Research Article

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Extension of optimal homotopy asymptotic method with use of Daftardar–Jeffery polynomials to Hirota–Satsuma coupled system of Korteweg–de Vries equations

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Abstract: In this study, the Daftardar–Jeffery polynomials are incorporated in the homotopy of the optimal homotopy asymptotic method (OHAM) for solving the generalized Hirota–Satsuma coupled system of Korteweg–de Vries equations. The results are displayed through graphs and tables. The results obtained in this study are also compared with the published work on OHAM, which shows that OHAM-DJ is more explicit, reliable, and an efficient analytical technique. The exactness of the

developed method can be improved by performing the higher iterations.

Keywords: optimal homotopy asymptotic method, OHAM-DJ, generalized Hirota–Satsuma coupled system of KDV equations

1 Introduction

Differential equations (DEs) are ubiquitous in various disciplines, such as engineering, applied mathematics, and physics, where they are used to model the physical systems in which physical quantities change in space and/or time.

Among DEs, partial differential equations (PDEs) are frequently encountered in modeling various physical processes. Some of the examples of PDEs are Schrodinger equation (govern the evolution of microscopic system), Laplace equation (describing the potential due to charge distribution), Maxwell's equations (govern the propagation of electromagnetic waves), heat equation (monitor the flow of heat energy due to the existence of temperature gradient), and so on.

All these PDEs are linear, in which the principle of superposition holds in contrast to the nonlinear PDEs. Majority of the physical processes are nonlinear in nature; therefore, they are modeled through nonlinear PDEs. Nonlinear Schrodinger equation, Korteweg–de Vries (KdV) equation, Burger equation, Navier–Stoke equations, and so on are some of the important nonlinear PDEs. C. Cattani [1] implemented the wavelet symmetries in modeling the archaea DNA. Recently, fractional operators are introduced in the field of engineering and technology [2–4]. İlhan and Kıymaz [5] extended the fractional operators with applications to DEs. The idea of fractional calculus is implemented to the Navier–Stokes equations by Jena et al. [6]. Baleanu et al [7] used the fractional operators to the planner system masses in an equilateral triangle. A more detailed survey on the fractional calculus can be found in ref. [8].

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The nonlinear coupled system of PDEs plays a significant role in fluid mechanics, plasma physics, chemical kinetics, optics, optical fibers, and geochemistry in modeling different complex physical phenomena. The solution of such type of coupled nonlinear PDEs plays an important role in describing and predicting the behavior of these complex phenomena. The non-linear coupled system of PDEs cannot always be solved exactly. For this purpose, the researchers had developed various techniques for the approximate solutions of such types of problems. The well-established numerical techniques are Runge–Kutta (R–K) method [9], variational iteration algorithm-I [10,11], shooting method (SM) [12], finite difference method (FDM) [13,14], variational iteration algorithm-II [15,16], collocation method [17–20], and expansion methods [21–23]. In 1992, Liao developed homotopy analysis method (HAM) by implementing the homotopy for the approximate solution of nonlinear mathematical problems [24]. Then, Herisanu et al. [25] presented optimal homotopy asymptotic method (OHAM) for solving the nonlinear mathematical problems. A detailed study on the KDV equation is presented by Hosseini et al. [26] using fractional operators. Gao et al. [27] used the fractional operators to solve numerically the falling film problems. Jajarmi and Baleanu [28] used the iterative technique for the fractional order boundary value problems. A more recent numerical study of the Burgers equation is carried out by Sweilam et al. [29]. OHAM makes the perturbative techniques to be free of the initial guess for the small embedding parameter and from the extensive computational process. OHAM is the generalized form of homotopy perturbation method (HPM) due to the usage of general auxiliary function $H(p)$. Different researchers have used numerical methods in order to solve linear and nonlinear problems [30–34]. Jafari and Daftardar-Gejji have developed an innovative method for solving the nonlinear problems in 2006 [35], whose convergence has been proved in ref. [36]. This method was termed as Daftardar-Jafari method (JDM) [37]. Afterward, Ali et al. developed another method based on the modification of OHAM with Daftardar-Jeffery polynomials for solving the nonlinear problems called OHAM-DJ [38,39]. Abbasbandy [40] used HAM to solve the linear as well as the nonlinear system of Klein–Gordon equations.

