Research Article

Rohul Amin, Kamal Shah, Imran Khan, Muhammad Asif, Kholod M. Abualnaja, Emad E. Mahmoud, and Abdel-Haleem Abdel-Aty*

A powerful numerical technique for treating twelfth-order boundary value problems

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Abstract: In this article, a fast algorithm is developed for the numerical solution of twelfth-order boundary value problems (BVPs). The Haar technique is applied to both linear and nonlinear BVPs. In Haar technique, the twelfth-order derivative in BVP is approximated using Haar functions, and the process of integration is used to obtain the expression of lower-order derivatives and approximate solution for the unknown function. Three linear and two nonlinear examples are taken from literature for checking the convergence of the proposed technique. A comparison of the results obtained by the present technique with results obtained by other techniques reveals that the present method is more effective and efficient. The maximum absolute and root mean square errors are compared with the exact solution at different collocation and Gauss points. The convergence rate using different numbers of collocation points is also calculated, which is approximately equal to 2.

* **Corresponding author: Abdel-Haleem Abdel-Aty**, Department of Physics, College of Sciences, University of Bisha, P.O. Box 344, Bisha, 61922, Saudi Arabia; Physics Department, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt, e-mail: amabdelaty@ub.edu.sa

Rohul Amin: Department of Mathematics, University of Peshawar, Peshawar, 25120, Khyber Pakhtunkhwa, Pakistan, e-mail: raminmath@uop.edu.pk

Imran Khan: Department of Mathematics, University of Peshawar, Peshawar, 25120, Khyber Pakhtunkhwa, Pakistan, e-mail: lec.imran@uop.edu.pk

Muhammad Asif: Department of Mathematics, University of Peshawar, Peshawar, 25120, Khyber Pakhtunkhwa, Pakistan, e-mail: asif.tangi@uop.edu.pk

Kamal Shah: Department of Mathematics, University of Malakand, Dir(L), Chakdara, 18000, Khyber Pakhtunkhwa, Pakistan, e-mail: kamalshah408@gmail.com

Kholod M. Abualnaja: Department of Mathematics, Faculty of Applied Science, Umm Al-Qura University, P.O. Box 14949, Makkah, Saudi Arabia

Emad E. Mahmoud: Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia; Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt, e-mail: e.mahmoud@tu.edu.sa

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1 Introduction

Boundary value problems (BVPs) with higher order arises in the field of astrophysics, hydrodynamics and hydro-magnetic stability, fluid dynamics, astronomy, beam and long wave theory, and applied physics and engineering. Many researcher studied higher-order BVPs because of mathematical importance and their uses in various field of applied sciences [1–5]. These equations modeled physics of stability problems in hydrodynamics. In the presence of magnetic field and in the direction of gravity, when an infinite horizontal layer of fluid is heated from below, subject to the axis of rotation, instability takes place inside. When this phenomena of instability take place as ordinary convection, then it is modeled by tenth-order BVPs [1], and when the aforesaid instability take place as over stability, then this phenomena is modeled using twelfth-order BVPs [2]. Because of the widespread applications of higher-orders BVPs, many researchers showed more interest in the numerical solution of these equations. Bishop et al. [6] modeled the phenomena of torsional vibration of beams as eighth-order BVP. Siddiqi and Akram [7,8] introduced nonic spline and non-polynomial spline methods for the numerical solution of special linear eighth-order BVP models. In ref. [7,8], Siddiqi also proved that the convergence of the aforementioned techniques is of second order. Wazwaz [9] established an efficient technique using the Adomian decomposition technique to solve some special eighth-order BVPs numerically. Recently, a numerical technique based on polynomial splines of degree six was introduced by Siddiqi and Twizell in ref. [10] for the numerical solution of some special types of eighth-order BVPs. However, it was found that at the points adjacent to the boundary, the method diverges. Further, in ref. [11,12], Twizell et al. observed the same problems by solving some higher-

order BVPs numerically. It was investigated that the results diverge because of the utilization of test functions with lower order in the aforesaid technique. In ref. [10], the authors applied differential quadrature methods (DQM) that use the higher-order test functions in the entire domain. However, the shortcoming of the DQM method is that it dealt only with second-order BVPs. For higher-order BVPs, a δ -point method should be used [11]. The details of improvement and applications of the DQM technique can be seen in ref. [10,11]. The current improvement in the DQM leads to the establishment of generalized DQM [13,14]. Theoretically, the generalized DQM can be used for any higher-order BVPs coupled with conventional δ -point method. In structure mechanics, the generalized DQM is used for the numerical solution of fourth- and sixth-order BVPs [14–18]. Furthermore, in fluid mechanics, it is used for the numerical solution of linear Onsager and nonlinear Blasius problems with third and sixth orders, respectively [13]. The initial value problems of second, third, and fourth orders are also solved using the aforesaid method [14]. The generalized DQM is extended by Liu et al. in ref. [19] for the numerical solution of eighth-order BVPs. In ref. [20], a reproducing kernel method is introduced by Geng et al. for the solution linear tenth-order BVPs. A variational iterative method is used by Siddiqi et al. for the numerical solution of tenth-order BVPs in ref. [21]. Akram et al. developed a numerical technique based on eleventh degree spline for linear special case of BVPs [22]. A tenth degree spline was used by Twizell et al. for the numerical solution of tenth-order BVPs and faced some problems in getting results near boundaries of the interval in ref. [23]. A new numerical technique was introduced by Boutayeb et al. for the solution of twelfth-order BVPs that arise in thermal instability [30]. Siddiqi et al. used tenth and twelfth degree spline for the numerical simulation of tenth- and twelfth-order BVPs [12,23]. Mustahsan et al. [24] used higher-order B-spline differential quadrature rule to approximate the generalized Rosenau-RLW equation. Harim et al. [25] gives a new scheme for positivity preserving interpolation using rational quartic spline of (quartic/quadratic) with three parameters. Tassaddiq et al. [26] found the numerical solution of the nonlinear differential equations arising in the study of hydrodynamics and hydro-magnetic stability problems using a new cubic B-spline scheme. Ghaffar et al. [27] proposed and analyzed a tensor product of nine-tic B-spline subdivision scheme to reduce the execution time needed to compute the subdivision process of quad meshes. Mustafa et al. [28] developed a numerical approach for solving second-order singularly perturbed BVPs. Khalid

et al. [29] used the cubic B-splines to find the numerical solution of linear and nonlinear eighth-order BVPs. There is a meager literature available on the numerical simulation of twelfth-order BVPs and the related eigenvalue problems. Agarwal et al. worked on the existence and uniqueness of twelfth-order BVPs [30].

In this article, we developed a collocation method based on Haar wavelet for the numerical simulation of twelfth-order BVPs. The following nonlinear problem of twelfth order will be considered in this article:

$$u^{(12)}(t) = F(t, u, u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}, u^{(8)}, u^{(9)}, u^{(10)}, u^{(11)}) \quad 0 \leq t \leq 1, \quad (1)$$

where F is given function, whereas in case of linear, the following general form is considered:

$$u^{(12)}(t) + f(t)u(t) = g(t), \quad t \in [0, 1], \quad (2)$$

with the following BCs:

$$\begin{aligned} u(0) &= \alpha_0, & u(1) &= \beta_0 \\ u^{(1)}(0) &= \alpha_1, & u^{(1)}(1) &= \beta_1, \\ u^{(2)}(0) &= \alpha_2, & u^{(2)}(1) &= \beta_2, \\ u^{(3)}(0) &= \alpha_3, & u^{(3)}(1) &= \beta_3, \\ u^{(4)}(0) &= \alpha_4, & u^{(4)}(1) &= \beta_4, \\ u^{(5)}(0) &= \alpha_5, & u^{(5)}(1) &= \beta_5, \end{aligned} \quad (3)$$

where $f(t)$ and $g(t)$ are known functions.

