a casual inspection reveals that these results, unlike those obtained by the Noether approach are trivial conservation laws of the first kind. See ref. [15] for a detailed discussion.

4 Concluding remarks

In this work, we performed an extensive study of a fifth-order integrable NLPDE (1.2), which was lately established in the literature and consisted of two dispersive terms. We obtained group transformations under which the equation (its solutions) remained invariant. Furthermore, we deduced an optimal system of one-dimensional subalgebras culminating in several group-invariant solutions. This resulted in parabolic, trigonometric, hyperbolic, elliptic and power series solutions. The corresponding solution profiles depict topological kink soliton and periodic behaviour. Moreover, we investigated the existence of a variational principle in relation to Helmholtz conditions and went on to derive nonlocal conserved vectors corresponding to the variational principle obtained. Local and low order conserved quantities were computed using a homotopy integral formula and multipliers.

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