The KdV equation is a nonlinear third-order PDE. The solution of the KdV equation describes an important class of wave pattern called solitons. A soliton results due to the cancellation of the nonlinear and dispersion phenomena. The nonlinear term in the KdV equation tries to steepen the amplitude of the wave, while the third-order term tries to widen the wave structure due to the

different phase speed of the component waves. The balance between these two effects gives rise to solitons, which are stable wave structures and maintain their shapes to larger distance in the propagating medium. Solitons are very useful in transferring the energy with very small dissipation in the travelling medium. The generalized Hirota–Satsuma system of coupled KdV equation is a set of three nonlinear third-order PDEs whose soliton solution was given by Fan [41]. The generalized Hirota–Satsuma system of coupled KdV equation has been examined by different researchers through various techniques. Some of the applied techniques are HAM [42], HPM [39], Jacobi elliptic method [43], ADM [44], and projective Riccati equation method [45]. OHAM-DJ [46,47] is a very useful technique for solving the nonlinear problem having some peculiar advantages over the conventional techniques. Here, OHAM-DJ is applied to solve the Hirota–Satsuma coupled system of KdV equations and the results are compared with OHAM. The OHAM-DJ results are more reliable and converge rapidly. Therefore, one shall be more confident in applying the OHAM-DJ for solving the nonlinear problems.

The structure of the present article is developed as follows. The main objective of this article is the application of OHAM-DJ to Hirota–Satsuma coupled system of KdV equations. The basics of OHAM-DJ are developed in Section 2. OHAM-DJ is applied to find the approximate solutions of the Hirota–Satsuma coupled system of KdV equations in Section 3. The results are discussed with the help of different tables and graphs in Section 4. The study is concluded finally in Section 5.

2 Basic concept of OHAM-DJ

The basic concept of OHAM-DJ is developed as follows:

Consider a general nonlinear DE:

$$\begin{aligned} M(\alpha(\rho, t)) + F(\alpha(\rho, t)) + l(\rho, t) &= 0, \\ \rho \in \Omega, \quad B\left(\alpha, \frac{\partial \alpha}{\partial t}\right) &= 0, \end{aligned} \quad (1)$$

where M is a linear operator, F is a nonlinear operator, l is a known function, and B represents the boundary operator. By OHAM-DJ formulation, the optimal homotopy $Q(\phi(\rho, t; p)) : \Psi \times [0, 1] \rightarrow R$ satisfies the following nonlinear equation:

$$\begin{aligned} (1 - p)[M(\phi(\rho, t; p)) + l(\rho, t)] \\ = Q(p)[M(\phi(\rho, t; p)) + F(\phi(\rho, t; p)) + l(\rho, t)], \end{aligned} \quad (2)$$

where $p \in [0, 1]$ is the inserting parameter, $Q(p)$ is an auxiliary function for $p \neq 0$, its value is zero, when

$p = 0$. The function $\alpha(\rho, t; p)$ takes the following form at the extreme values of p :

$$\alpha(\rho, t; 0) = \alpha_0(\rho, t)$$

and

$$\alpha(\rho, t; 1) = \alpha(\rho, t).$$

The auxiliary function $Q(p)$ is defined as:

$$Q(p) = \sum_{i=1}^q p^i C_i. \quad (3)$$

where C_i is the conjunction parameter which we will find latter. Consequently, the function $\phi(\rho, t; p)$ can further be improved about p as

$$\phi(\rho, t; p) = \alpha_0(\rho, t) + \sum_{j=1}^{\infty} \alpha_j(\rho, t; C_i) p^j. \quad (4)$$

