Research Article

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Role of fractal-fractional operators in modeling of rubella epidemic with optimized orders

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Abstract: Fractal-fractional (FF) differential and integral operators having the capability to subsume features of retaining memory and self-similarities are used in the present research analysis to design a mathematical model for the rubella epidemic while taking care of dimensional consistency among the model equations. Infectious diseases have history in their transmission dynamics and thus non-local operators such as FF play a vital role in modeling dynamics of such epidemics. Monthly actual rubella incidence cases in Pakistan for the years 2017 and 2018 have been used to validate the FF rubella model and such a data set also helps for parameter estimation. Using nonlinear least-squares estimation with MATLAB function *lsqcurvefit*, some parameters for the classical and the FF model are obtained. Upon comparison of error norms for both models (classical and FF), it is found that the FF produces the smaller error. Locally asymptotically stable points (rubella-free and rubella-present) of the model are computed when the basic reproduction number $\mathcal{R}_0$ is less and greater than unity and the sensitivity is investigated. Moreover, solution of the FF rubella system is shown to exist. A new iterative method is proposed to carry out numerical simulations which resulted in getting insights for the transmission dynamics of the rubella epidemic.

Keywords: Caputo, Riemann–Liouville, parameter estimation, -norm, infectious, transmission dynamics

1 Introduction and model formulation

A number of infectious diseases prevail in human society with some of them having minor impacts on our health

while others are fatal. Rubella also known as German measles is one such fatal epidemics commonly found in children and young adults. This contagious viral infection is mostly dangerous for humans' skin and lymph nodes. Fetal death and congenital rubella syndrome are most common happenings in pregnant women. Being viral, the only prevention available for rubella-infected individuals is the use of vaccination (MMR vaccine). Rubella is found to have several transmission modes, including direct contact with the infected individuals or via airborne droplets when an infected one sneezes, coughs, or talks. Those who catch it may not even realize it for about a week or two. However, the lasting period of the epidemic is from 3 to 5 days.

In order to figure out the way the infection behaves, mathematical modeling for the infection can be helpful. Numerous mathematical models have been designed to understand dynamics of infectious diseases using the tools from differential calculus [1–5]. Classical (integer-order) derivatives have been used for deterministic modeling of the epidemics which consists of first-order ordinary differential equations having autonomous type of nature such as the one given below for dynamics of the rubella epidemic [6].

$$\begin{aligned}\frac{dS(t)}{dt} &= \nu - BS(t)I(t) - \mu S(t), & S(0) &= S_0 \geq 0, \\ \frac{dE(t)}{dt} &= BS(t)I(t) - (\mu + \lambda)E(t), & E(0) &= E_0 \geq 0, \\ \frac{dI(t)}{dt} &= \lambda E(t) - (\mu + \gamma)I(t), & I(0) &= I_0 \geq 0, \\ \frac{dR(t)}{dt} &= \gamma I(t) - \mu R(t), & R(0) &= R_0 \geq 0.\end{aligned}\quad (1)$$

However, such classical models make use of local differential and integral operators having no characteristics of retaining memory of the epidemic under consideration. Therefore, memory features of the underlying epidemic are not taken into consideration within classical calculus. In order to subsume such memory effects within the deterministic model of the epidemic, nonlocal operators must be used because of their superiority over the classical ones as proved in various recently conducted research studies [7–19]. Among different operators, a newly

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proposed concept of fractal-fractional (FF) differential and integral operators first proposed by Atangana in ref. [28] has been used, in the present research study, for modeling rubella epidemic. It may also be noted that such FF concept has not been employed before in the existing literature to model dynamics of infectious disease called rubella. This research analysis is the first step in the direction of such FF modeling within mathematical epidemiology of the rubella disease. There are a few studies recently conducted in the direction of FF modeling of infectious diseases which are proved to have found characteristics to capture transmission dynamics under the realm of these new operators, see for example, refs. [20–25] for diverse applications of FF operators in the Caputo sense. When it comes to physical problems, then applications of fractals are found for the dark energy in fractal spacetime, fractal boundary of carbon nano-tube, nano-scale hydrodynamics, fractal-Cantorian spacetime, fractal wave equation, and many more as described in refs. [26,27] and most of the references cited therein.

