Research Article

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Numerical solution of two-term time-fractional PDE models arising in mathematical physics using local meshless method

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Abstract: Multi-term time-fractional partial differential equations (PDEs) have become a hot topic in the field of mathematical physics and are used to improve the modeling accuracy in the description of anomalous diffusion processes compared to the single-term PDE results. This research includes the numerical solutions of two-term time-fractional PDE models using an efficient and accurate local meshless method. Due to the advantages of the meshless nature and ease of applicability in higher dimensions, the demand for meshless techniques is increasing. This approach approximates the solution on a uniform or scattered set of nodes, resulting in well-conditioned and sparse coefficient matrices. Numerical tests are performed to demonstrate the efficacy and accuracy of the proposed local meshless technique.

Keywords: meshless method, radial basis function, Caputo derivative, irregular domain

1 Introduction

In the past decade, fractional partial differential equations (PDEs) have received a lot of attention. Today, it is an active research area among scientists and engineers. PDEs have the ability to formulate many complex phenomena in various fields such as biology, fluid mechanics, plasma physics, fluid mechanics, optics and so on, and many exact and numerical schemes have been being derived such as those in refs. [1–18]. However, many researchers have not managed to derive and formulate many complex phenomena in the nonlinear PDEs with an integer order [19]. Thus, fractional is considered as a suitable solution for this issue where it contains a nonlocal property that is not found in nonlinear PDEs with an integer order [20]. In the present work, we consider two-term two-dimensional time-fractional Sobolev equation which is defined as follows:

$$\begin{aligned} & \frac{\partial^{\alpha_1} U}{\partial t^{\alpha_1}} + \frac{\partial^{\alpha_2} U}{\partial t^{\alpha_2}} - \frac{\partial}{\partial t} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \\ &= \beta \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) - \gamma \left(U \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \right) \\ & - \gamma \left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \right)^2 - \delta U + F(\bar{z}, t), \end{aligned} \quad (1)$$

$$\bar{z} \in \Omega \subset \mathbb{R}^n, \quad 0 < \alpha_2 \leq \alpha_1 \leq 1, \quad t > 0,$$

with the following initial and boundary conditions:

$$U(\bar{z}, 0) = U_0(\bar{z}), \quad (2)$$

$$U(\bar{z}, t) = g_1(\bar{z}, t), \quad \bar{z} \in \partial\Omega, \quad (3)$$

where β , γ and δ are known constants. Moreover, $\frac{\partial^{\alpha_1}}{\partial t^{\alpha_1}}$ and $\frac{\partial^{\alpha_2}}{\partial t^{\alpha_2}}$ are the Caputo fractional derivative operator of order $0 < \alpha_2 \leq \alpha_1 \leq 1$ for the function $U(\bar{z}, t)$.

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Recently, a great effort has been expended to develop the exact and approximate behavior of fractional PDEs [21–35], and a variety of meshless algorithms, in almost all engineering disciplines, have gained increasing interest for the solution of various models of PDEs. In particular, the radial basis function (RBF)-based meshless methods [36–43] are the most common of these methods. Contrast to mesh-based algorithms, meshless algorithms do not need mesh in the computational domains and take into account a number of uniform or scattered collocation points. Additionally, the RBF only depends on the Euclidean distance between two points in the spatial domain, so it increases the preferences and advantages of the meshless technique. As indicated by these realities, the meshless method is a truly adaptable and helpful numerical technique and can be applied to enormous practical problems [44,45]. Hardy [46] developed the RBF algorithm in 1971, which introduced a multiquadric (MQ) RBF as a meshless interpolation method. In 1982, Franke [47] further worked on this algorithm and popularized it. Franke conducted a series of extensive tests and concluded that the MQ technique had a better performance than other RBFs. In addition, Kansa [48] recommended the MQ method for approximating elliptical and parabolic PDEs. The convergence, existence and uniqueness of RBF approximations have been described in detail in several studies [49,50].