The function $F(\alpha(\rho, t; p))$ decomposes as

$$\begin{aligned} F(\alpha(\rho, t; p)) &= F(\alpha_0(\rho, t)) + p[F(\alpha_0(\rho, t) + \alpha_1(\rho, t)) \\ &\quad - F(\alpha_0(\rho, t))] + p^2[F(\alpha_0(\rho, t) \\ &\quad + \alpha_1(\rho, t) + \alpha_2(\rho, t)) \\ &\quad - F(\alpha_0(\rho, t) + \alpha_1(\rho, t))] + \dots \end{aligned} \quad (5)$$

The expression on the R.H.S. of equation (5)

$$\begin{aligned} &F(\alpha_0(\rho, t)), [F(\alpha_0(\rho, t) + \alpha_1(\rho, t)) - F(\alpha_0(\rho, t))], \\ &[F(\alpha_0(\rho, t) + \alpha_1(\rho, t) + \alpha_2(\rho, t)) \\ &\quad - F(\alpha_0(\rho, t) + \alpha_1(\rho, t))] + \dots \end{aligned} \quad (6)$$

is the DJ polynomial. The convergence of DJ polynomial has been found by Bhalekar and Daftardar-Gejji [24,36]. For generalization, we consider the polynomial as

$$F(\alpha(\rho, t; p)) = F_0 + \sum_{k=1}^{\infty} p^k F_k, \quad (7)$$

where

$$F_0 = F(\alpha_0(\rho, t)) \quad (8)$$

and

$$F_k = F\left(\sum_{i=0}^q \alpha_i(\rho, t)\right) - F\left(\sum_{i=0}^{q-1} \alpha_i(\rho, t)\right). \quad (9)$$

Using equations (7)–(9) in equation (2), then equating identical expressions of p , we obtain the different order equations as:

Zero-order problem:

$$F(\alpha_0(\rho, t)) + g(\rho, t) = 0, \quad \beta\left(\alpha_0, \frac{\partial \alpha_0}{\partial t}\right) = 0. \quad (10)$$

First-order problem:

$$M(\alpha_1(\rho, t)) = C_1 F_0(\alpha_0(\rho, t)), \quad \beta\left(\alpha_1, \frac{\partial \alpha_1}{\partial t}\right) = 0. \quad (11)$$

Second-order problem:

$$\begin{aligned} &(\alpha_2(\rho, t)) - M(\alpha_1(\rho, t)) \\ &= C_2 F_0(\alpha_0(\rho, t)) + C_1 [F(\alpha_1(\rho, t)) \\ &\quad + F_1(\alpha_0(\rho, t), \alpha_1(\rho, t))] \beta\left(\alpha_2, \frac{\partial \alpha_2}{\partial t}\right) = 0. \end{aligned} \quad (12)$$

The governing relation for $\alpha_j(\rho, t)$ is:

$$\begin{aligned} &M(\alpha_j(\rho, t)) - M(\alpha_{j-1}(\rho, t)) \\ &= C_j F_0(\alpha_0(\rho, t)) \\ &\quad + F_{j-1}(\alpha_0(\rho, t), \alpha_1(\rho, t), \dots, \alpha_{j-1}(\rho, t)) \\ &\quad + \sum_{i=1}^{j-1} C_i [M(\alpha_{j-i}(\rho, t), \beta\left(\alpha_j, \frac{\partial \alpha_j}{\partial t}\right)) = 0, \\ &\quad j = 2, 3, \dots \end{aligned} \quad (13)$$

The solutions of the third and even higher order equations can be easily computed, but the solution up to the second-order problem provides us the required results.