This article is organized as follows: Haar functions are defined in Section 2. Numerical Haar wavelet collocation (HWC) technique for the solution of both nonlinear and linear twelfth-order BVPs is given in Section 3. In Section 4, some problems from literature are given for the validation of HWC method. Conclusion is given in the final Section 5.

2 Haar wavelet

The Haar functions are piecewise constant functions having three values 1, -1 , and 0. The Haar scaling function on interval $[\gamma_1, \gamma_2]$ is given by [31]:

$$h_1(t) = \begin{cases} 1 & \text{for } t \in [\gamma_1, \gamma_2], \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

The mother wavelet for the HW functions on $[\gamma_1, \gamma_2]$ is

$$h_2(t) = \begin{cases} 1 & \text{for } t \in \left[\gamma_1, \frac{\gamma_1 + \gamma_2}{2}\right), \\ -1 & \text{for } t \in \left[\frac{\gamma_1 + \gamma_2}{2}, \gamma_2\right), \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

All the other terms in the HW series can be represented in $t \in [\rho_1, \rho_2]$ except the scaling function

$$h_i(t) = \begin{cases} 1 & \text{for } t \in [\rho_1, \rho_2), \\ -1 & \text{for } t \in [\rho_2, \rho_3), \\ 0 & \text{otherwise,} \end{cases} \quad (6)$$

where

$$\begin{aligned} \rho_1 &= \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta}{m}, \\ \rho_2 &= \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta + 0.5}{m}, \\ \rho_3 &= \gamma_1 + (\gamma_2 - \gamma_1) \frac{\zeta + 1}{m}, \end{aligned}$$

where integer $m = 2^r$ and $r = 0, 1, \dots, r'$, and let the integer $\zeta = 0, 1, \dots, m - 1$. The number i can be obtained as $i = m + \zeta + 1$. In the interval $[0, 1]$, ρ_1 , ρ_2 , and ρ_3 are defined as:

$$\rho_1 = \frac{\zeta}{m}, \quad \rho_2 = \frac{\zeta + 0.5}{m}, \quad \rho_3 = \frac{\zeta + 1}{m}. \quad (7)$$

Any function of $L^2[0, 1]$ space of square integrable function is expressed as:

$$u(t) = \sum_{i=1}^{\infty} \lambda_i h_i(t). \quad (8)$$

This series is truncated at finite N terms for approximation purpose, i.e.

$$u(t) \approx \sum_{i=1}^N \lambda_i h_i(t).$$

We use the symbol

$$p_{i,1}(t) = \int_0^t h_i(t) dt, \quad (9)$$

and the value of the above integral is calculated by definition of h_i and is given by

$$p_{i,1}(t) = \begin{cases} t - \rho_1 & \text{for } t \in [\rho_1, \rho_2), \\ \rho_3 - t & \text{for } t \in [\rho_2, \rho_3), \\ 0 & \text{elsewhere.} \end{cases} \quad (10)$$

Thus value of $p_{i,2}$ is

$$p_{i,2}(t) = \int_0^t p_{i,1}(s) ds,$$

and by simplifying this integral, we have

$$p_{i,2}(t) = \begin{cases} \frac{1}{2}(t - \rho_1)^2 & \text{if } t \in [\rho_1, \rho_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\rho_3 - t)^2 & \text{if } t \in [\rho_2, \rho_3), \\ \frac{1}{4m^2} & \text{if } t \in [\rho_3, 1), \\ 0 & \text{elsewhere.} \end{cases} \quad (11)$$

In addition, the value of $p_{i,3}$ is given by

$$p_{i,3}(t) = \int_0^t p_{i,2}(s) ds,$$

and by simplifying this integral, we obtain

$$p_{i,3}(t) = \begin{cases} \frac{1}{6}(t - \rho_1)^3 & \text{if } t \in [\rho_1, \rho_2), \\ \frac{1}{4m^2}(t - \rho_2) - \frac{1}{6}(\rho_3 - t)^3 & \text{if } t \in [\rho_2, \rho_3), \\ \frac{1}{4m^2}(t - \rho_2) & \text{if } t \in [\rho_3, 1) \\ 0 & \text{elsewhere.} \end{cases} \quad (12)$$

Similarly, the value of $p_{i,4}$ is given by

$$p_{i,4}(t) = \int_0^t p_{i,3}(s) ds,$$

and by simplifying this integral, we obtain

$$p_{i,4}(t) = \begin{cases} \frac{1}{24}(t - \rho_1)^4 & \text{if } t \in [\rho_1, \rho_2), \\ \frac{1}{8m^2}(t - \rho_2)^2 - \frac{1}{24}(\rho_3 - t)^4 & \text{if } t \in [\rho_2, \rho_3), \\ + \frac{1}{192m^4}(\rho_3 - t)^3 & \\ \frac{1}{8m^2}(t - \rho_2)^2 + \frac{1}{192m^4} & \text{if } t \in [\rho_3, 1) \\ 0 & \text{elsewhere.} \end{cases} \quad (13)$$

Generally,

$$p_{i,n}(t) = \int_0^t p_{i,n-1}(t) dt. \quad (14)$$

Thus $p_{i,n}(t)$ is obtained as follows [10]:

$$p_{i,n}(t) = \begin{cases} 0 & \text{for } t \in [0, \rho_1), \\ \frac{(t - \rho_1)^n}{n!} & \text{for } t \in [\rho_1, \rho_2), \\ \frac{[(t - \rho_1)^n - 2(\rho_1 - \rho_2)^n]}{n!} & \text{for } t \in [\rho_2, \rho_3), \\ \frac{1}{n!}[(t - \rho_1)^n - 2(\rho_1 - \rho_2)^n + (t - \rho_3)^n], & \text{for } t \in [\rho_3, 1). \end{cases} \quad (15)$$

We also introduce the following notation:

$$C_{i,v} = \int_0^1 p_{i,v}(t) dt. \quad (16)$$

For HWC technique, the interval $[\alpha, \beta]$ is discretized using the formula:

$$t_m = \alpha + (\beta - \alpha) \frac{m - 1/2}{2M} \quad m = 1, 2, 3, 4, \dots, 2M. \quad (17)$$

Equation (17) is known as collocation point (CP). Gauss points (GPs) are also known as integration points because numerical integration is carried out at these points. These points are represented as:

$$G_j = h \left(\frac{j-1}{2} + \frac{3-\sqrt{3}}{6} \right),$$

$$G_{j+1} = h \left(\frac{j-1}{2} + \frac{3+\sqrt{3}}{6} \right), \quad j = 1, 2, 3, 4, \dots, N-1.$$

3 Haar collocation technique

In this section, HWC scheme is developed for solution of twelfth-order both linear and nonlinear BVPs. We developed HWC method for interval $[0, 1]$. Let $u^{(12)}(t) \in L_2[0, 1]$, then

$$u^{(12)}(t) = \sum_{i=1}^N a_i h_i(t). \quad (18)$$

Integrating equation (18) from 0 to t and using BCs, we obtain the values of $u^{(11)}, u^{(10)}, u^{(9)}, u^{(8)}, u^{(7)}, u^{(6)}, u^{(5)}, u^{(4)}, u^{(3)}, u^{(2)}, u^{(1)}$, and $u(t)$ as follows:

$$u^{(11)}(t) = u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,1}(t), \quad (19)$$

$$u^{(10)}(t) = u^{(10)}(0) + tu^{(11)}(0) + \sum_{i=1}^N a_i p_{i,2}(t), \quad (20)$$

$$u^{(9)}(t) = u^{(9)}(0) + tu^{(10)}(0) + \frac{t^2}{2}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,3}(t), \quad (21)$$

$$u^{(8)}(t) = u^{(8)}(0) + tu^{(9)}(0) + \frac{t^2}{2}u^{(10)}(0) + \frac{t^3}{6}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,4}(t), \quad (22)$$