After the use of FF concepts with the Riemann–Liouville operator on each first-order ordinary equation given in (1), we obtain the following new model for the rubella epidemic while taking care for dimensional consistency among biological parameters of the model and the orders of FF differential equations:

$$\begin{aligned}
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} S(\tau) d\tau \\
 &= v^{\min(\alpha,\beta)} - B^{\min(\alpha,\beta)} S(t)I(t) - \mu^{\min(\alpha,\beta)} S(t), \quad S(0) = S_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} E(\tau) d\tau \\
 &= B^{\min(\alpha,\beta)} S(t)I(t) - (\mu^{\min(\alpha,\beta)} + \lambda^{\min(\alpha,\beta)}) E(t), \quad E(0) = E_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} I(\tau) d\tau \\
 &= \lambda^{\min(\alpha,\beta)} E(t) - (\mu^{\min(\alpha,\beta)} + \gamma^{\min(\alpha,\beta)}) I(t), \quad I(0) = I_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} R(\tau) d\tau \\
 &= \gamma^{\min(\alpha,\beta)} I(t) - \mu^{\min(\alpha,\beta)} R(t), \quad R(0) = R_0 \geq 0,
 \end{aligned} \tag{2}$$

where $\alpha, \beta \in (0, 1)$ denote, respectively, fractional order and fractal dimension of the rubella epidemic model represented by (2). Later, it has been found via parameter estimation technique that the minimum order is the fractional order α . So, the proposed FF rubella epidemiological model will be of the following structure:

$$\begin{aligned}
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} S(\tau) d\tau \\
 &= v^\alpha - B^\alpha S(t)I(t) - \mu^\alpha S(t), \quad S(0) = S_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} E(\tau) d\tau \\
 &= B^\alpha S(t)I(t) - (\mu^\alpha + \lambda^\alpha) E(t), \quad E(0) = E_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} I(\tau) d\tau \\
 &= \lambda^\alpha E(t) - (\mu^\alpha + \gamma^\alpha) I(t), \quad I(0) = I_0 \geq 0, \\
 & \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau^\beta} \int_0^t (t-\tau)^{-\alpha} R(\tau) d\tau \\
 &= \gamma^\alpha I(t) - \mu^\alpha R(t), \quad R(0) = R_0 \geq 0.
 \end{aligned} \tag{3}$$

One of the challenging tasks in mathematical epidemiology is the issue of parameter estimation, especially when real statistical data for the epidemic incidence are available. Among various existing techniques for this purpose, we have employed nonlinear least squares estimation technique which provides minimum value of the sum of squared errors (SSEs) between the approximate solution of the system and the data values as follows:

$$\begin{aligned}
 \text{SSE}(\varepsilon) &= d(f(x(t, \varepsilon)), y)^2 \\
 &= \sum_{k=1}^q \|f(x(t, \varepsilon)) - f(y^{(k)})\|^2,
 \end{aligned} \tag{4}$$

where

$$y = (f(x^{(1)}), f(x^{(2)}), \dots, f(x^{(q)})), \quad x^{(k)} \in \mathbb{R}^d, \tag{5}$$

are real statistical data points and the function $f(x)$: $\mathbb{R}^d \rightarrow \mathbb{R}^n$ denotes observational time points of the state variable x obtained from numerical simulations of either classical or FF rubella model (3) and $\varepsilon \in \mathbb{R}^m$ is an m -dimensional parameter. Thus, some parameters of the rubella model have been estimated following the aforementioned approach. The rubella transmission rate B , fractional order α , and the fractal dimension β are among those estimated, whereas the remaining biological parameters are held fixed as shown in Table 1.

Table 1: Fixed and fitted biological and non-biological parameters for the rubella epidemic model equation (2)

Parameter	Description	Value	Source
ν	Recruitment rate	3,74,125 (fixed)	[29]
B (classical)	Rubella transmission rate	9.644727×10^{-9}	Fitted
B (FF)	Rubella transmission rate	4.700991×10^{-10}	Fitted
μ	Natural death rate	$1/67.7$ (fixed)	[29]
λ	Exposure rate for rubella virus	2 (fixed)	[30]
γ	Recovery rate of humans	1.579 (fixed)	[30]
α	Fractional order parameter	8.567422×10^{-1}	Fitted
β	Fractal order parameter	1	Fitted

2 Equilibrium points of the FF rubella model

The FF rubella model (3) is found to have two types of equilibrium points. One of them is called disease-free equilibrium (wherein the rubella is considered to be completely absent) and the other one is said to be endemic (also called rubella persistence) equilibrium. The rubella-free equilibrium point is computed to be $Q_0 = \left(\frac{\nu^\alpha}{\mu^\alpha}, 0, 0, 0\right)$ by equating right hand sides of the model (3) to zero. For the endemic situation wherein the rubella is assumed to be present, the FF rubella model is simplified for its state variables as follows:

$$\begin{aligned} S^+(t) &= \frac{(\mu^\alpha + \gamma^\alpha)(\mu^\alpha + \lambda^\alpha)}{(B\lambda)^\alpha}, \\ E^+(t) &= \frac{\mu^\alpha + \gamma^\alpha}{\lambda^\alpha} I^+(t), \\ I^+(t) &= \frac{\nu^\alpha - \mu^\alpha S^+(t)}{B^\alpha S^+(t)}. \end{aligned} \quad (6)$$