When $p = 1$, equation (4) gives us

$$\tilde{\alpha}(\rho, t; C_i) = \alpha_0(\rho, t) + \sum_{j \geq 1} \alpha_j(\rho, t; C_i). \quad (14)$$

Replacing equation (14) in equation (2), the residual is obtained as

$$R(\rho, t; C_i) = M(\tilde{\alpha}(\rho, t; C_i)) + F(\tilde{\alpha}(\rho, t; C_i)) + l(\rho, t). \quad (15)$$

If $R(\rho, t; C_i) = 0$, then the given problem is solved exactly. The value of the constant C_i can be determined by using different techniques. The most efficient one is the least square method. By using this technique, we construct the function:

$$\aleph(C_i) = \int_0^t \int_{\Omega} R^2(\rho, t; C_i) d\rho dt. \quad (16)$$

The minimization condition is given by

$$\frac{\partial \aleph}{\partial C_1} = \frac{\partial \aleph}{\partial C_2} = \dots = \frac{\partial \aleph}{\partial C_i} = 0. \quad (17)$$

The approximate solution is obtained by replacing the C_i , for $i = 1, 2, 3, \dots, j$ in equation (14).

3 Applications of OHAM-DJ

In this section, OHAM-DJ is applied for solving the Hirota–Satsuma coupled system of KdV equations. The different order equations are obtained and then solved through OHAM-DJ. The first-order solutions are calculated and then compared with the results obtained through OHAM.

3.1 Problem: Hirota–Satsuma coupled system of KdV equations

The Hirota–Satsuma coupled system of KdV equations is given as

$$\begin{aligned} \frac{\partial \alpha(\rho, t)}{\partial t} - \frac{1}{2} \frac{\partial^3 \alpha(\rho, t)}{\partial \rho^3} + 3\alpha(\rho, t) \frac{\partial \alpha(\rho, t)}{\partial \rho} \\ - 3 \frac{\partial}{\partial \rho} \beta(\rho, t) \omega(\rho, t) = 0, \\ \frac{\partial \beta(\rho, t)}{\partial t} + \frac{\partial^3 \beta(\rho, t)}{\partial \rho^3} - 3\alpha(\rho, t) \frac{\partial \beta(\rho, t)}{\partial \rho} = 0, \\ \frac{\partial \omega(\rho, t)}{\partial t} + \frac{\partial^3 \omega(\rho, t)}{\partial \rho^3} - 3\alpha(\rho, t) \frac{\partial \omega(\rho, t)}{\partial \rho} = 0. \end{aligned} \quad (26)$$

Take equation (26) with the following initial conditions:

$$\begin{aligned} \alpha(\rho, 0) &= \frac{1}{3}(\chi - 2^2) + 2q^2 \tanh(q\rho), \\ \beta(\rho, 0) &= \frac{-4d_0(\chi + q^2)}{3d_1^2} + \frac{-4dq^2(\rho + q^2)}{3d_1} \tanh(q\rho), \\ \omega(\rho, 0) &= d_0 + d_1 \tanh(q\rho), \end{aligned} \quad (27)$$

where $\chi = 1.5$, $q = 0.1$, $d_0 = 1.5$, and $d_1 = 0.1$. Equation (26) has the following exact solution:

$$\begin{aligned} \alpha(\rho, t) &= \frac{1}{3}(\chi - 2q^2) + 2q^2 \tanh(q(\rho + \chi t)), \\ \beta(\rho, t) &= \frac{-4d_0(\chi + q^2)}{3d_1^2} + \frac{4q^2(\rho + q^2)}{3d_1} \\ &\quad \times \tanh(q(\rho + \chi t)), \\ \omega(\rho, t) &= d_0 + d_1 \tanh(q(\rho + \chi t)). \end{aligned} \quad (28)$$

Applying OHAM-DJ, we get the following different order problems:

3.1.1 Zero-order problem

$$\begin{aligned} \frac{\partial \alpha_0(\rho, t)}{\partial t} = 0, \quad \alpha_0(\rho, 0) &= \frac{1}{3}(\rho - 2q^2) \\ &\quad + 2q^2 \tanh(q(\rho + \chi t)), \\ \frac{\partial \beta_0(\rho, t)}{\partial t} = 0, \quad \beta_0(\rho, 0) &= \frac{-4d_0(\rho + q^2)}{3d_1^2} \\ &\quad + \frac{4q^2(\rho + q^2)}{3d_1} \tanh(q(\rho + \chi t)), \\ \frac{\partial \omega_0(\rho, t)}{\partial t} = 0, \quad \omega_0(\rho, 0) &= d_0 + d_1 \tanh(q(\rho + \chi t)). \end{aligned} \quad (29)$$