In the RBF-based meshless methods, the RBFs usually have a free parameter c called the shape parameter. Choosing the right value (in a sense) or the optimal value for the shape parameter has been a major problem for a long time. In fact with the RBF technique, it is realized that the determination of the shape parameters has a great influence on the results in terms of accuracy and stability (see ref. [51]). In short, shape parameters are typically user-defined and depend on the problem as well as the geometry of the problem. The determination of the optimum/ideal shape parameter in the RBF method has been studied extensively and a lot of research studies have been carried out over the past 20 years. In the beginning phases of improvement, the researchers utilized their expertise for the best choice of this parameter, whereas others came to some methodologies. For instance, Hardy chose $c = 0.815d$, where d denotes the average distance between the close-by nodes [46]. Franke utilized $c = 1.25D/\sqrt{N}$, where N and D denote the total number of nodes and diameter of smallest circle containing given nodes, respectively [47]. Fasshaeur [52] has suggested $c = 2/\sqrt{N}$, whereas interested readers are advised to read the text [51] for more information on these choices.

In the above discussed cases, the referenced strategies were impractical for this longstanding problem. Luckily, this issue has generated a lot of interest, and several techniques have been proposed in recent literature. Rippa [53] recommended a technique known as leave-one-out cross-validation (LOOCV)-based algorithm to determine an appropriate shape parameter, which is modified by Fasshaeur and Zhang [54]. Fasshaeur's modification was examined further by Uddin in ref. [55]. Significant improvements in the use of adaptive LOOCV-based algorithms were reported in ref. [56]. Several other algorithms have also been proposed for the selection of the optimal value of shape parameter [57].

In light of the above discussed shortcomings such as sensitivity to the shape parameters value and ill-conditioned and dense system of algebraic equations, the researchers recommend the local meshless method (LMM) [58,59]. The local RBF-based methods utilize neighboring collocation points to approximate the differential operator and make the system sparse and well-conditioned.

In this article, the explicit, implicit and Crank–Nicolson time discretization schemes are coupled with the LMM for the numerical solution of two-term time-fractional model (1). MQ, inverse quadric (IQ) and inverse multiquadric (IMQ) RBFs are considered. Furthermore, one irregular puncture domain is also considered in numerical examinations.

2 Proposed methodology

Utilizing the suggested local meshless methodology, the derivatives of $U(\bar{\mathbf{z}}, t)$ are approximated at the centers $\bar{\mathbf{z}}_h$ by the neighborhood of $\bar{\mathbf{z}}_h$, $\{\bar{\mathbf{z}}_{h1}, \bar{\mathbf{z}}_{h2}, \bar{\mathbf{z}}_{h3}, \dots, \bar{\mathbf{z}}_{hn_h}\} \subset \{\bar{\mathbf{z}}_1, \bar{\mathbf{z}}_2, \dots, \bar{\mathbf{z}}_{N^n}\}$, $n_h \ll N^n$, where $h = 1, 2, \dots, N^n$. In one-, two- and three-dimensional case, $\bar{\mathbf{z}} = x$ and $\bar{\mathbf{z}} = (x, y)$, respectively.

Now in one-dimensional case, we have

$$U^{(m)}(x_h) \approx \sum_{k=1}^{n_h} \lambda_k^{(m)} U(x_{hk}), \quad h = 1, 2, \dots, N. \quad (4)$$

Substituting RBF $\psi(\|x - x_p\|)$ in (4)

$$\psi^{(m)}(\|x_h - x_p\|) = \sum_{k=1}^{n_h} \lambda_{hk}^{(m)} \psi(\|x_{hk} - x_p\|), \quad (5)$$

$$p = h1, h2, \dots, hn_h.$$

Matrix form of (5) is

$$\underbrace{\begin{bmatrix} \psi_{h1}^{(m)}(x_h) \\ \psi_{h2}^{(m)}(x_h) \\ \vdots \\ \psi_{hn_h}^{(m)}(x_h) \end{bmatrix}}_{\mathbf{y}_{n_h}^{(m)}} = \underbrace{\begin{bmatrix} \psi_{h1}(x_{h1}) & \psi_{h2}(x_{h1}) & \cdots & \psi_{hn_h}(x_{h1}) \\ \psi_{h1}(x_{h2}) & \psi_{h2}(x_{h2}) & \cdots & \psi_{hn_h}(x_{h2}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{h1}(x_{hn_h}) & \psi_{h2}(x_{hn_h}) & \cdots & \psi_{hn_h}(x_{hn_h}) \end{bmatrix}}_{\mathbf{A}_{n_h}} \underbrace{\begin{bmatrix} \lambda_{h1}^{(m)} \\ \lambda_{h2}^{(m)} \\ \vdots \\ \lambda_{hn_h}^{(m)} \end{bmatrix}}_{\mathbf{l}_{n_h}^{(m)}},$$