It may also be noted that $S(t) + E(t) + I(t) + R(t) = 1 \Rightarrow R(t) = 1 - (S(t) + E(t) + I(t))$. Thus, $R(t)$ can be analyzed using dynamic profiles for the remaining state variables. For the endemic point Q_1 to lie within a bounded region Φ , the following condition has to be satisfied:

$$Q_1 \leftrightarrow \nu^\alpha - \mu^\alpha S^+(t) > 0. \quad (7)$$

Simplification of the aforementioned inequality yields the following:

$$\left(\frac{\nu B \lambda}{\mu}\right)^\alpha \frac{1}{(\mu^\alpha + \gamma^\alpha)(\mu^\alpha + \lambda^\alpha)} > 1. \quad (8)$$

The left side of the aforementioned inequality keeps an integral position in the study of infectious diseases. This is what we call basic reproduction number $\mathcal{R}_0$ (the single most important quantity which measures number of individuals, on average, likely to be infected when an

infectious person is introduced within a completely susceptible population). The quantity $\mathcal{R}_0$ is tried to be under 1 so that the underlying disease can be got rid of. The following theorems are the immediate results from the stability theory of infectious diseases:

Theorem 2.1. *The rubella-free equilibrium point $Q_0 = \left(\frac{\nu^\alpha}{\mu^\alpha}, 0, 0, 0\right)$ is locally asymptotically stable if $\mathcal{R}_0 < 1$ and the rubella-present equilibrium point $Q_1 = (S^+(t), E^+(t), I^+(t))$ is locally asymptotically stable if $\mathcal{R}_0 > 1$ wherein Q_0 gets unstable.*

Proof. The proof of the aforementioned theorem is straightforward and a recently published research [31] can be consulted with for related details. $\square$

2.1 Sensitivity analysis

The principle of sensitivity analysis is used in this portion to discover the robust significance of the generic parameters present in the $\mathcal{R}_0$ base reproduction number. In addition, both analytic and numerical values of the parameters in $\mathcal{R}_0$ are obtained with the aid of parameter values from accurate assumptions. The analytical expressions obtained can be used if and only if the dynamics obey the model (1), to shed some light on how to monitor the model's onset in variant locations. The threshold value $\mathcal{R}_0$ is a quantity which is considered the main way to curtail and abort the spread of the ailment by lowering the number to less than unity. To calculate the most sensitive parameters in the model, the sensitivity index technique is used, those with positive sign are considered to be extremely and proportionally sensitive to the value of $\mathcal{R}_0$ while those with negative sign are less sensitive to decreasing $\mathcal{R}_0$ and the other group (with zero

relative sensitivity) is neutrally sensitive. It is generally recognized that the cause of the violation transmission is directly related to the basic reproduction number $\mathcal{R}_0$. Elasticity indices of $\mathcal{R}_0$ are specified as ref. [32]: for the associated parameters in the model, we can write.

$$\Upsilon_{\mathbf{P}_i}^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial \mathbf{P}_i} \times \frac{\mathbf{P}_i}{\mathcal{R}_0}, \quad (9)$$

where $\mathcal{R}_0$ denotes the basic reproduction ratio and $\mathbf{P}_i$ is as stated above. Following the described formula, we obtain:

$$\begin{aligned} \Upsilon_\nu &= \alpha, \\ \Upsilon_B &= \alpha, \\ \Upsilon_\lambda &= \lambda(\mu^\alpha + \xi^\alpha)(\mu^\alpha \\ &\quad + \lambda^\alpha) \left(\left(\frac{\nu B \lambda}{\mu} \right)^\alpha \alpha \lambda^{-1} (\mu^\alpha + \xi^\alpha)^{-1} (\mu^\alpha + \lambda^\alpha)^{-1} \right. \\ &\quad \left. - \left(\frac{\nu B \lambda}{\mu} \right)^\alpha \lambda^\alpha \alpha (\mu^\alpha + \xi^\alpha)^{-1} (\mu^\alpha + \lambda^\alpha)^{-2} \lambda^{-1} \right) \left(\left(\frac{\nu B \lambda}{\mu} \right)^\alpha \right)^{-1}, \\ \Upsilon_\mu &= \mu(\mu^\alpha + \xi^\alpha)(\mu^\alpha \\ &\quad + \lambda^\alpha) \left(- \left(\frac{\nu B \lambda}{\mu} \right)^\alpha \alpha \mu^{-1} (\mu^\alpha + \xi^\alpha)^{-1} (\mu^\alpha + \lambda^\alpha)^{-1} \right. \\ &\quad \left. - \left(\frac{\nu B \lambda}{\mu} \right)^\alpha \mu^\alpha \alpha (\mu^\alpha + \xi^\alpha)^{-2} (\mu^\alpha + \lambda^\alpha)^{-1} \mu^{-1} - \left(\frac{\nu B \lambda}{\mu} \right)^\alpha \right. \\ &\quad \left. \times \mu^\alpha \alpha (\mu^\alpha + \xi^\alpha)^{-1} (\mu^\alpha + \lambda^\alpha)^{-2} \mu^{-1} \right) \left(\left(\frac{\nu B \lambda}{\mu} \right)^\alpha \right)^{-1}, \\ \Upsilon_\gamma &= -\frac{\gamma^\alpha \alpha}{\mu^\alpha + \gamma^\alpha}. \end{aligned} \quad (10)$$