$$u^{(7)}(t) = u^{(7)}(0) + tu^{(8)}(0) + \frac{t^2}{2}u^{(9)}(0) + \frac{t^3}{6}u^{(10)}(0) + \frac{t^4}{24}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,5}(t), \quad (23)$$

$$u^{(6)}(t) = u^{(6)}(0) + tu^{(7)}(0) + \frac{t^2}{2}u^{(8)}(0) + \frac{t^3}{6}u^{(9)}(0) + \frac{t^4}{24}u^{(10)}(0) + \frac{t^5}{120}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,6}(t), \quad (24)$$

$$u^{(5)}(t) = \alpha_5 + tu^{(6)}(0) + \frac{t^2}{2}u^{(7)}(0) + \frac{t^3}{6}u^{(8)}(0) + \frac{t^4}{24}u^{(9)}(0) + \frac{t^5}{120}u^{(10)}(0) + \frac{t^6}{720}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,7}(t), \quad (25)$$

$$u^{(4)}(t) = \alpha_4 + t\alpha_5 + \frac{t^2}{2}u^{(6)}(0) + \frac{t^3}{6}u^{(7)}(0) + \frac{t^4}{24}u^{(8)}(0) + \frac{t^5}{120}u^{(9)}(0) + \frac{t^6}{720}u^{(10)}(0) + \frac{t^7}{5040}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,8}(t), \quad (26)$$

$$u^{(3)}(t) = \alpha_3 + t\alpha_4 + \frac{t^2}{2}\alpha_5 + \frac{t^3}{6}u^{(6)}(0) + \frac{t^4}{24}u^{(7)}(0) + \frac{t^5}{120}u^{(8)}(0) + \frac{t^6}{720}u^{(9)}(0) + \frac{t^7}{5040}u^{(10)}(0) + \frac{t^8}{40320}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,9}(t), \quad (27)$$

$$u^{(2)}(t) = \alpha_2 + t\alpha_3 + \frac{t^2}{2}\alpha_4 + \frac{t^3}{6}\alpha_5 + \frac{t^4}{24}u^{(6)}(0) + \frac{t^5}{120}u^{(7)}(0) + \frac{t^6}{720}u^{(8)}(0) + \frac{t^7}{5040}u^{(9)}(0) + \frac{t^8}{40320}u^{(10)}(0) + \frac{t^9}{362880}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,10}(t), \quad (28)$$

$$u^{(1)}(t) = \alpha_1 + t\alpha_2 + \frac{t^2}{2}\alpha_3 + \frac{t^3}{6}\alpha_4 + \frac{t^4}{24}\alpha_5 + \frac{t^5}{120}u^{(6)}(0) + \frac{t^6}{720}u^{(7)}(0) + \frac{t^7}{5040}u^{(8)}(0) + \frac{t^8}{40320}u^{(9)}(0) + \frac{t^9}{362880}u^{(10)}(0) + \frac{t^{10}}{3628800}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,11}(t), \quad (29)$$

$$u(t) = \alpha_0 + t\alpha_1 + \frac{t^2}{2}\alpha_2 + \frac{t^3}{6}\alpha_3 + \frac{t^4}{24}\alpha_4 + \frac{t^5}{120}\alpha_5 + \frac{t^6}{720}u^{(6)}(0) + \frac{t^7}{5040}u^{(7)}(0) + \frac{t^8}{40320}u^{(8)}(0) + \frac{t^9}{362880}u^{(9)}(0) + \frac{t^{10}}{3628800}u^{(10)}(0) + \frac{t^{11}}{39916800}u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,12}(t). \quad (30)$$

Now to find the unknown $u^{(6)}(0)$, $u^{(7)}(0)$, $u^{(8)}(0)$, $u^{(9)}(0)$, $u^{(10)}(0)$, and $u^{(11)}(0)$ in equation (30), we integrate equations (24)–(29) each from 0 to 1, and using BCs we obtain the following expressions:

$$\beta_5 - \alpha_5 = u^{(6)}(0) + \frac{1}{2}u^{(7)}(0) + \frac{1}{6}u^{(8)}(0) + \frac{1}{24}u^{(9)}(0) + \frac{1}{120}u^{(10)}(0) + \frac{1}{720}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,6}, \quad (31)$$

$$\beta_4 - \alpha_4 - \alpha_5 = \frac{1}{2}u^{(6)}(0) + \frac{1}{6}u^{(7)}(0) + \frac{1}{24}u^{(8)}(0) + \frac{1}{120}u^{(9)}(0) + \frac{1}{720}u^{(10)}(0) + \frac{1}{5040}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,7}, \quad (32)$$

$$\beta_3 - \alpha_3 - \alpha_4 - \frac{1}{2}\alpha_5 = \frac{1}{6}u^{(6)}(0) + \frac{1}{24}u^{(7)}(0) + \frac{1}{120}u^{(8)}(0) + \frac{1}{720}u^{(9)}(0) + \frac{1}{5040}u^{(10)}(0) + \frac{1}{40320}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,8}, \quad (33)$$

$$\begin{aligned} \beta_2 - \alpha_2 - \alpha_3 - \frac{1}{2}\alpha_4 - \frac{1}{6}\alpha_5 &= \frac{1}{24}u^{(6)}(0) + \frac{1}{120}u^{(7)}(0) + \frac{1}{720}u^{(8)}(0) \\ &+ \frac{1}{5040}u^{(9)}(0) + \frac{1}{40320}u^{(10)}(0) \\ &+ \frac{1}{362880}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,9}, \end{aligned} \quad (34)$$

$$\begin{aligned} \beta_1 - \alpha_1 - \alpha_2 - \frac{1}{2}\alpha_3 - \frac{1}{6}\alpha_4 - \frac{1}{24}\alpha_5 &= \frac{1}{120}u^{(6)}(0) + \frac{1}{720}u^{(7)}(0) + \frac{1}{5040}u^{(8)}(0) \\ &+ \frac{1}{40320}u^{(9)}(0) + \frac{1}{362880}u^{(10)}(0) \\ &+ \frac{1}{3628800}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,10}, \end{aligned} \quad (35)$$

$$\begin{aligned} \beta_0 - \alpha_0 - \alpha_1 - \frac{1}{2}\alpha_2 - \frac{1}{6}\alpha_3 - \frac{1}{24}\alpha_4 - \frac{1}{120}\alpha_5 &= \frac{1}{720}u^{(6)}(0) + \frac{1}{5040}u^{(7)}(0) + \frac{1}{40320}u^{(8)}(0) \\ &+ \frac{1}{362880}u^{(9)}(0) + \frac{1}{3628800}u^{(10)}(0) \\ &+ \frac{1}{39916800}u^{(11)}(0) + \sum_{i=1}^N a_i c_{i,11}. \end{aligned} \quad (36)$$