where

$$\psi_p(x_k) = \psi(\|x_k - x_p\|), \quad p = h1, h2, \dots, hn_h,$$

for each $k = i1, h2, \dots, hn_h$. (6) can be written as

$$\mathbf{y}_{n_h}^{(m)} = \mathbf{A}_{n_h} \mathbf{l}_{n_h}^{(m)}. \quad (7)$$

From (7), we obtain

$$\mathbf{l}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \mathbf{y}_{n_h}^{(m)}.$$

(4) and (8) imply

$$\mathbf{U}^{(m)}(x_h) = (\mathbf{l}_{n_h}^{(m)})^T \mathbf{U}_{n_h},$$

where

$$\mathbf{U}_{n_h} = [U(x_{h1}), U(x_{h2}), \dots, U(x_{hn_h})]^T.$$

Find the derivatives of $U(x, y, t)$ w.r.t. x and y as follows:

$$U_x^{(m)}(x_h, y_h) \approx \sum_{k=1}^{n_h} \gamma_k^{(m)} U(x_{hk}, y_{hk}), \quad h = 1, 2, \dots, N^2,$$

$$U_y^{(m)}(x_h, y_h) \approx \sum_{k=1}^{n_h} \eta_k^{(m)} U(x_{hk}, y_{hk}), \quad h = 1, 2, \dots, N^2.$$

For $\gamma_k^{(m)}$ and $\eta_k^{(m)}$ ($k = 1, 2, \dots, n_h$), we continue as

$$\mathbf{y}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \mathbf{\Phi}_{n_h}^{(m)},$$

$$\mathbf{l}_{n_h}^{(m)} = \mathbf{A}_{n_h}^{-1} \mathbf{\Phi}_{n_h}^{(m)}.$$

The time derivative $\frac{\partial^{\alpha_1} U(\bar{\mathbf{z}}, t)}{\partial t^{\alpha_1}}$ is discretized by using the Caputo derivative [60] for $\alpha_1 \in (0, 1)$ as

$$\frac{\partial^{\alpha_1} U(\bar{\mathbf{z}}, t)}{\partial t^{\alpha_1}} = \begin{cases} \frac{1}{\Gamma(1 - \alpha_1)} \int_0^t \frac{\partial U(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t - \vartheta)^{-\alpha_1} d\vartheta, & 0 < \alpha_1 < 1 \\ \frac{\partial U(\bar{\mathbf{z}}, t)}{\partial t}, & \alpha_1 = 1. \end{cases}$$

Consider $t_q = q\tau$, $q = 0, 1, 2, \dots, Q$, for $[0, t]$ interval and the time step size is τ . To compute time-fractional derivative term, we proceed as follows:

$$\begin{aligned} \frac{\partial^{\alpha_1} U(\bar{\mathbf{z}}, t_{q+1})}{\partial t^{\alpha_1}} &= \frac{1}{\Gamma(1 - \alpha_1)} \int_0^{t_{q+1}} \frac{\partial U(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t_{q+1} - \vartheta)^{-\alpha_1} d\vartheta, \\ &= \frac{1}{\Gamma(1 - \alpha_1)} \sum_{s=0}^q \int_{s\tau}^{(s+1)\tau} \frac{\partial U(\bar{\mathbf{z}}, \vartheta)}{\partial \vartheta} (t_{s+1} - \vartheta)^{-\alpha_1} d\vartheta, \\ &\approx \frac{1}{\Gamma(1 - \alpha_1)} \sum_{s=0}^q \int_{s\tau}^{(s+1)\tau} \frac{\partial U(\bar{\mathbf{z}}, \vartheta_s)}{\partial \vartheta} (t_{s+1} - \vartheta)^{-\alpha_1} d\vartheta. \end{aligned} \quad (8)$$