Table 2 offers the numerical values showing the relative importance of the $\mathcal{R}_0$. Positive relationship parameters are n , B , λ , while μ , γ are negative relationships. A negative relationship suggests that an increase in the value of these parameters will help minimize the disease's brutality. A positive relationship implies that an increase in that parameter's values will have a major effect on the frequency of the ailment spread. The numerical signs mentioned in Table 2 below, are shown schematically in Figure 1.

Table 2: Elasticity indices for $\mathcal{R}_0$ to the parameters of the model

Parameter	Baseline value	Elasticity index
ν	374,125	0.8567422000
B	$4.700991 \times 10^{+10}$	0.8567422000
λ	2	0.01259421014
μ	1/67.7	-0.8847067365
γ	1.579	-0.8413718738

Figure 1: Elasticity indices for significance of parameters in $\mathcal{R}_0$.

2.2 Existence and uniqueness

In this subsection, we report the Cauchy problem with power law in order to justify the existence and uniqueness of solution [20]. We start with the following result:

$$G(t) = G(0) + \frac{\beta \alpha}{\Gamma(\beta)} \int_0^t \lambda^{\alpha-1} F(\lambda, G(\lambda)) d\lambda. \quad (11)$$

Employ the following map:

$$\Lambda \phi(t) = G(0) + \frac{\beta \alpha}{\Gamma(\beta)} \int_0^t \lambda^{\alpha-1} F(\lambda, G(\lambda)) d\lambda, \quad (12)$$

so that,

$$\|\Lambda \phi(t) - G(0)\| < cU, \quad (13)$$

and this stands for $\sup_{\Pi_a^b} |f| = U$ and $U < \frac{c\Gamma(\beta)}{\alpha\beta a^{\alpha+\beta-3}E(\alpha, \beta)}$. By taking into account $\phi_1(t), \phi_2(t) \in C[I_a(t_n), A_b(t_n)]$, we obtain the following:

$$\|\Lambda \phi(t) - G(0)\| \leq \frac{\beta \alpha L}{\Gamma(\beta)} a^{\alpha+\beta-3} E(\alpha, \beta). \quad (14)$$

Therefore, the property for contraction is attained only if

$$L < \frac{\Gamma(\beta)}{\alpha\beta a^{\alpha+\beta-3}E(\alpha, \beta)}. \quad (15)$$

If immediate condition holds and that

$$U < \frac{c\Gamma(\beta)}{\alpha\beta a^{\alpha+\beta-3}E(\alpha, \beta)}, \quad (16)$$

then the proposed rubella system possesses a unique solution. In this way, it has been shown that the existence and uniqueness of solution of the system having power law kernel under the FF operator is achieved.

3 Numerical results and discussion

To observe dynamics of state variables including susceptible, exposed, infectious, and recovered individuals in the FF rubella epidemic model (3), we need to numerically simulate the model with the help of an iterative technique since the underlying model (3) is of nonlinear nature. In order to achieve this, we use biological parameters (fitted and estimated) as given in Table 1 along with the initial conditions which are estimated to be $S(0) = 204989885$, $E(0) = 10$, $I(0) = 7$, and $R(0) = 0$. Thus, the iterative technique for the purpose of simulations is organized as follows:

$$\begin{aligned} & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-v)^{-\alpha} \left(\frac{d}{dv^\beta} S(v) \right) dv \\ &= \mathbb{J}_1(t, S(t), E(t), I(t), R(t)), \\ & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-v)^{-\alpha} \left(\frac{d}{dv^\beta} E(v) \right) dv \\ &= \mathbb{J}_2(t, S(t), E(t), I(t), R(t)), \\ & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-v)^{-\alpha} \left(\frac{d}{dv^\beta} I(v) \right) dv \\ &= \mathbb{J}_3(t, S(t), E(t), I(t), R(t)), \\ & \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-v)^{-\alpha} \left(\frac{d}{dv^\beta} R(v) \right) dv \\ &= \mathbb{J}_4(t, S(t), E(t), I(t), R(t)), \end{aligned} \quad (17)$$

where $\mathbb{J}_1(t, S(t), E(t), I(t), R(t))$, $\mathbb{J}_2(t, S(t), E(t), I(t), R(t))$, $\mathbb{J}_3(t, S(t), E(t), I(t), R(t))$, and $\mathbb{J}_4(t, S(t), E(t), I(t), R(t))$ are provided on right sides of the FF rubella epidemic model (3). The fractal dimension β is introduced in order to capture self-similarities within transmission dynamics of the rubella epidemic. As it is known that the fractional integral is differentiable, equation (17) is to be reduced into the Volterra integral equation and the FF derivative in the Riemann–Liouville sense is to convert as follows:

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-v)^\alpha \mathbb{J}(v) dv = \frac{1}{\beta t^{\beta-1}}, \quad (18)$$

in such a way that the FF rubella epidemic model (3) takes the following form:

$$\begin{aligned} {}^{\text{RL}}D_{0,t}^\alpha(S(t)) &= \beta t^{\beta-1} \mathbb{J}_1(t, S(t), E(t), I(t), R(t)), \\ {}^{\text{RL}}D_{0,t}^\alpha(E(t)) &= \beta t^{\beta-1} \mathbb{J}_2(t, S(t), E(t), I(t), R(t)), \\ {}^{\text{RL}}D_{0,t}^\alpha(I(t)) &= \beta t^{\beta-1} \mathbb{J}_3(t, S(t), E(t), I(t), R(t)), \\ {}^{\text{RL}}D_{0,t}^\alpha(R(t)) &= \beta t^{\beta-1} \mathbb{J}_4(t, S(t), E(t), I(t), R(t)). \end{aligned} \quad (19)$$

Now, the RL operator is replaced by the Caputo operator so that integer order initial conditions having clear physical interpretations can be used. Having done so, the RL fractional integral is applied on both sides of equation (19) to obtain the following:

$$\begin{aligned} S(t) &= S(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t v^{\beta-1} (t-v)^{\alpha-1} \mathbb{J}_1(v, S, E, I, R) dv, \\ E(t) &= E(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t v^{\beta-1} (t-v)^{\alpha-1} \mathbb{J}_2(v, S, E, I, R) dv, \\ I(t) &= I(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t v^{\beta-1} (t-v)^{\alpha-1} \mathbb{J}_3(v, S, E, I, R) dv, \\ R(t) &= R(0) + \frac{\beta}{\Gamma(\alpha)} \int_0^t v^{\beta-1} (t-v)^{\alpha-1} \mathbb{J}_4(v, S, E, I, R) dv. \end{aligned} \quad (20)$$

At this stage, the required iterative technique is to be presented using a new approach. At the grid point $t = t_{n+1}$, the rubella epidemic model (3) turns into the following:

$$\begin{aligned} S_{n+1} &= S_0 + \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} v^{\beta-1} (t_{n+1}-v)^{\alpha-1} \mathbb{J}_1(v, S, E, I, R) dv, \\ E_{n+1} &= E_0 + \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} v^{\beta-1} (t_{n+1}-v)^{\alpha-1} \mathbb{J}_2(v, S, E, I, R) dv, \\ I_{n+1} &= I_0 + \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} v^{\beta-1} (t_{n+1}-v)^{\alpha-1} \mathbb{J}_3(v, S, E, I, R) dv, \\ R_{n+1} &= R_0 + \frac{\beta}{\Gamma(\alpha)} \int_0^{t_{n+1}} v^{\beta-1} (t_{n+1}-v)^{\alpha-1} \mathbb{J}_4(v, S, E, I, R) dv. \end{aligned} \quad (21)$$

Then we approximate the above obtained integrals to

$$\begin{aligned}
S_{n+1} &= S_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} \mathbb{J}_1(\nu, S, E, I, R) d\nu, \\
E_{n+1} &= E_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} \mathbb{J}_2(\nu, S, E, I, R) d\nu, \\
I_{n+1} &= I_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} \mathbb{J}_3(\nu, S, E, I, R) d\nu, \\
R_{n+1} &= R_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} \mathbb{J}_4(\nu, S, E, I, R) d\nu.
\end{aligned} \quad (22)$$