Finally solving these equations simultaneously from equations (31)–(36), we get the below unknown $u^{(6)}(0)$, $u^{(7)}(0)$, $u^{(8)}(0)$, $u^{(9)}(0)$, $u^{(10)}(0)$, and $u^{(11)}(0)$, respectively:

$$\begin{aligned} u^{(6)}(0) = & -\frac{6470201353035691}{6757837294824}\beta_5 + \frac{118516580759761}{32010808238640}\alpha_5 + \frac{2151202903913401}{3201080823864}\beta_4 - \frac{27713230633076743}{7602566956677}\alpha_4 \\ & - \frac{270012512586410}{98734635801}\beta_3 - \frac{19984894766555}{76793605623}\alpha_3 - \frac{128319829296376810}{7602566956677}\beta_2 - \frac{210931438483313885}{2172161987622}\alpha_2 \\ & - \frac{577747094319923377549975}{70107941048334922089}\beta_1 - \frac{2207122273538488435}{15205133913354}\alpha_0 - \frac{35091125295843245}{400135102983}\alpha_1 \\ & - \frac{2207122273538488435}{15205133913354}\beta_0 - \frac{42990435239}{215108139000}\sum_{i=1}^N a_i c_{i,6} - \frac{2343267753345241}{1600540411932}\sum_{i=1}^N a_i c_{i,7} \\ & - \frac{566218149728848055}{15205133913354}\sum_{i=1}^N a_i c_{i,8} + \frac{290305544850841105}{15205133913354}\sum_{i=1}^N a_i c_{i,9} + \frac{78079399025187}{1792567825}\sum_{i=1}^N a_i c_{i,10} \\ & + \frac{20731003199424}{358513565}\sum_{i=1}^N a_i c_{i,11}, \end{aligned} \quad (37)$$

$$\begin{aligned}
u^{(7)}(0) = & \frac{216602183598847}{12702701682} \beta_5 - \frac{1340551706356039}{114324315138} \beta_4 + \frac{3689401654484092}{57162157569} \alpha_4 + \frac{7331969120981}{1143243151380} \alpha_5 \\
& + \frac{3746059237983320}{230380816869} \beta_3 + \frac{5756575050500}{577395531} \alpha_3 + \frac{17760563974515640}{57162157569} \beta_2 + \frac{101959974465237070}{57162157569} \alpha_2 \\
& + \frac{950775109242156354562900}{23369313682778307363} \beta_1 + \frac{151757773725749150}{57162157569} \alpha_0 + \frac{93164723291813900}{57162157569} \alpha_1 \\
& + \frac{151757773725749150}{57162157569} \beta_0 + \frac{91157358413}{2830370250} \sum_{i=1}^N a_i c_{i,6} + \frac{1498319261246479}{57162157569} \sum_{i=1}^N a_i c_{i,7} \\
& + \frac{72589568721481930}{1086080993811} \sum_{i=1}^N a_i c_{i,8} - \frac{20268761916539210}{57162157569} \sum_{i=1}^N a_i c_{i,9} - \frac{92127552012888}{358513565} \sum_{i=1}^N a_i c_{i,10} \\
& - \frac{69772030495776}{71702713} \sum_{i=1}^N a_i c_{i,11},
\end{aligned} \tag{38}$$

$$\begin{aligned}
u^{(8)}(0) = & -\frac{82753633919459575}{563153107902} \beta_5 + \frac{26280691607835565}{266756735322} \beta_4 - \frac{6441191282440120}{230380816869} \beta_3 - \frac{1403002294599812260}{2534188985559} \alpha_4 \\
& - \frac{323345554427839}{533513470644} \alpha_5 - \frac{2416572966468620}{25597868541} \alpha_3 - \frac{6884103002774936200}{2534188985559} \beta_2 - \frac{5629521122144896450}{362026997937} \alpha_2 \\
& + \frac{294793943471530806860500}{23369313682778307363} \beta_1 - \frac{58631564241173563550}{2534188985559} \alpha_0 - \frac{1905938466108421700}{133378367661} \alpha_1 \\
& - \frac{58631564241173563550}{2534188985559} \beta_0 - \frac{1774331045959}{3585135650} \sum_{i=1}^N a_i c_{i,6} - \frac{29977939959398485}{133378367661} \sum_{i=1}^N a_i c_{i,7} \\
& + \frac{137394719262534170}{2534188985559} \sum_{i=1}^N a_i c_{i,8} + \frac{7926586457424639050}{2534188985559} \sum_{i=1}^N a_i c_{i,9} + \frac{166738121772516}{358513565} \sum_{i=1}^N a_i c_{i,10} \\
& + \frac{676798603391232}{71702713} \sum_{i=1}^N a_i c_{i,11},
\end{aligned} \tag{39}$$

$$\begin{aligned}
u^{(9)}(0) = & \frac{190386159422417555}{281576553951} \beta_5 - \frac{58990309819975385}{133378367661} \beta_4 - \frac{8674505759448400}{230380816869} \beta_3 \\
& + \frac{31867182012494208400}{2534188985559} \beta_2 + \frac{6432400181871939640}{2534188985559} \alpha_4 + \frac{1243141312426859}{266756735322} \alpha_5 \\
& + \frac{10439997376280600}{25597868541} \alpha_3 + \frac{25990558745826853300}{362026997937} \alpha_2 - \frac{1946079602919231779603000}{3338473383254043909} \beta_1 \\
& + \frac{270957847562001413900}{2534188985559} \alpha_0 + \frac{8824481642824250600}{133378367661} \alpha_1 + \frac{270957847562001413900}{2534188985559} \beta_0 \\
& + \frac{5707992885539}{1792567825} \sum_{i=1}^N a_i c_{i,6} + \frac{136802974884271090}{133378367661} \sum_{i=1}^N a_i c_{i,7} - \frac{442502191489463300}{2534188985559} \sum_{i=1}^N a_i c_{i,8} \\
& - \frac{36967696475892148100}{2534188985559} \sum_{i=1}^N a_i c_{i,9} + \frac{112480197536952}{358513565} \sum_{i=1}^N a_i c_{i,10} - \frac{3577685847232896}{71702713} \sum_{i=1}^N a_i c_{i,11},
\end{aligned} \tag{40}$$

$$\begin{aligned}
u^{(10)}(0) = & \frac{-54329321975175680}{40225221993} \beta_5 + \frac{16470508603713200}{19054052523} \beta_4 + \frac{18436417520000}{215108139} \beta_3 - \frac{9112831100007568000}{362026997937} \beta_2 \\
& + \frac{2244515359000000}{1927317413} \beta_1 - \frac{77417447408684518400}{362026997937} \alpha_0 - \frac{2521304667636953600}{19054052523} \alpha_1 \\
& - \frac{1830556686934688800}{362026997937} \alpha_4 - \frac{225581421589120}{19054052523} \alpha_5 - \frac{2613911678271200}{3656838363} \alpha_3 \\
& - \frac{51908694786208086400}{362026997937} \alpha_2 - \frac{77417447408684518400}{362026997937} \beta_0 - \frac{2885469830656}{358513565} \sum_{i=1}^N a_i c_{i,6} \\
& - \frac{38702962690381600}{19054052523} \sum_{i=1}^N a_i c_{i,7} + \frac{89949647479520000}{362026997937} \sum_{i=1}^N a_i c_{i,8} + \frac{10635309524455760000}{362026997937} \sum_{i=1}^N a_i c_{i,9} \\
& - \frac{243552012850944}{358513565} \sum_{i=1}^N a_i c_{i,10} + \frac{8047769347763712}{71702713} \sum_{i=1}^N a_i c_{i,11},
\end{aligned} \tag{41}$$

$$\begin{aligned}
u^{(11)}(0) = & \frac{328017836400}{71702713}\beta_5 - \frac{33777580400}{11321481}\beta_4 + \frac{18436417520000}{215108139}\beta_3 + \frac{13068852928000}{215108139}\beta_2 + \frac{3393234313600}{215108139}\alpha_4 \\
& - \frac{3377758040}{11321481}\alpha_5 - \frac{1415792400000}{71702713}\alpha_3 + \frac{71878691080000}{215108139}\alpha_2 + \frac{3095254640000}{11321481}\beta_1 \\
& + \frac{117619676360000}{215108139}\alpha_0 + \frac{3095254640000}{11321481}\alpha_1 + \frac{117619676360000}{215108139}\beta_0 - \frac{6482565936}{358513565} \sum_{i=1}^N a_i c_{i,6} \\
& + \frac{67555160800}{11321481} \sum_{i=1}^N a_i c_{i,7} - \frac{1633606616000}{215108139} \sum_{i=1}^N a_i c_{i,8} - \frac{16102693784000}{215108139} \sum_{i=1}^N a_i c_{i,9} \\
& + \frac{816803307936}{358513565} \sum_{i=1}^N a_i c_{i,10} - \frac{39206558780928}{71702713} \sum_{i=1}^N a_i c_{i,11}.
\end{aligned} \tag{42}$$