Its solution is

$$\begin{aligned} \alpha_0(\rho, t) &= 0.020000000000000004 \\ &\quad (24.666666666666666 + 1. \tanh(0.1\rho)^2), \\ \beta_0(\rho, t) &= 0.20133333333333336 \\ &\quad (-1499.9999999999999 + 1. \tanh(0.1\rho)), \\ \omega_0(\rho, t) &= 0.1(1 \tanh(0.1\rho) + 15) \end{aligned} \quad (30)$$

3.1.2 First-order problem

$$\begin{aligned} -C_1 \frac{\partial \alpha_0(\rho, t)}{\partial t} - \frac{\partial \alpha_0(\rho, t)}{\partial t} \\ + -3C_1 \alpha_0(\rho, t) \frac{\partial \alpha_0(\rho, t)}{\partial \rho} + \frac{\partial \alpha_1(\rho, t)}{\partial t} \\ + C_1 \omega_0(\rho, t) \frac{\partial \beta_0(\rho, t)}{\partial \rho} + \frac{\partial \omega_0(\rho, t)}{\partial \rho} C_1 \beta_0(\rho, t) \\ + \frac{1}{2} C_1 \frac{\partial^3 \alpha_0(\rho, t)}{\partial \rho^3} = 0, \quad \alpha_1(\rho, 0) = 0 \\ 3C_3 \alpha_0(\rho, t) \frac{\partial \beta_0(\rho, t)}{\partial \rho} - \frac{\partial \beta_0(\rho, t)}{\partial t} \\ - C_3 \frac{\partial \beta_0(\rho, t)}{\partial t} - C_3 \frac{\partial^3 \beta_0(\rho, t)}{\partial \rho^3} \\ + \frac{\partial \beta_1(\rho, t)}{\partial t} = 0, \quad \beta_1(\rho, 0) = 0, \\ - \frac{\partial \omega_0(\rho, t)}{\partial t} - C_5 \frac{\partial \omega_0(\rho, t)}{\partial t} \\ + \frac{\partial \omega_1(\rho, t)}{\partial t} + 3C_5 \alpha_0(\rho, t) \frac{\partial \omega_0(\rho, t)}{\partial \rho} \\ - C_5 \frac{\partial^3 \omega_0(\rho, t)}{\partial \rho^3} = 0, \quad \omega_1(\rho, 0) = 0. \end{aligned} \quad (31)$$

Its solution is

$$\begin{aligned} \alpha_1(\rho, t, C_1) &= 0.0001600007t186862.24999996C_1 \\ &\quad \times \operatorname{sech}^2(0.1\rho) + 1. C_1 \operatorname{sech}^4(0.1\rho) \tanh(0.1\rho) \\ &\quad + 0.999999998C_1 \operatorname{sech}^2(0.1\rho) \tanh^3(0.1\rho) \\ &\quad + 11.833325C_1 \operatorname{sech}^2(0.1\rho) \tanh(0.1\rho), \\ \beta_1(\rho, t, C_3) &= -(74. C_3 \operatorname{sech}^2(0.1\rho) \\ &\quad \times 0.0004026666673t + 1.000000002C_3 \\ &\quad \times \operatorname{sech}^2(0.1\rho) \tanh^2(0.1\rho))1. C_3 \operatorname{sech}^4(0.1\rho), \\ \omega_1(\rho, t, C_5) &= -(73.9999999C_5 \operatorname{sech}^2(0.1\rho) \\ &\quad \times \operatorname{sech}^2(0.1\rho) + 1.000000004C_5 \operatorname{sech}^2(0.1\rho) \\ &\quad \times \tanh^2(0.1\rho) + 1.000000004C_5 \operatorname{sech}^2(0.1\rho) \\ &\quad \times \tanh^2(0.1\rho) + 1. C_5 \operatorname{sech}^4(0.1\rho)) \\ &\quad \times 0.000200000000000000006t. \end{aligned} \quad (32)$$