Within the finite interval $[t_j, t_{j+1}]$, we approximate the function $\nu^{\beta-1} \mathbb{J}_i(\nu, S, E, I, R)$, where $i = 1, 2, 3, 4$ using the Lagrangian piece-wise interpolation such that

$$\begin{aligned}
P_j(\nu) &= \frac{\nu - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} \mathbb{J}_1(t_j, S_j, E_j, I_j, R_j) \\
&\quad - \frac{\nu - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} \mathbb{J}_1(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}), \\
Q_j(\nu) &= \frac{\nu - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} \mathbb{J}_2(t_j, S_j, E_j, I_j, R_j) \\
&\quad - \frac{\nu - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} \mathbb{J}_2(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}), \\
U_j(\nu) &= \frac{\nu - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} \mathbb{J}_3(t_j, S_j, E_j, I_j, R_j) \\
&\quad - \frac{\nu - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} \mathbb{J}_3(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}), \\
V_j(\nu) &= \frac{\nu - t_{j-1}}{t_j - t_{j-1}} t_j^{\beta-1} \mathbb{J}_4(t_j, S_j, E_j, I_j, R_j) \\
&\quad - \frac{\nu - t_j}{t_j - t_{j-1}} t_{j-1}^{\beta-1} \mathbb{J}_4(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}).
\end{aligned} \quad (23)$$

Therefore, one obtains

$$\begin{aligned}
S_{n+1} &= S_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} P_j(\nu) d\nu, \\
E_{n+1} &= E_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} Q_j(\nu) d\nu, \\
I_{n+1} &= I_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} U_j(\nu) d\nu, \\
R_{n+1} &= R_0 + \frac{\beta}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} \nu^{\beta-1} (t_{n+1} - \nu)^{\alpha-1} V_j(\nu) d\nu.
\end{aligned} \quad (24)$$

The right hand sides for the aforementioned equations are simplified to get the required iterative technique for the simulations of the FF rubella epidemic model as given by equation (3):

$$\begin{aligned}
S_{n+1} &= S_0 + \frac{\beta(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n [t_j^{\beta-1} \mathbb{J}_1(t_j, S_j, E_j, I_j, R_j) \\
&\quad \times ((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)) \\
&\quad - t_{j-1}^{\beta-1} \mathbb{J}_1(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}) ((n+1-j)^{\alpha+1} \\
&\quad - (n-j)^\alpha (n-j+1+\alpha))], \\
E_{n+1} &= E_0 + \frac{\beta(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n [t_j^{\beta-1} \mathbb{J}_2(t_j, S_j, E_j, I_j, R_j) \\
&\quad \times ((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)) \\
&\quad - t_{j-1}^{\beta-1} \mathbb{J}_2(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}) ((n+1-j)^{\alpha+1} \\
&\quad - (n-j)^\alpha (n-j+1+\alpha))], \\
I_{n+1} &= I_0 + \frac{\beta(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n [t_j^{\beta-1} \mathbb{J}_3(t_j, S_j, E_j, I_j, R_j) \\
&\quad \times ((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)) \\
&\quad - t_{j-1}^{\beta-1} \mathbb{J}_3(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}) ((n+1-j)^{\alpha+1} \\
&\quad - (n-j)^\alpha (n-j+1+\alpha))], \\
R_{n+1} &= R_0 + \frac{\beta(\Delta t)^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n [t_j^{\beta-1} \mathbb{J}_4(t_j, S_j, E_j, I_j, R_j) \\
&\quad \times ((n+1-j)^\alpha (n-j+2+\alpha) - (n-j)^\alpha (n-j+2+2\alpha)) \\
&\quad - t_{j-1}^{\beta-1} \mathbb{J}_4(t_{j-1}, S_{j-1}, E_{j-1}, I_{j-1}, R_{j-1}) ((n+1-j)^{\alpha+1} \\
&\quad - (n-j)^\alpha (n-j+1+\alpha))],
\end{aligned} \quad (25)$$

where $\mathbb{J}_i$, ($i = 1, 2, 3, 4$) are the right hand sides of the FF rubella epidemic model (3), wherein α is the fractional order and β denotes the fractal dimension of the FF Caputo operator.

Thus, using the proposed iterative technique as given by (25), we have obtained various simulation results for the FF rubella epidemic model (3). For instance, the model under consideration has first been simulated under the classical case, that is, when $\alpha = \beta = 1$. Figure 2(a) shows comparison of the real statistical points (actual monthly rubella incidence cases in Pakistan for the years 2017 and 2018) with the simulation results (for infectious individuals) obtained via MATLAB solver ode23s while using the best estimated value of the rubella transmission rate ($B = 9.644727 \times 10^{-9}$), whereas the remaining parameters are taken from Table 1. The respective residual errors are also obtained in Figure 2(b), wherein the data points lying above the classical curve constitute larger amount of error where the L_2 -error norm is computed to be about 1.928230×10^2 .

Figure 2: (a) Best fitted classical curve using the estimated value of the rubella transmission rate $B = 9.644727 \times 10^{-9}$ and the (b) residual errors.