3.1 Linear case

For linear case substituting the values from equations (18) and (30) to equation (2), simplification leads the following system of equations:

$$\begin{aligned}
\sum_{i=1}^N a_i \left(h_i(t_j) + f(t_j) \right) & \left(\frac{t_j^6}{720} \left(-\frac{42990435239}{215108139000} c_{i,6} - \frac{2343267753345241}{1600540411932} c_{i,7} - \frac{566218149728848055}{15205133913354} c_{i,8} \right. \right. \\
& + \frac{290305544850841105}{15205133913354} c_{i,9} + \frac{78079399025187}{1792567825} c_{i,10} + \left. \frac{20731003199424}{358513565} c_{i,11} \right) \\
& + \frac{t_j^7}{5040} \left(\frac{91157358413}{2830370250} c_{i,6} + \frac{1498319261246479}{57162157569} c_{i,7} + \frac{72589568721481930}{1086080993811} c_{i,8} \right. \\
& - \frac{20268761916539210}{57162157569} c_{i,9} - \frac{92127552012888}{358513565} - \frac{69772030495776}{71702713} c_{i,11} \left. \right) + \frac{t_j^8}{40320} \left(-\frac{1774331045959}{3585135650} c_{i,6} \right. \\
& - \frac{29977939959398485}{133378367661} c_{i,7} + \frac{137394719262534170}{2534188985559} c_{i,8} + \frac{7926586457424639050}{2534188985559} c_{i,9} + \frac{166738121772516}{358513565} c_{i,10} \\
& + \frac{676798603391232}{71702713} c_{i,11} \left. \right) + \frac{t_j^9}{362880} \left(\frac{5707992885539}{1792567825} c_{i,6} + \frac{136802974884271090}{133378367661} c_{i,7} - \frac{442502191489463300}{2534188985559} c_{i,8} \right. \\
& - \frac{36967696475892148100}{2534188985559} c_{i,9} + \frac{112480197536952}{358513565} c_{i,10} - \frac{3577685847232896}{71702713} c_{i,11} \left. \right) \\
& + \frac{t_j^{10}}{3628800} \left(-\frac{2885469830656}{358513565} c_{i,6} - \frac{38702962690381600}{19054052523} c_{i,7} + \frac{89949647479520000}{362026997937} c_{i,8} \right. \\
& + \frac{10635309524455760000}{362026997937} c_{i,9} - \frac{243552012850944}{358513565} c_{i,10} + \frac{8047769347763712}{71702713} c_{i,11} \left. \right) \\
& + \frac{t_j^{11}}{39916800} \left(-\frac{6482565936}{358513565} c_{i,6} + \frac{67555160800}{11321481} c_{i,7} - \frac{1633606616000}{215108139} c_{i,8} - \frac{16102693784000}{215108139} c_{i,9} \right. \\
& + \frac{816803307936}{358513565} c_{i,10} - \frac{39206558780928}{71702713} c_{i,11} \left. \right) + a_i p_{i,12}(t_j) \Bigg).
\end{aligned}$$

$$\begin{aligned}
&= h(t_j) - f(t_j) \left(\alpha_0 + t_j \alpha_1 + \frac{t_j^2}{2} \alpha_2 + \frac{t_j^3}{6} \alpha_3 + \frac{t_j^4}{24} \alpha_4 + \frac{t_j^5}{120} \alpha_5 + \frac{t_j^6}{720} \left(-\frac{6470201353035691}{6757837294824} \beta_5 + \frac{118516580759761}{32010808238640} \alpha_5 \right. \right. \\
&\quad + \frac{2151202903913401}{3201080823864} \beta_4 - \frac{27713230633076743}{7602566956677} \alpha_4 - \frac{270012512586410}{98734635801} \beta_3 - \frac{19984894766555}{76793605623} \alpha_3 \\
&\quad - \frac{128319829296376810}{7602566956677} \beta_2 - \frac{210931438483313885}{2172161987622} \alpha_2 - \frac{577747094319923377549975}{70107941048334922089} \beta_1 - \frac{2207122273538488435}{15205133913354} \alpha_0 \\
&\quad - \frac{35091125295843245}{400135102983} \alpha_1 - \frac{2207122273538488435}{15205133913354} \beta_0 \Big) + \frac{t_j^7}{5040} \left(\frac{216602183598847}{12702701682} \beta_5 - \frac{1340551706356039}{114324315138} \beta_4 \right. \\
&\quad + \frac{3689401654484092}{57162157569} \alpha_4 + \frac{7331969120981}{1143243151380} \alpha_5 + \frac{3746059237983320}{230380816869} \beta_3 + \frac{5756575050500}{577395531} \alpha_3 \\
&\quad + \frac{17760563974515640}{57162157569} \beta_2 + \frac{101959974465237070}{57162157569} \alpha_2 + \frac{950775109242156354562900}{23369313682778307363} \beta_1 \\
&\quad + \frac{151757773725749150}{57162157569} \alpha_0 + \frac{93164723291813900}{57162157569} \alpha_1 + \frac{151757773725749150}{57162157569} \beta_0 \Big) + \frac{t_j^8}{40320} \left(-\frac{82753633919459575}{563153107902} \beta_5 \right. \\
&\quad + \frac{26280691607835565}{266756735322} \beta_4 - \frac{6441191282440120}{230380816869} \beta_3 - \frac{1403002294599812260}{2534188985559} \alpha_4 \\
&\quad - \frac{323345554427839}{533513470644} \alpha_5 - \frac{2416572966468620}{25597868541} \alpha_3 - \frac{6884103002774936200}{2534188985559} \beta_2 - \frac{5629521122144896450}{362026997937} \alpha_2 \\
&\quad + \frac{294793943471530806860500}{23369313682778307363} \beta_1 - \frac{58631564241173563550}{2534188985559} \alpha_0 \\
&\quad - \frac{1905938466108421700}{133378367661} \alpha_1 - \frac{58631564241173563550}{2534188985559} \beta_0 \Big) \\
&\quad + \frac{t_j^9}{362880} \left(\frac{190386159422417555}{281576553951} \beta_5 - \frac{58990309819975385}{133378367661} \beta_4 - \frac{8674505759448400}{230380816869} \beta_3 \right. \\
&\quad + \frac{31867182012494208400}{2534188985559} \beta_2 + \frac{6432400181871939640}{2534188985559} \alpha_4 + \frac{1243141312426859}{266756735322} \alpha_5 \\
&\quad + \frac{10439997376280600}{25597868541} \alpha_3 + \frac{25990558745826853300}{362026997937} \alpha_2 - \frac{1946079602919231779603000}{3338473383254043909} \beta_1 \\
&\quad + \frac{270957847562001413900}{2534188985559} \alpha_0 + \frac{8824481642824250600}{133378367661} \alpha_1 + \frac{270957847562001413900}{2534188985559} \beta_0 \Big) \\
&\quad + \frac{t_j^{10}}{3628800} \left(-\frac{54329321975175680}{40225221993} \beta_5 + \frac{16470508603713200}{19054052523} \beta_4 + \frac{18436417520000}{215108139} \beta_3 - \frac{9112831100007568000}{362026997937} \beta_2 \right. \\
&\quad + \frac{2244515359000000}{1927317413} \beta_1 - \frac{77417447408684518400}{362026997937} \alpha_0 - \frac{2521304667636953600}{19054052523} \alpha_1 - \frac{1830556686934688800}{362026997937} \alpha_4 \\
&\quad - \frac{225581421589120}{19054052523} \alpha_5 - \frac{2613911678271200}{3656838363} \alpha_3 \\
&\quad - \frac{51908694786208086400}{362026997937} \alpha_2 - \frac{77417447408684518400}{362026997937} \beta_0 \Big) \\
&\quad + \frac{t_j^{11}}{39916800} \left(\frac{328017836400}{71702713} \beta_5 - \frac{3377580400}{11321481} \beta_4 + \frac{18436417520000}{215108139} \beta_3 \right. \\
&\quad + \frac{13068852928000}{215108139} \beta_2 + \frac{3393234313600}{215108139} \alpha_4 - \frac{337758040}{11321481} \alpha_5 - \frac{1415792400000}{71702713} \alpha_3 + \frac{71878691080000}{215108139} \alpha_2 \\
&\quad + \frac{3095254640000}{11321481} \beta_1 + \frac{117619676360000}{215108139} \alpha_0 + \frac{3095254640000}{11321481} \alpha_1 + \frac{117619676360000}{215108139} \beta_0 \Big) \Big). \tag{43}
\end{aligned}$$

Gauss elimination technique is used for the solution of this $N \times N$ linear system. The solution gives the values of unknown coefficients a_i 's. The solution at discrete CPs is obtained using these coefficients a_i 's in equation (30).