Adding equation (30) and (32), we obtain the first-order estimated result as

$$\begin{aligned}\tilde{\alpha}(\rho, t) &= \alpha_0(\rho, t) + \alpha_1(\rho, t, C_1), \\ \tilde{\beta}(\rho, t) &= \beta_0(\rho, t) + \beta_1(\rho, t, C_3), \\ \tilde{\omega}(\rho, t) &= \omega_0(\rho, t) + \omega_1(\rho, t, C_5).\end{aligned}\quad (33)$$

The different constants in equation (33) are computed through the collocation method.

These constants have the following values:

$$C_1 = -0.0002033, C_3 = -0.60294521, C_5 = -0.6029452.$$

The results obtained are given in Section 4.

4 Results and discussion

This section discusses the results obtained in this study. The OHAM-DJ method is applied to solve the Hirota–Satsuma coupled system of KdV equations. Tables 1–3 display the first-order approximate solution comparison obtained through OHAM-DJ and OHAM for $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ at $t = 0.01$, respectively. Tables 3–6 display the computation of absolute errors in $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ for different time intervals obtained through OHAM-DJ. From Tables 1–3, it is observed that the OHAM-DJ approximate solutions are more consistent with the exact values for the first-order quantities as compared with OHAM results. Tables 3–6 show that the absolute errors in first order $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ mitigate with the decrease in the values of the temporal variable t . The residuals in $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ are plotted in Figures 1–3. From Figure 1, it is evident that the residual in $\alpha(\rho, t)$ enhances abruptly with augmenting ρ , reaches maximum at about $\rho = 07$, then drops and attains a constant value beyond $x = 32$. Figures 2 and 3 display almost the same behavior in the

Table 1: First-order OHAM-DJ results comparison with OHAM for $\alpha(\rho, t)$ at $t = 0.01$

ρ	OHAM [34]	OHAM-DJ	Exact	Absolute error
0	0.493419	0.493327	0.493338	0.0000105793
10	0.504977	0.504931	0.505124	0.000193059
20	0.511928	0.51192	0.511961	0.0000407318
30	0.513137	0.513136	0.513142	5.86375×10^{-6}
40	0.513307	0.513307	0.513307	8.00264×10^{-7}
50	0.51333	0.51333	0.51333	1.08427×10^{-7}
60	0.513333	0.513333	0.513333	1.46763×10^{-8}
70	0.513333	0.513333	0.513333	1.98626×10^{-9}
80	0.513333	0.513333	0.513333	2.68812×10^{-10}
90	0.513333	0.513333	0.513333	3.63796×10^{-11}
100	0.513333	0.513333	0.513333	4.92351×10^{-12}

Table 2: First-order OHAM-DJ results comparison with OHAM for $\beta(\rho, t)$ at $t = 0.01$

ρ	OHAM [34]	OHAM-DJ	Exact	Absolute error
0	−30.1999	−302.0	−302.0	0.00011991
10	−27.5998	−301.847	−301.847	0.0000502145
20	−27.2886	−301.806	−301.806	8.44095×10^{-6}
30	−27.1949	−301.8	−301.8	1.1786×10^{-6}
40	−27.182	−301.799	−301.799	1.60188×10^{-7}
50	−27.1803	−301.799	−301.799	2.16916×10^{-8}
60	−27.18	−301.799	−301.799	2.93585×10^{-9}
70	−27.18	−301.799	−301.799	3.97335×10^{-10}
80	−27.18	−301.799	−301.799	5.37739×10^{-11}
90	−27.18	−301.799	−301.799	7.27596×10^{-12}
100	−27.18	−301.799	−301.799	9.66338×10^{-11}