It has been observed in Figure 3(a) that the FF curve for the infectious individuals fit the real statistical data points better than the classical case. This curve has been obtained using the above derived iterative technique while using the best estimated values of the rubella

transmission rate ($B = 4.700991 \times 10^{-10}$), fractional order operator ($\alpha = 8.567422 \times 10^{-1}$), and the fractal dimension ($\beta = 1$), whereas the remaining parameters are taken from Table 1 under the FF Caputo operator. The respective residual errors are also obtained in Figure 3(b), wherein

Figure 3: (a) Best fitted FF curve using the estimated value of the rubella transmission rate $B = 4.700991 \times 10^{-10}$ and best fitted values for the FF orders $\alpha = 8.567422 \times 10^{-1}$, $\beta = 1$ and the (b) residual errors.

Figure 4: Dynamics of infectious individuals for (a) increasing values of the recovery rate γ and (b) increasing values of the rubella transmission rate B .

Figure 5: Profile for basic reproduction number R_0 in terms of λ and μ .

the data points lying above the FF curve constitute larger amount of error where the $\mathbb{L}_2$ -error norm is computed to be about 1.902852×10^2 .

To observe behavior of the infectious individuals under variations of some important biological parameters, we have, in Figure 4, simulations of the infectious individuals for increasing values of the human recovery rate γ and also for increasing rate for the rubella

transmission rate B . It can be seen in Figure 4(a) that the recovery rate plays an important role to decrease the rubella epidemic substantially and likewise a slightest increase in the B value would increase the epidemic as illustrated in Figure 4(b). The profile of the basic reproduction number against some parameters and their respective contour plots are demonstrated in Figures 5 and 6.

Figure 6: Profile for basic reproduction number R_0 in terms of γ and λ .

4 Concluding remarks

Present research findings reveal that the FF differential and integral operators play a better role to comprehend transmission dynamics of an infectious disease. These operators have first time been used in the present research analysis to propose a new rubella epidemic model while taking care of dimensional consistency among the dynamical equations of the model. Moreover, real monthly rubella cases of Pakistan for the years 2017 and 2018 are taken into consideration to validate the FF rubella model. Not only this but this data set is also used to obtain the best fitted values of some parameters including rubella transmission rate B , fractional order α , and the fractal dimension β while using nonlinear least squares estimation technique with MATLAB function *lsqcurvefit*. It has been observed that the smaller L_2 -error norm is computed in case of FF rubella model when compared with the classical ($\alpha = \beta = 1$) rubella model. Local stability analysis for the rubella-free and rubella-present points has also been discussed for the FF rubella model with sensitivity of parameters checked. Existence and uniqueness of special solution of the proposed system is also presented. Numerical simulations confirmed these observations with the best fitted curve obtained for the FF rubella model. Two biological parameters for control and spread of the rubella infection play a vital role. These parameters are found to be the human recovery rate γ which needs to be improved and the rubella transmission rate B wherein a slight decline causes a substantial decline in the

rubella epidemic. Thus, it is to be suggested that along with vaccination it is important that the infected individuals should be kept at isolation for about a week or two in order to stop the epidemic to reach its human hosts. In order to continue the present research work in future, we will employ optimal control strategies on the rubella model under FF operators for the control of the epidemic.

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References

- [1] Jajarmi A, Ghanbari B, Baleanu D. A new and efficient numerical method for the fractional modeling and optimal control of diabetes and tuberculosis co-existence. *Chaos*. 2019;29(9):093111.
- [2] Iqbal Z, Ahmed N, Baleanu D, Adel W, Rafiq M, Rehman MAU, et al. Positivity and boundedness preserving numerical algorithm for the solution of fractional nonlinear epidemic model of HIV/AIDS transmission. *Chaos Solitons Fractals*. 2020;134:109706.
- [3] Ameen I, Baleanu D, Ali HM. An efficient algorithm for solving the fractional optimal control of SIRV epidemic model with a combination of vaccination and treatment. *Chaos Solitons Fractals*. 2020;137:109892.
- [4] Raza A, Rafiq M, Baleanu D, Arif MS. Numerical simulations for stochastic meme epidemic model. *Adv Differ Equ*. 2020;2020(1):1–16.
- [5] Arshad S, Deftferli O, Baleanu D. A second order accurate approximation for fractional derivatives with singular and non-