The term $\frac{\partial U(\bar{\mathbf{z}}, \vartheta_s)}{\partial \vartheta}$ is approximated as follows:

$$\frac{\partial U(\bar{\mathbf{z}}, \vartheta_s)}{\partial \vartheta} = \frac{U(\bar{\mathbf{z}}, \vartheta_{s+1}) - U(\bar{\mathbf{z}}, \vartheta_s)}{\vartheta} + O(\tau).$$

Then

$$\begin{aligned} \frac{\partial^{\alpha_1} U(\bar{\mathbf{z}}, t_{q+1})}{\partial t^{\alpha_1}} &\approx \frac{1}{\Gamma(1 - \alpha_1)} \sum_{s=0}^q \frac{U(\bar{\mathbf{z}}, t_{s+1}) - U(\bar{\mathbf{z}}, t_s)}{\tau} \int_{s\tau}^{(s+1)\tau} (t_{s+1} - \vartheta)^{-\alpha_1} d\vartheta, \\ &= \frac{1}{\Gamma(1 - \alpha_1)} \sum_{s=0}^q \frac{U(\bar{\mathbf{z}}, t_{q+1-s}) - U(\bar{\mathbf{z}}, t_{q-s})}{\tau} \int_{s\tau}^{(s+1)\tau} (t_{s+1} - \vartheta)^{-\alpha_1} d\vartheta, \\ &= \begin{cases} \frac{\tau^{-\alpha_1}}{\Gamma(2 - \alpha_1)} (U^{q+1} - U^q) + \frac{\tau^{-\alpha_1}}{\Gamma(2 - \alpha_1)} \sum_{s=1}^q (U^{q+1-s} - U^{q-s}) [(s+1)^{1-\alpha_1} - s^{1-\alpha_1}], & q \geq 1 \\ \frac{\tau^{-\alpha_1}}{\Gamma(2 - \alpha_1)} (U^1 - U^0), & q = 0. \end{cases} \end{aligned}$$

Letting $a_0 = \frac{\tau^{-\alpha_1}}{\Gamma(2-\alpha_1)}$ and $b_s = (s+1)^{1-\alpha_1} - s^{1-\alpha_1}$, $s = 0, 1, \dots, q$, we have

$$\frac{\partial^{\alpha_1} U(\bar{z}, t_{q+1})}{\partial t^{\alpha_1}} \approx \begin{cases} a_0(U^{q+1} - U^q) + a_0 \sum_{s=1}^q b_s(U^{q+1-s} - U^{q-s}), & q \geq 1 \\ a_0(U^1 - U^0), & q = 0. \end{cases} \quad (9)$$

A similar methodology is employed for fractional derivative of order α_2 .

3 Numerical discussion

In this section, we validate the applicability and accuracy of the proposed computational technique on two-term time-fractional model (1). In this computational process, the uniform and scattered nodes with regular and one irregular domain are considered. Throughout the paper, we have used three RBFs such as IQ, MQ and IMQ with shape parameter value $c = 10$. The local stencil five with spatial domain $[0, 2]$ unless mentioned explicitly. The accuracy is measured through Max-error, L2 and RMS error norms which are defined as follows:

$$L_{\text{absolute}} = |\hat{U} - U|, \quad L2 = \sqrt{\Delta h \sum_{h=1}^{N^n} (\hat{U}_h - U_h)^2},$$

$$\text{Max-error} = \max(L_{\text{absolute}}), \quad \text{RMS} = \sqrt{\frac{\sum_{h=1}^{N^n} (\hat{U}_h - U_h)^2}{N}}, \quad (10)$$

where $\hat{U}$ is the exact solution, U is the approximate solution and Δh is the space step size.