3.2 Nonlinear case

For nonlinear twelfth-order BVP, substituting the values of $u(t_j)$, $u^{(1)}(t_j)$, $u^{(2)}(t_j)$, $u^{(3)}(t_j)$, $u^{(4)}(t_j)$, $u^{(5)}(t_j)$, $u^{(6)}(t_j)$, $u^{(7)}(t_j)$, $u^{(8)}(t_j)$, $u^{(9)}(t_j)$, $u^{(10)}(t_j)$, $u^{(11)}(t_j)$, and $u^{(12)}(t_j)$ in (1) and the CPs, simplification leads to the following nonlinear system:

$$\begin{aligned}
 & F(t_j, u, u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}, u^{(5)}, u^{(6)}, u^{(7)}, u^{(8)}, u^{(9)}, u^{(10)}, u^{(11)}) \\
 &= \sum_{i=1}^N a_i h_i(t_j) - f \left(t_j, \alpha_0 + t_j \alpha_1 + \frac{t_j^2}{2} \alpha_2 + \frac{t_j^3}{6} \alpha_3 \right. \\
 &+ \frac{t_j^4}{24} \alpha_4 + \frac{t_j^5}{120} \alpha_5 + \frac{t_j^6}{720} u^{(6)}(0) + \frac{t_j^7}{5040} u^{(7)}(0) + \frac{t_j^8}{40320} u^{(8)}(0) + \frac{t_j^9}{362880} u^{(9)}(0) \\
 &+ \frac{t_j^{10}}{3628800} u^{(10)}(0) + \frac{t_j^{11}}{39916800} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,12}(t_j), \alpha_1 + t_j \alpha_2 + \frac{t_j^2}{2} \alpha_3 + \frac{t_j^3}{6} \alpha_4 + \frac{t_j^4}{24} \alpha_5 \\
 &+ \frac{t_j^5}{120} u^{(6)}(0) + \frac{t_j^6}{720} u^{(7)}(0) + \frac{t_j^7}{5040} u^{(8)}(0) + \frac{t_j^8}{40320} u^{(9)}(0) + \frac{t_j^9}{362880} u^{(10)}(0) \\
 &+ \frac{t_j^{10}}{3628800} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,11}(t_j), \alpha_2 + t_j \alpha_3 + \frac{t_j^2}{2} \alpha_4 + \frac{t_j^3}{6} \alpha_5 + \frac{t_j^4}{24} u^{(6)}(0) + \frac{t_j^5}{120} u^{(7)}(0) + \frac{t_j^6}{720} u^{(8)}(0) \\
 &+ \frac{t_j^7}{5040} u^{(9)}(0) + \frac{t_j^8}{40320} u^{(10)}(0) + \frac{t_j^9}{362880} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,10}(t_j), \alpha_3 + t_j \alpha_4 + \frac{t_j^2}{2} \alpha_5 + \frac{t_j^3}{6} u^{(6)}(0) \\
 &+ \frac{t_j^4}{24} u^{(7)}(0) + \frac{t_j^5}{120} u^{(8)}(0) + \frac{t_j^6}{720} u^{(9)}(0) + \frac{t_j^7}{5040} u^{(10)}(0) + \frac{t_j^8}{40320} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,9}(t_j), \alpha_4 \\
 &+ t_j \alpha_5 + \frac{t_j^2}{2} u^{(6)}(0) + \frac{t_j^3}{6} u^{(7)}(0) + \frac{t_j^4}{24} u^{(8)}(0) + \frac{t_j^5}{120} u^{(9)}(0) + \frac{t_j^6}{720} u^{(10)}(0) + \frac{t_j^7}{5040} u^{(11)}(0) \\
 &+ \sum_{i=1}^N a_i p_{i,8}(t_j), \alpha_5 + t_j u^{(6)}(0) + \frac{t_j^2}{2} u^{(7)}(0) + \frac{t_j^3}{6} u^{(8)}(0) + \frac{t_j^4}{24} u^{(9)}(0) + \frac{t_j^5}{120} u^{(10)}(0) + \frac{t_j^6}{720} u^{(11)}(0) \\
 &+ \sum_{i=1}^N a_i p_{i,7}(t_j), u^{(6)}(0) + t_j u^{(7)}(0) + \frac{t_j^2}{2} u^{(8)}(0) + \frac{t_j^3}{6} u^{(9)}(0) + \frac{t_j^4}{24} u^{(10)}(0) + \frac{t_j^5}{120} u^{(11)}(0) \\
 &+ \sum_{i=1}^N a_i p_{i,6}(t_j), + t_j u^{(8)}(0) + \frac{t_j^2}{2} u^{(9)}(0) + \frac{t_j^3}{6} u^{(10)}(0) + \frac{t_j^4}{24} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,5}(t_j), u^{(8)}(0) \\
 &+ t_j u^{(9)}(0) + \frac{t_j^2}{2} u^{(10)}(0) + \frac{t_j^3}{6} u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,4}(t_j), u^{(9)}(0) + t_j u^{(10)}(0) + \frac{t_j^2}{2} u^{(11)}(0) \\
 &+ \sum_{i=1}^N a_i p_{i,3}(t_j), u^{(10)}(0) + t_j u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,2}(t_j), u^{(11)}(0) + \sum_{i=1}^N a_i p_{i,1}(t_j), \sum_{i=1}^N a_i h_i(t_j) \Big).
 \end{aligned}$$

The unknown values of $u^{(6)}(0)$, $u^{(7)}(0)$, $u^{(8)}(0)$, $u^{(9)}(0)$, $u^{(10)}(0)$, and $u^{(11)}(0)$, which are used in the above nonlinear system, are already calculated above in equations (37)–(42). In addition, we can easily calculate the Jacobian of the above very long-lasting equation by taking their partial derivatives with respect to a_i . Broyden's technique is used for solution of this $N \times N$ nonlinear system of algebraic equations. The solution gives the values of unknown Haar coefficients a_i 's. The approximate solution at discrete CPs can be easily obtained using these unknown coefficients a_i 's in equation (30).