Table 3: First-order OHAM-DJ results comparison with OHAM for $\omega(\rho, t)$ at $t = 0.01$

ρ	OHAM [34]	OHAM-DJ	Exact	Absolute error
	1.50002	1.50015	1.5009	0.0000595581
10	1.57617	1.57622	1.5762	0.000024941
20	1.5964	1.59641	1.59641	4.19253×10^{-6}
30	1.59951	1.59951	1.59951	5.85397×10^{-7}
40	1.59993	1.59993	1.59993	7.95634×10^{-8}
50	1.59999	1.59999	1.59999	1.0774×10^{-8}
60	1.6	1.6	1.6	1.45821×10^{-9}
70	1.6	1.6	1.6	1.97349×10^{-10}
80	1.6	1.6	1.6	2.67082×10^{-11}
90	1.6	1.6	1.6	3.61444×10^{-12}
100	1.6	1.6	1.6	4.89164×10^{-13}

residuals of $\beta(\rho, t)$ and $\omega(\rho, t)$. The plots show that the residuals in $\beta(\rho, t)$ and $\omega(\rho, t)$ drop to minimum at about $\rho = 09$. Table 7 displays the absolute error in $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ for varying values of ρ . It is observed that the absolute error in these quantities decreases with the augmenting values of ρ . Thus, it is concluded that the OHAM-DJ gives convergent as well as stable solution of the Hirota–Satsuma coupled system of KdV equations. These results show that OHAM-DJ is a well-organized effective analytical technique to solve the coupled systems of PDEs. It is also important to mention that the OHAM-DJ gives fast convergent solutions.

5 Conclusion

The OHAM-DJ method is successfully applied for the solution of generalized Hirota–Satsuma coupled system

Table 4: Absolute error in $\alpha(\rho, t)$ for different time intervals

ρ	$t = 20$	$t = 10$	$t = 5$	$t = 0.1$
0	0.021018	0.016993	0.0083722	0.000010579
10	0.0088836	0.0081231	0.0062509	0.00019305
20	0.0014953	0.0013831	0.0011102	0.000040731
30	0.00020883	0.00019345	0.00015612	0.0000058637
40	0.000028384	0.000026298	0.000021239	8.0026×10^{-7}
50	0.0000038436	0.0000035613	0.0000028765	1.0842×10^{-7}
60	5.2021×10^{-7}	4.8201×10^{-7}	3.8933×10^{-7}	1.4676×10^{-8}
70	7.0404×10^{-8}	6.5233×10^{-8}	5.2690×10^{-8}	1.9862×10^{-9}
80	9.5282×10^{-9}	8.8284×10^{-9}	7.1309×10^{-9}	2.6881×10^{-10}
90	1.2895×10^{-9}	1.1948×10^{-9}	9.6506×10^{-10}	3.6379×10^{-11}
100	1.7451×10^{-10}	1.6169×10^{-10}	1.3060×10^{-10}	4.9235×10^{-12}

Table 5: Absolute error in $\beta(\rho, t)$ for different time intervals

ρ	$t = 20$	$t = 10$	$t = 5$	$t = 0.1$
0	0.163841	0.00014706	3.68319×10^{-2}	1.19887×10^{-3}
10	0.105082	0.0311688	2.04046×10^{-3}	4.89174×10^{-4}
20	0.0185054	0.00598917	8.28846×10^{-4}	8.16608×10^{-5}
30	0.00259983	0.000850545	1.25193×10^{-4}	1.13900×10^{-6}
40	0.000353646	0.000115864	1.71902×10^{-5}	1.54783×10^{-6}
50	0.0000478939	0.000156943	2.33099×10^{-6}	2.09594×10^{-7}
60	6.48234×10^{-6}	2.1242×10^{-6}	3.15549×10^{-7}	2.83675×10^{-8}
70	8.773×10^{-7}	2.8749×10^{-7}	4.27065×10^{-8}	3.83920×10^{-9}
80	1.1873×10^{-7}	3.8907×10^{-8}	5.77966×10^{-9}	5.19548×10^{-10}
90	1.60684×10^{-8}	5.2656×10^{-9}	7.82222×10^{-10}	7.03153×10^{-11}
100	2.17466×10^{-9}	7.1264×10^{-10}	1.05842×10^{-10}	9.49285×10^{-12}