Example 1. Consider the model equation (1) with $\beta = 1$, $\gamma = \delta = 0$ having the exact solution:

$$U(\bar{z}, t) = e^{-t} \sin(\pi x) \sin(\pi y), \quad \bar{z} = (x, y) \in \Omega. \quad (11)$$

In Example 1, numerical results are obtained by the LMM utilizing MQ, IQ and IMQ RBFs. These results are displayed in Table 1, and the error stands for Max-error. These results are computed using different values of time step size τ , fractional order $\alpha_1 = \alpha_2 = 0.3$, nodal points $N^2 = 20$ and final time $t = 1$ and $t = 5$. Furthermore, explicit, implicit and Crank–Nicolson schemes are used. It is observed from the table that the results of the LMM are in very good agreement with the exact solution and also the accuracy increases when τ decreases. The results of the Crank–Nicolson scheme utilizing IQ RBF are more accurate among other RBFs and time integration schemes. Figure 1 shows numerical results for different values of N^2 and fractional order. It can be seen that the

Table 1: Example 1, simulation results of the LMM

Max-error							
Method	τ	$t = 1$			$t = 5$		
		MQ	IQ	IMQ	MQ	IQ	IMQ
Explicit	1.0000×10^{-1}	1.8985×10^{-2}	1.8025×10^{-2}	1.9894×10^{-2}	1.5971×10^{-3}	1.6438×10^{-3}	1.2824×10^{-2}
	5.0000×10^{-2}	9.2780×10^{-3}	8.8557×10^{-3}	9.6846×10^{-3}	8.2371×10^{-4}	8.4253×10^{-4}	1.8936×10^{-3}
	2.5000×10^{-2}	4.5880×10^{-3}	4.3939×10^{-3}	4.7808×10^{-3}	4.1804×10^{-4}	4.2641×10^{-4}	6.2922×10^{-4}
	1.2500×10^{-2}	2.2815×10^{-3}	2.1902×10^{-3}	2.3752×10^{-3}	2.1057×10^{-4}	2.1450×10^{-4}	2.7806×10^{-4}
	6.2500×10^{-3}	1.1377×10^{-3}	1.0941×10^{-3}	1.1838×10^{-3}	1.0567×10^{-4}	1.1075×10^{-4}	1.3158×10^{-4}
Implicit	1.0000×10^{-1}	1.7500×10^{-2}	1.6812×10^{-2}	1.8127×10^{-2}	1.7904×10^{-3}	1.8238×10^{-3}	4.2065×10^{-3}
	5.0000×10^{-2}	8.9133×10^{-3}	8.5647×10^{-3}	9.2442×10^{-3}	8.7217×10^{-4}	8.8880×10^{-4}	1.3934×10^{-3}
	2.5000×10^{-2}	4.4982×10^{-3}	4.3247×10^{-3}	4.6693×10^{-3}	4.3024×10^{-4}	4.3836×10^{-4}	5.8666×10^{-4}
	1.2500×10^{-2}	2.2594×10^{-3}	2.1739×10^{-3}	2.3468×10^{-3}	2.1364×10^{-4}	2.1760×10^{-4}	2.7162×10^{-4}
	6.2500×10^{-3}	1.1323×10^{-3}	1.0903×10^{-3}	1.1765×10^{-3}	1.0645×10^{-4}	1.1169×10^{-4}	1.3021×10^{-4}
Crank–Nicolson	1.0000×10^{-1}	3.0659×10^{-4}	2.0459×10^{-4}	3.9556×10^{-4}	3.0957×10^{-5}	1.5536×10^{-5}	6.6946×10^{-4}
	5.0000×10^{-2}	8.4249×10^{-5}	4.5236×10^{-5}	1.0332×10^{-4}	8.4718×10^{-6}	3.2714×10^{-6}	8.3605×10^{-5}
	2.5000×10^{-2}	2.4213×10^{-5}	9.5438×10^{-6}	2.7555×10^{-5}	2.3901×10^{-6}	6.4059×10^{-7}	1.6480×10^{-5}
	1.2500×10^{-2}	7.1414×10^{-6}	1.8555×10^{-6}	7.4599×10^{-6}	6.8938×10^{-7}	1.1316×10^{-7}	3.9518×10^{-6}
	6.2500×10^{-3}	2.1392×10^{-6}	3.4572×10^{-7}	2.1154×10^{-6}	2.0203×10^{-7}	9.5540×10^{-8}	1.0463×10^{-6}

Figure 1: Example 1, (left) nodes versus error norms and (right) fractional order α s versus error norms.

Figure 2: Example 1, (left) c versus RMS and (right) c versus condition number.