- singular kernel applied to a HIV model. *Appl Math Comput.* 2020;374:125061.
- [6] Li MY. An introduction to mathematical modeling of infectious diseases, vol. 2. Springer International Publishing AG; 2018.
 - [7] Atangana A, Qureshi S. Modeling attractors of chaotic dynamical systems with fractal–fractional operators. *Chaos Solitons Fractals.* 2019;123:320–37.
 - [8] Atangana A. Application of fractional calculus to epidemiology. In: *Fractional Dynamics.* Sciendo Migration. De Gruyter; 2015. p. 174–90.
 - [9] Ahmed N, Rafiq M, Baleanu D, Alshomrani AS, Aziz-ur Rehman M. Positive explicit and implicit computational techniques for reaction–diffusion epidemic model of dengue disease dynamics. *Adv Differ Equ.* 2020;2020:202. doi: 10.1186/s13662-020-02622-z.
 - [10] Qureshi S, Atangana A. Mathematical analysis of dengue fever outbreak by novel fractional operators with field data. *Phys A Stat Mech Appl.* 2019;526:121127.
 - [11] Raza A, Rafiq M, Baleanu D, Arif MS. Numerical simulations for stochastic meme epidemic model. *Adv Differ Equ.* 2020;2020:176. doi: 10.1186/s13662-020-02593-1.
 - [12] Qureshi S, Yusuf A. Modeling chickenpox disease with fractional derivatives: From caputo to atangana–baleanu. *Chaos Solitons Fractals.* 2019;122:111–8.
 - [13] Jajarmi A, Yusuf A, Baleanu D, Inc M. A new fractional HRSV model and its optimal control: a non-singular operator approach. *Phys A.* 2020;547:123860.
 - [14] Qureshi S, Yusuf A. Fractional derivatives applied to MSEIR problems: comparative study with real world data. *Eur Phys J Plus.* 2019;134(4):171.
 - [15] Aydogan SM, Baleanu D, Mohammadi H, Rezapour S. On the mathematical model of Rabies by using the fractional Caputo–Fabrizio derivative. *Adv Differ Equ.* 2020;2020:382. doi: 10.1186/s13662-020-02798-4.
 - [16] Qureshi S, Yusuf A. Mathematical modeling for the impacts of deforestation on wildlife species using Caputo differential operator. *Chaos Solitons Fractals.* 2019;126:32–40.
 - [17] Jajarmi A, Baleanu D. A new fractional analysis on the interaction of HIV with CD4+ T-cells. *Chaos Solitons Fractals.* 2018;113:221–9.
 - [18] Qureshi S, Yusuf A, Shaikh AA, Inc M. Transmission dynamics of varicella zoster virus modeled by classical and novel fractional operators using real statistical data. *Phys A Stat Mech Appl.* 2019;534:122149.
 - [19] Qureshi S, Bonyah E, Shaikh AA. Classical and contemporary fractional operators for modeling diarrhea transmission dynamics under real statistical data. *Phys A Stat Mech Appl.* 2019;535:122496.
 - [20] Li Z, Liu Z, Altaf Khan M. Fractional investigation of bank data with fractal-fractional Caputo. *Chaos Solitons Fractals.* 2020;131:109528.
 - [21] Wang W, Khan MA. Analysis and numerical simulation of fractional model of bank data with fractal–fractional Atangana–Baleanu derivative. *J Comput Appl Math.* 2020;369:112646.
 - [22] Atangana A, Khan MA. Modeling and analysis of competition model of bank data with fractal-fractional Caputo–Fabrizio operator. *Alex Eng J.* 2020;59(4):1985–98.
 - [23] Khan MA. The dynamics of dengue infection through fractal-fractional operator with real statistical data. *Alex Eng J.* 2020. doi: 10.1016/j.aej.2020.08.018.
 - [24] Khan MA, Alfiniyah C, Alzahrani E. Analysis of dengue model with fractal-fractional Caputo–Fabrizio operator. *Adv Differ Equ.* 2020;2020(1):1–23.
 - [25] Razminia K, Razminia A, Baleanu D. Fractal-fractional modeling of partially penetrating wells. *Chaos Solitons Fractals.* 2019;119:135–42.
 - [26] He JH. A tutorial review on fractal spacetime and fractional calculus. *Int J Theor Phys.* 2014;53(11):3698–718.
 - [27] Yang YJ, Baleanu D, Yang XJ. Analysis of fractal wave equations by local fractional Fourier series method. *Adv Math Phys.* 2013;2013:632309.
 - [28] Atangana A. Fractal-fractional differentiation and integration: connecting fractal calculus and fractional calculus to predict complex system. *Chaos Solitons Fractals.* 2017;102:396–406.
 - [29] IndexMundi, https://www.indexmundi.com/pakistan/demographics_profile.html accessed: 2019-09-01.
 - [30] Huang J, Ruan S, Wu X, Zhou X. Seasonal transmission dynamics of measles in China. *Theory Biosci.* 2018;137:185–95.
 - [31] Qureshi S, Memon Z. Monotonically decreasing behavior of measles epidemic well captured by Atangana–Baleanu–Caputo fractional operator under real measles data of Pakistan. *Chaos Solitons Fractals.* 2019;131:109478. (Available Online 18 October: In press).
 - [32] Asamoah JKK, Jin Z, Sun GQ, Li MY. A deterministic model for QFever transmission dynamics within dairy cattle herds: using sensitivity analysis and optimal controls. *Comput Math Methods Med.* 2020;2020:6820608. doi: 10.1155/2020/6820608.