4 Numerical examples

In this section, some examples are given to show the performance of the HWC technique. Three linear and two nonlinear twelfth-order BVPs are tested using the proposed HWC algorithm. If u_{ap} represents the approximate and u_{ex} represents the exact solution at CPs and GPs, respectively, then the maximum absolute errors E_{cp} and E_{gp} are defined as:

$$E_{\text{cp}} = \max |u_{\text{exc}} - u_{\text{apc}}|,$$

$$E_{\text{gp}} = \max |u_{\text{exg}} - u_{\text{apg}}|.$$

The root mean square root errors M_{cp} and M_{gp} at CPs and GPs are defined as:

$$M_{\text{cp}} = \sqrt{\frac{1}{N} \left(\sum_{i=1}^N |u_{\text{exc}} - u_{\text{apc}}|^2 \right)},$$

$$M_{\text{gp}} = \sqrt{\frac{1}{N} \left(\sum_{i=1}^N |u_{\text{exg}} - u_{\text{apg}}|^2 \right)}.$$

The convergence rate at CPs is denoted by R_{cp} and defined as [32,33]:

$$R_{\text{cp}} = \frac{\log [u_{\text{apc}}(N/2)/u_{\text{apc}}(N)]}{\log 2}. \quad (44)$$

Problem 1. Consider the twelfth-order BVP [34]

$$\begin{cases} u^{(12)}(t) + tu(t) = -(120 + 23t + t^3)e^t, & t \in [0, 1], \\ u(0) = 0, & u(1) = 0, & u^{(1)}(0) = 1, & u^{(1)}(1) = -e, \\ u^{(2)}(0) = 0, & u^{(2)}(1) = -4e, \\ u^{(3)}(0) = -3, \\ u^{(3)}(1) = -9e, \\ u^{(4)}(0) = -8, & u^{(4)}(1) = -16e, \\ u^{(5)}(0) = -15, \\ u^{(5)}(1) = -25e. \end{cases} \quad (45)$$

The analytical solution is

$$u(t) = t(1 - t)e^t. \quad (46)$$

The maximum absolute errors are shown in Table 1. It is observed that the errors in absolute values are better than those of Siddiqi and Akram [34] and Siddiqi and Twizell [35] as shown in Table 2. From Table 2, it is observed that HWC results are better than other methods. The convergence rate is also calculated, which is approximately equal to two. The comparison of approximate and exact solution is given in Figure 1.

Problem 2. Next, we have twelfth-order BVP [34]

$$\begin{cases} u^{(12)}(t) - u(t) = -12(2t \cos t + 11 \sin t), & -1 \leq t \leq 1, \\ u(-1) = u(1) = 0, & u^{(1)}(-1) = u^{(1)}(1) = 2 \sin 1, \\ u^{(2)}(-1) = -u^{(2)}(1) = -4 \cos 1 - 2 \sin 1, \\ u^{(3)}(-1) = u^{(3)}(1) = 6 \cos 1 - 6 \sin 1, \\ u^{(4)}(-1) = -u^{(4)}(1) = 8 \cos 1 + 12 \sin 1, \\ u^{(5)}(-1) = u^{(5)}(1) = -20 \cos 1 + 10 \sin 1. \end{cases} \quad (47)$$

The analytical solution is

$$u(t) = (t^2 - 1) \sin t. \quad (48)$$

The comparison of approximate and exact solution is given in Figure 2. It is observed that the errors in absolute values are better than those of Siddiqi and Akram [34] and Siddiqi and Twizell [35] as shown in Table 3. From Table 4, it is observed that HWC results are better than the

Table 1: E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} for Problem 1

J	$N = 2^{J+1}$	E_{cp}	R_{cp}	E_{gp}	M_{cp}	M_{gp}
1	4	4.080066×10^{-09}	—	1.890321×10^{-12}	2.040527×10^{-09}	9.452123×10^{-13}
2	8	2.429096×10^{-09}	0.7482	1.218525×10^{-12}	8.771540×10^{-10}	4.361947×10^{-13}
3	16	8.924874×10^{-10}	1.4445	4.515832×10^{-13}	2.532597×10^{-10}	1.256058×10^{-13}
4	32	2.675610×10^{-10}	1.7380	1.339484×10^{-13}	6.567113×10^{-11}	3.228519×10^{-14}
5	64	7.306394×10^{-11}	1.8726	3.624878×10^{-14}	1.656898×10^{-11}	8.121789×10^{-15}
6	128	1.907864×10^{-11}	1.9372	9.325873×10^{-15}	4.151756×10^{-12}	2.031743×10^{-15}
7	256	4.873948×10^{-12}	1.9688	2.386979×10^{-15}	1.038536×10^{-12}	5.138567×10^{-16}
8	512	1.231670×10^{-12}	1.9845	6.661338×10^{-16}	2.596703×10^{-13}	1.335675×10^{-16}
9	1024	3.096424×10^{-13}	1.9919	2.775557×10^{-16}	6.492086×10^{-14}	4.701457×10^{-17}

Table 2: Comparison of maximum absolute errors of present method with other methods

Present method	Siddiqi and Akram [34]	Siddiqi and Twizell [35]
3.10×10^{-13}	7.38×10^{-9}	2.07×10^{-3}

Figure 1: Comparison of numerical and analytical solutions for 32 CPs of Problem 1. **Figure 2:** Comparison of numerical and analytical solutions for 32 CPs of Problem 2.

other methods. The convergence rate is also calculated, which is approximately equal to two (Table 5). The comparison of approximate and exact solution is given in Figure 3 for 32 number of CPs.

Problem 3. Consider the linear twelfth-order BVP [34]

Table 3: E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} for Problem 2

J	$N = 2^{J+1}$	E_{cp}	R_{cp}	E_{gp}	M_{cp}	M_{gp}
1	4	4.087801×10^{-9}	—	1.887768×10^{-12}	2.044393×10^{-9}	9.439349×10^{-13}
2	8	2.358883×10^{-9}	0.7932	1.162847×10^{-12}	8.515840×10^{-10}	4.162510×10^{-13}
3	16	8.609238×10^{-10}	1.4541	4.243827×10^{-13}	2.440498×10^{-10}	1.180074×10^{-13}
4	32	2.578136×10^{-10}	1.7396	1.253442×10^{-13}	6.316870×10^{-11}	3.020509×10^{-14}
5	64	7.040099×10^{-11}	1.8727	3.402834×10^{-14}	1.593048×10^{-11}	7.602254×10^{-15}
6	128	1.838557×10^{-11}	1.9370	8.881784×10^{-15}	8.881784×10^{-15}	1.906115×10^{-15}
7	256	4.697249×10^{-12}	1.9687	2.220446×10^{-15}	9.983748×10^{-13}	4.762571×10^{-16}
8	512	1.187088×10^{-12}	1.9844	6.106227×10^{-16}	2.496279×10^{-13}	1.254793×10^{-16}
9	1,024	2.983625×10^{-13}	1.9923	2.775558×10^{-16}	6.240902×10^{-14}	5.778539×10^{-17}

$$\begin{aligned}
 & \left\{ \begin{aligned}
 u^{(12)}(t) - u(t) &= \frac{1}{4096} e^{\sin t} (-1823912 + 3439408 \cos(2t) + 527012068 \cos(4t) - 1060549512 \cos(6t) \\
 &+ 59278120 \cos(8t) - 313256 \cos(10t) + 156 \cos(12t) + 8338252 \sin t + 171393617 \sin(3t) \\
 &+ 1451213951 \sin(5t) - 346542218 \sin(7t) + 5688330 \sin(9t) - 9713 \sin(11t) \\
 &+ \sin(13t)), \quad t \in [0, 1], \\
 u(0) &= 0, \\
 u(1) &= e^{\sin(1)} \sin(1), \\
 u^{(1)}(0) &= 1, \\
 u^{(1)}(1) &= e^{\sin(1)} \cos(1) + e^{\sin(1)} \cos(1) \sin(1), \\
 u^{(2)}(0) &= 2, \\
 u^{(2)}(1) &= 2e^{\sin(1)} \cos^2(1) - e^{\sin(1)} \sin(1) + e^{\sin(1)} \cos^2(1) \sin(1) - e^{\sin(1)} \sin^2(1), \\
 u^{(3)}(0) &= 2, \\
 u^{(3)}(1) &= -e^{\sin 1} \cos 1 + 3e^{\sin 1} \cos^3 1 - 7e^{\sin 1} \cos 1 \sin 1 + e^{\sin 1} \cos^3 1 \sin 1 - 3e^{\sin 1} \cos 1 \sin^2 1, \\
 u^{(4)}(0) &= -4, \\
 u^{(4)}(1) &= -8e^{\sin(1)} \cos^2(1) + 4e^{\sin(1)} \cos^4(1) + e^{\sin(1)} \sin(1) - 22e^{\sin(1)} \cos^2(1) \sin(1) + e^{\sin(1)} \cos^4(1) \sin(1) \\
 &+ 7e^{\sin(1)} \sin^2(1) - 6e^{\sin(1)} \cos^2(1) \sin^2(1) + 3e^{\sin(1)} \sin^3(1), \\
 u^{(5)}(0) &= -24, \\
 u^{(5)}(1) &= e^{\sin(1)} \cos(1) - 30e^{\sin(1)} \cos^3(1) + 5e^{\sin(1)} \cos^5(1) + 31e^{\sin(1)} \cos(1) \sin(1) - 50e^{\sin(1)} \cos^3(1) \sin(1) \\
 &+ e^{\sin(1)} \cos^5(1) \sin(1) + 60e^{\sin(1)} \cos(1) \sin^2(1) - 10e^{\sin(1)} \cos^3(1) \sin^2(1) + 15e^{\sin(1)} \cos(1) \sin^3(1).
 \end{aligned} \right. \quad (49)
 \end{aligned}$$