Figure 3: Example 1, (left) exact versus approximate solution and (right) exact versus absolute error.

proposed technique produced better results on courser grid and various values of $\alpha_1 = \alpha_2$. The accuracy and stability of the meshless based on RBFs fully depend on the value of shape parameter c as well as the number of

nodes N . It is observed from the literature that the accuracy and conditional number of the global meshless method are extremely sensitive to the values of c . In contrast, the recommended LMM is checked for a wide range

Table 2: Example 2, simulation results of the LMM

Method	τ	L2			
		$t = 1$		$t = 2$	
		$\alpha_1 = \alpha_2 = 0.3$	$\alpha_1 = \alpha_2 = 0.6$	$\alpha_1 = \alpha_2 = 0.3$	$\alpha_1 = \alpha_2 = 0.6$
Explicit	1.0000×10^{-1}	5.2290×10^{-2}	5.1872×10^{-2}	3.7158×10^{-2}	3.7523×10^{-2}
	5.0000×10^{-2}	2.5622×10^{-2}	2.5444×10^{-2}	1.8438×10^{-2}	1.8637×10^{-2}
	2.5000×10^{-2}	1.2687×10^{-2}	1.2612×10^{-2}	9.1845×10^{-3}	9.2941×10^{-3}
	1.2500×10^{-2}	6.3133×10^{-3}	6.2822×10^{-3}	4.5840×10^{-3}	4.6433×10^{-3}
	6.2500×10^{-3}	3.1494×10^{-3}	3.1364×10^{-3}	2.2900×10^{-3}	2.3216×10^{-3}
Implicit	1.0000×10^{-1}	4.8471×10^{-2}	4.8894×10^{-2}	3.6200×10^{-2}	3.7120×10^{-2}
	5.0000×10^{-2}	2.4691×10^{-2}	2.4862×10^{-2}	1.8215×10^{-2}	1.8657×10^{-2}
	2.5000×10^{-2}	1.2461×10^{-2}	1.2528×10^{-2}	9.1339×10^{-3}	9.3447×10^{-3}
	1.2500×10^{-2}	6.2589×10^{-3}	6.2845×10^{-3}	4.5728×10^{-3}	4.6732×10^{-3}
	6.2500×10^{-3}	3.1364×10^{-3}	3.1458×10^{-3}	2.2877×10^{-3}	2.3357×10^{-3}
Crank–Nicolson	1.0000×10^{-1}	7.0289×10^{-4}	3.5095×10^{-4}	5.0136×10^{-4}	2.5348×10^{-4}
	5.0000×10^{-2}	1.6547×10^{-4}	9.2813×10^{-5}	1.1769×10^{-4}	7.1868×10^{-5}
	2.5000×10^{-2}	3.8330×10^{-5}	4.9749×10^{-5}	2.7198×10^{-5}	3.8280×10^{-5}
	1.2500×10^{-2}	8.6918×10^{-6}	2.4637×10^{-5}	6.1554×10^{-6}	1.8688×10^{-5}
	6.2500×10^{-3}	1.9200×10^{-6}	1.0941×10^{-5}	1.3581×10^{-6}	8.2362×10^{-6}

of c (up to 200) and observed from Figure 2 that the method shows stable behavior. Also, Figure 2 (right) shows the condition number of the IQ, MQ and IMQ RBFs and noted that MQ and IMQ RBFs have less condition numbers as compared to IQ RBF. Figure 3 (left) shows the good agreement between the exact and numerical solution of the LMM using MQ RBF for $N^2 = 30$, $\tau = 6.2500 \times 10^{-3}$, $\alpha_1 = \alpha_2 = 0.5$, $t = 1$, $t = 2$ and $t = 3$, whereas Figure 3 (right) shows the absolute error obtained by the recommended LMM.

Example 2. Consider the model equation (1) with $\beta = 1$, $\gamma = \delta = 0$ having the exact solution

Figure 4: Example 2, fractional order α 's versus L_2 error norms. **Figure 5:** Example 2, (left) c versus RMS and (right) c versus condition number.