Table 6: Absolute error in $\omega(\rho, t)$ for different time intervals

ρ	$t = 20$	$t = 10$	$t = 5$	$t = 0.1$
0	0.081378	0.018294	0.018294	0.00059546
10	0.052192	0.0010134	0.0010134	0.00024296
20	0.0091914	0.00041167	0.00041167	0.000040560
30	0.0012913	0.000062182	0.000062182	0.0000056573
40	0.000175652	0.0000085382	0.0000085382	7.6879×10^{-7}
50	0.000023788	0.0000011577	0.0000011577	1.0410×10^{-7}
60	0.00000321977	0.5672×10^{-7}	0.5672×10^{-7}	1.4089×10^{-8}
70	4.3574×10^{-7}	2.1211×10^{-8}	2.1211×10^{-8}	1.9068×10^{-9}
80	5.8971×10^{-8}	2.8707×10^{-9}	2.8707×10^{-9}	2.5806×10^{-10}
90	7.9809×10^{-9}	3.8851×10^{-10}	3.8851×10^{-10}	3.4925×10^{-11}
100	1.0801×10^{-9}	5.2579×10^{-11}	5.2579×10^{-11}	4.7266×10^{-12}

of KDV equations. At the start, the concept of the OHAM-DJ is developed by introducing the DJ polynomials to OHAM. Then the different order equations are obtained,

and their solutions are computed. The comparison of the results obtained through OHAM-DJ and OHAM with the exact values of the first-order quantities is presented

Figure 1: The residual in $\alpha(\rho, t)$ by OHAM-DJ solution at $t = 0.1$.

Figure 3: The residual in $\omega(\rho, t)$ by OHAM-DJ solution at $t = 0.1$.

Figure 2: The residual of $\beta(\rho, t)$ by OHAM-DJ solution at $t = 0.1$.

Table 7: Absolute error in the state variables for $t = 0.1$

$\alpha(\rho, t)$	$\beta(\rho, t)$	$\omega(\rho, t)$
2.1444×10^{-5}	3.9698×10^{-6}	3.9703×10^{-5}
3.8234×10^{-4}	1.6206×10^{-6}	1.5823×10^{-5}
7.9566×10^{-5}	2.7057×10^{-7}	2.6262×10^{-6}
1.1427×10^{-5}	3.7740×10^{-8}	3.6598×10^{-7}
1.5590×10^{-6}	5.1287×10^{-9}	4.9729×10^{-8}
2.1122×10^{-7}	6.9448×10^{-10}	6.7337×10^{-9}
2.8590×10^{-8}	9.3995×10^{-11}	9.1138×10^{-10}
3.8693×10^{-9}	1.2721×10^{-11}	1.2334×10^{-10}
5.2365×10^{-10}	1.7215×10^{-12}	1.6692×10^{-11}
7.0869×10^{-11}	2.3298×10^{-13}	2.2590×10^{-12}
9.5912×10^{-12}	3.1454×10^{-14}	3.0572×10^{-13}

through different tables. The comparison shows that OHAM-DJ results are in agreement with the exact values as compared with the OHAM results. The absolute errors in the first-order quantities through OHAM-DJ are computed for different time intervals, which shows that the absolute errors are small when the time intervals are small. The residuals in $\alpha(\rho, t)$, $\beta(\rho, t)$, and $\omega(\rho, t)$ are depicted through different figures, which display that the residual decreases with increasing displacement. These results ascertain that OHAM-DJ is an explicit, reliable, and efficient technique for the solution of coupled system of PDEs. It is planned for the future study to apply OHAM-DJ to different control problems, for example, the research work undertaken in refs. [48–60].

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