Table 4: Comparison of maximum absolute errors of present method with other methods

Present method	Siddiqi and Akram [34]	Siddiqi and Twizell [35]
2.98×10^{-13}	4.69×10^{-5}	2.07×10^{-3}

The exact solution is

$$u(t) = e^{\sin t} \sin t. \quad (50)$$

Gauss elimination technique is used for solution of this linear problem. The E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} are given in Table 3.

Problem 4. Consider the nonlinear twelfth-order BVP [36]

Figure 3: Comparison of numerical and analytical solutions for 32 CPs of Problem 3.

Table 5: E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} for Problem 3

J	$N = 2^{J+1}$	E_{cp}	R_{cp}	E_{gp}	M_{cp}	M_{gp}
1	4	4.791599×10^{-5}	—	2.233135×10^{-8}	2.396389×10^{-5}	1.116627×10^{-8}
2	8	2.464102×10^{-5}	0.9594	1.196611×10^{-8}	8.894615×10^{-6}	4.283299×10^{-9}
3	16	8.712958×10^{-6}	1.4998	4.095127×10^{-9}	2.467474×10^{-6}	1.137446×10^{-9}
4	32	2.589810×10^{-6}	1.7503	1.188665×10^{-9}	6.334443×10^{-7}	2.857365×10^{-10}
5	64	7.060396×10^{-7}	1.8750	3.199413×10^{-10}	1.594197×10^{-7}	7.144727×10^{-11}
6	128	1.843290×10^{-7}	1.9375	8.298362×10^{-11}	3.992147×10^{-8}	1.786133×10^{-11}
7	256	4.709231×10^{-8}	1.9687	2.113087×10^{-11}	9.984531×10^{-9}	4.465273×10^{-12}
8	512	1.190143×10^{-8}	1.9844	5.331291×10^{-12}	2.496393×10^{-9}	1.116319×10^{-12}
9	1,024	2.991534×10^{-9}	1.9922	1.338707×10^{-12}	6.241146×10^{-10}	2.790845×10^{-13}

Table 6: E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} for Problem 4

J	$N = 2^{J+1}$	E_{cp}	R_{cp}	E_{gp}	M_{cp}	M_{gp}
1	4	2.422818×10^{-11}	—	1.121325×10^{-14}	1.211702×10^{-11}	5.607176×10^{-15}
2	8	1.480859×10^{-11}	0.7103	7.438494×10^{-15}	5.347973×10^{-12}	2.654397×10^{-15}
3	16	5.478839×10^{-12}	1.4345	2.775558×10^{-15}	1.555463×10^{-12}	7.913990×10^{-16}
4	32	1.645017×10^{-12}	1.7358	8.881784×10^{-16}	4.040732×10^{-13}	2.338726×10^{-16}
5	64	4.493628×10^{-13}	1.8721	3.330669×10^{-16}	1.019971×10^{-13}	1.279469×10^{-16}
6	128	1.172951×10^{-13}	1.9377	3.330669×10^{-16}	2.555701×10^{-14}	1.118863×10^{-16}
7	256	2.997602×10^{-14}	1.9683	4.440892×10^{-16}	6.391976×10^{-15}	1.275700×10^{-16}

Table 7: E_{cp} , R_{cp} , E_{gp} , M_{cp} , and M_{gp} for Problem 5

J	$N = 2^{J+1}$	E_{cp}	R_{cp}	E_{gp}	M_{cp}	M_{gp}
1	4	5.640377×10^{-11}	—	2.531308×10^{-14}	2.820869×10^{-11}	1.266044×10^{-14}
2	8	3.268852×10^{-11}	0.7870	1.598721×10^{-14}	1.180128×10^{-11}	5.751769×10^{-15}
3	16	1.194245×10^{-11}	1.4527	6.217249×10^{-15}	3.385745×10^{-12}	1.807312×10^{-15}
4	32	3.577583×10^{-12}	1.7390	1.332268×10^{-15}	8.766552×10^{-13}	4.775249×10^{-16}
5	64	9.769963×10^{-13}	1.8726	8.881784×10^{-16}	2.211255×10^{-13}	3.964289×10^{-16}
6	128	2.549072×10^{-13}	1.9384	2.332268×10^{-16}	5.538092×10^{-14}	4.807407×10^{-16}
7	256	6.572520×10^{-14}	1.9555	1.142386×10^{-16}	1.372139×10^{-14}	4.466837×10^{-16}

Figure 4: Comparison of numerical and analytical solutions for 32 CPs of Problem 4.

$$\begin{cases} u^{(12)}(t) = 2e^t u^2(t) + u^{(3)}(t), & t \in [0, 1], \\ u^{(2k)}(0) = 1, & u^{(2k)}(1) = e^{-1}, \quad k = 0, 1, 2, 3, 4, 5. \end{cases} \quad (51)$$

The analytical solution is

$$u(t) = e^{-t}. \quad (52)$$

Two errors E_{cp} and M_{cp} for distinct number of CPs and GPs are shown in Table 6. Figure 4 represent the comparison of approximate and exact solution for 32 number of CPs. The E_{cp} is decreased up to order $\times 10^{-14}$, whereas the results of Wazwaz [36] are decreased up to $\times 10^{-09}$. The convergence rate is also calculated, which is approximately equal to two (Table 7). The comparison of approximate and exact solution is given in Figure 5 for 32 number of CPs.

Problem 5. Consider the nonlinear twelfth-order BVP [37]

$$\begin{cases} u^{(12)}(t) = \frac{1}{2}e^{-t}u^2(t), & t \in [0, 1], \\ u(0) = u^{(2)}(0) = u^{(4)}(0) = u^{(6)}(0) = u^{(8)}(0) = u^{(10)}(0) = 2, \\ u(1) = u^{(2)}(1) = u^{(4)}(1) = u^{(6)}(1) = u^{(8)}(1) = u^{(10)}(1) = 2. \end{cases} \quad (53)$$

The analytical solution is

$$u(t) = 2e^t. \quad (54)$$

Broyden's technique is used for solution of this problem. Two errors E_{cp} and M_{cp} for distinct number of CPs and GPs are shown in Table 4. From this table, it is evident that by increasing the number of CPs the errors E_{cp} and M_{cp} are decreased. The convergence rate is also calculated, which is approximately equal to two.

Figure 5: Comparison of numerical and analytical solutions for 32 CPs of Problem 5.

5 Conclusion

In this article, the HWC technique is used to find the solution of linear and nonlinear twelfth-order BVPs. The maximum absolute errors for distant number of discrete CPs and GPs are shown for each example in tables. The convergence rate is also calculated, which is approximately equal to two. The maximum absolute errors of present method is compared with Siddiqi and Akram [34], Siddiqi and Twizell [35], and Wazwaz [36]. The results show that the HWC technique is better than the other techniques available in the literature. MATLAB software is used for all computational work.

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