Figure 6: Computational domain.

$$U(\bar{\mathbf{z}}, t) = e^{x-y-t} \sin(\pi x) \sin(\pi y), \quad \bar{\mathbf{z}} = (x, y) \in \Omega. \quad (12)$$

In Example 2, numerical results are obtained by the LMM utilizing MQ RBF. These results are displayed in Table 2, and the error stands for L2. These results are computed using different values of τ , fractional order $\alpha_1 = \alpha_2 = 0.3$ and $\alpha_1 = \alpha_2 = 0.6$, $N^2 = 6$, $t = 1$ and $t = 2$. Moreover, explicit, implicit and Crank–Nicolson schemes

are utilized and observed that the accuracy increases when τ decreases in this case as well. Also, the results of the Crank–Nicolson scheme are more accurate among other time integration schemes. The numerical results of Example 2 utilizing explicit, implicit and Crank–Nicolson schemes for various values of fractional orders α 's are shown in Figure 4. It can be noted that Crank–Nicolson scheme is more effective as compared to explicit and implicit schemes. Like previous example, the accuracy and stability of the suggested method are tested in terms of shape parameter value for a wide range (up to 200) as shown in Figure 5. The MQ and IMQ RBFs are more stable in this case in comparison to IQ RBF, also the conation number of IQ RBF is also high than MQ and IMQ RBFs. One of the advantages of the meshless methods over mesh-based methods is the ease of implementation in irregular domain. We have considered a challenging irregular punctured domain which is shown in Figure 6. The numerical results of the LMM corresponding to the irregular domain are tabulated in Table 3. It can be revealed from this table that the suggested LMM gives good results in irregular domain as well. The accuracy of the MQ and IMQ is better than that of IQ RBF in this case.

Table 3: Example 2, simulation results of the LMM utilizing computational domain given in Figure 6

RBFs	Error norm	$t = 1$	$t = 2$	$t = 3$
IQ	Max-error	2.8503×10^{-3}	6.9177×10^{-3}	1.9053×10^{-2}
	RMS	5.3969×10^{-4}	1.1019×10^{-3}	2.6854×10^{-3}
	L2	5.8179×10^{-3}	1.1879×10^{-2}	2.8949×10^{-2}
MQ	Max-error	1.0762×10^{-4}	7.4827×10^{-5}	3.9556×10^{-5}
	RMS	1.2498×10^{-5}	8.9038×10^{-6}	4.8116×10^{-6}
	L2	1.3473×10^{-4}	9.5983×10^{-5}	5.1869×10^{-5}
IMQ	Max-error	1.6486×10^{-4}	1.2021×10^{-4}	6.5942×10^{-5}
	RMS	1.6465×10^{-5}	1.2001×10^{-5}	6.5818×10^{-6}
	L2	1.7749×10^{-4}	1.2937×10^{-4}	7.0953×10^{-5}

Figure 7: Example 3, (left) exact solution and (right) numerical solution.

Figure 8: Example 3, absolute error at (left) $t = 0.01$ and (right) $t = 0.05$.

Example 3. Consider the model equation (1) with $\beta = 1$, $\gamma = 1$, $\delta = \pi^2$ having the exact solution

$$U(\bar{\mathbf{z}}, t) = e^t \sin(\pi x) \sin(\pi y), \quad \bar{\mathbf{z}} = (x, y) \in \Omega. \quad (13)$$

The comparison of approximate solution obtained by the LMM with the exact solution for Example 3 is shown in Figure 7 using $N^2 = 20$, $\alpha_1 = \alpha_2 = 0.25$ and $t = 0.1$. One can see that the numerical solution is in good agreement with the exact solution, whereas the absolute errors are shown in Figure 8.

4 Conclusion

The LMM based on RBFs is utilized for two-term time-fractional Sobolev equations. Three types of RBFs are used. The proposed algorithm framework leads to a sparse linear system of equations and approximated the solution with good accuracy. To check the accuracy of the proposed scheme, several examples have been considered using rectangular and one irregular computational domain. The numerical results demonstrate that the algorithm is reliable, effective and gives accurate solution. Considering the current work, we can say that the proposed strategy is amazing, powerful and successful instrument for the numerical solution of multi-term time-fractional PDEs, so it can be applied to a wide scope of complex problems that emerge in natural sciences and engineering.

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