Research Article

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Investigation of interactional phenomena and multi wave solutions of the quantum hydrodynamic Zakharov–Kuznetsov model

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Abstract: The symbolic computation with the ansatz function and the logarithmic transformation method are used to obtain a formula for certain exact solutions of the $(3 + 1)$ Zakharov–Kuznetsov ($Z$–$K$) equation. We use homoclinic breather, three waves method, and double exponential. There is a conflict of results with considerably known results, which indicates the solutions found in this study are new. By selecting appropriate parameter values, 3d representations are plotted to establish W-shaped, multi-peak, and kinky breathers solutions.

Keywords: 02.30Xx soliton solutions, $(3 + 1)$ Zakharov–Kuznetsov equation, the logarithmic transformation method

1 Introduction

Many physical phenomena originate in various fields of engineering and science, such as fluid dynamics, fiber optics, quantum mechanics, plasma, and physics. These physical phenomena were constructed in the form of nonlinear equations [1–7]. To understand the behavior of a phenomenon, we need to solve nonlinear equations that describe the phenomenon, which is often challenging. Many methods have been developed over the years to solve such nonlinear equations, many of which are based on assumptions. Investigating the exact solutions for these nonlinear equations is a major area of concern for many mathematicians and physicists because of their important role in understanding the behavior of nonlinear physical phenomena. The Zakharov–Kuznetsov ($Z$–$K$) equation supports stable solitary waves, see ref. [8,9]. This makes the $Z$–$K$ equation a very attractive model equation for investigating of vortices in geophysical ope. The $Z$–$K$ equation determines weak non-linear behavior of ion sound waves that contain electrons with hot and cold temperatures in a regular magnetic field, see ref. [9]. With regard to the exact solutions of the $Z$–$K$ and $(3 + 1)$-dimensional $Z$–$K$ equations, many solving methods have been used to solve it, see [2,9–14], and have many applications in numerical and analytical approaches in physical sciences [15–21]. An efficient analytical technique for fractional partial differential equations occurs in ion-acoustic waves in plasma and warm plasma [37,38].

Some new methods and important developments in the search for analysis have been done to investigate wave solutions for partial non-linear differential equations. The results of this manuscript may be completely complementary to in existence manuscripts of literature such as: direct algebraic method that is extended and modified; techniques’ Seadawy and extended mapping method to find solutions for some nonlinear partial differential equations [5]; showing the bi-directional propagation of small-amplitude long-capillary gravity waves on the surface of shallow water [22]; bright and dark solitons, solitary wave solutions of higher-order non-linear differential equations and the elliptic function [23]; the higher-order non-linear differential equations with cubic quintic nonlinearity [24]; solitary wave solutions to the nonlinear-modified KdV dynamical equation [25]; modified equal–width equations and dispersive traveling wave solutions of the equal–width [26]; the exact travelling wave solutions of the SRLW equation and modified Liouville equation [27]; fourth-order non-linear Ablowitz–Kaup–Newell–Segur water wave dynamical equation [28]; nonlinear wave solutions of the
2 Interactional phenomena and multi-waves solutions

Solutions for multiple waves and fractional phenomena logarithmic transformation can be shown as

\[ \phi(\xi) = 2\log(f(\xi))_\xi \]  

equation (3) has the form:

\[ k((\delta + 1)v^2 + k^2)f(\xi)\phi'f^2(\xi) + 2k((\delta + 1)v^2 + k^2) + 2k^2\phi(\xi)f^2(\xi) + 2\omega \phi(\xi)f^2(\xi) = 0 \]  

2.1 Type (I): Three waves hypothesis

In this subsection, for the next three waves hypothesis, multiple wave solutions can be expressed for nonlinear equation (5) as follows:

\[ f(\xi) = b_0 \cosh(a_1\xi + a_2) + b_1 \cos(a_3\xi + a_4) + b_2 \cosh(a_c\xi + a_5), \]  

where the real constants \( a_j, (j = 1, \ldots, 6) \) are determined later. Substituting (6) into (5) with the help of collecting all the parameters for all the exponents of \( \cosh(a_1\xi + a_2) \), \( \cos(a_3\xi + a_4) \), \( \sinh(a_c\xi + a_5) \), \( \sin(a_3\xi + a_4) \) and \( \sinh(a_c\xi + a_5) \) functions be zero, which establishes a set of algebraic equations with respect to above constants. With help of the Wolfram Mathematica program software, solving the determination algebraic equations, we conclude the following cases:

**Case 1**

\[ a_1 = \frac{\sqrt{\omega}}{\sqrt{2}\lambda^2k^2 - 6\lambda^2}, \quad a_5 = a_1, \quad a_3 = 0, \quad b_1 = \sqrt{b_0^2 + b_2^2}, \quad \nu = \frac{\sqrt{4\lambda^2 - k^2}}{\sqrt{\delta + 1}}, \]  

substituting from (6), (7), and (4). By taking \( \gamma = \frac{\sqrt{\omega}}{\sqrt{2}\lambda^2k^2 - 6\lambda^2} \) and \( \tau = \frac{\sqrt{2}\lambda^2}{\lambda^2k^2 - 6\lambda^2} \), we obtain,

\[ u_{11}(x, y, z, t) = \frac{\tau \left[ b_0 \sinh \left( a_2 + \gamma(kx + tw + vy + \lambda z) \right) + b_1 \sinh \left( a_6 + \gamma(kx + tw + vy + \lambda z) \right) \right]}{b_0 \cosh \left( a_2 + \gamma(kx + tw + vy + \lambda z) \right) + b_2 \cosh \left( a_6 + \gamma(kx + tw + vy + \lambda z) \right) + \sqrt{b_0^2 + b_2^2} \cos \left( a_4 \right)} \]  

Figure 1

**Case 2**

\[ a_1 = \frac{1}{\sqrt{2}}, \quad a_3 = -\frac{i}{\sqrt{2}}, \quad a_5 = 0, \quad \omega = -\lambda^2\sqrt{k^2 - 6\lambda^2}, \quad \delta = \frac{-k^2 + 4\lambda^2 - \nu^2}{\nu^2}, \quad b_0 = \nu b_1, \]  

substituting from (6), (9), and (4), we have:

\[ u_{12}(x, y, z, t) = \frac{\sqrt{2}b_1 \left( i \sinh \left( a_2 + \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} \right) + \sinh \left( \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} + ia_4 \right) \right)}{b_2 \cosh \left( a_6 \right) + b_1 \left( i \cosh \left( a_2 + \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} \right) + \cosh \left( \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} + ia_4 \right) \right)} \]  

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Figure 1

**Case 2**

\[ a_1 = \frac{1}{\sqrt{2}}, \quad a_3 = -\frac{i}{\sqrt{2}}, \quad a_5 = 0, \quad \omega = -\lambda^2\sqrt{k^2 - 6\lambda^2}, \quad \delta = \frac{-k^2 + 4\lambda^2 - \nu^2}{\nu^2}, \quad b_0 = \nu b_1, \]  

substituting from (6), (9), and (4), we have:

\[ u_{12}(x, y, z, t) = \frac{\sqrt{2}b_1 \left( i \sinh \left( a_2 + \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} \right) + \sinh \left( \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} + ia_4 \right) \right)}{b_2 \cosh \left( a_6 \right) + b_1 \left( i \cosh \left( a_2 + \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} \right) + \cosh \left( \frac{\nu(kx + tw + vy + \lambda z)}{\sqrt{2}} + ia_4 \right) \right)} \]
This figure for wave solutions (8) using parameters: $k = 3, \lambda = 1, \delta = 5, \omega = 4, a_3 = 1, a_4 = 30, a_5 = 2, b_0 = 1, b_1 = 1, z = 1, y = \frac{3}{\sqrt{7}}$.

(a) and (b) Represented wave solutions in three dimensions, (c) and (d) showed wave solutions in two dimensions and (e) and (f) a contour solutions.

Figure 2

Case 3

\[ a_1 = 0, \quad a_3 = \frac{\sqrt{k}}{\sqrt{\delta k^2 + 2k^2 + \lambda^2 v^2}}, \quad a_5 = \frac{i \sqrt{k}}{\sqrt{\delta k^2 + 2k^2 + \lambda^2 v^2}}, \quad b_1 = i b_2, \quad \omega = -\frac{4 k^2}{8 \nu^2 + 2 k^2 + v^2} \]
Figure 2: This figure for wave solutions (10) using parameters: \(a_2 = 1, a_4 = 1, a_6 = 2, b_2 = 2\sqrt{2}, b_4 = 2\sqrt{2}, y = 2, k = 4, \nu = 2\). Wave solutions appear in (a) and (b) as three dimensions but (c) and (d) represented wave solutions in two dimensions and (e) and (f) a contour solutions.

substituting from (6), (11), and (4). By taking \(y_2 = \frac{\sqrt{k}}{\sqrt{(\delta + 1)\lambda^2 + 2\lambda^2}}\), we obtain

\[
\frac{2ib_2y_2(\sin(a_x + y_2(kx + tw + vy + \lambda z)) - i\sin(y_2(kx + tw + vy + \lambda z) - ia_6))}{b_0 \cosh(a_2) + b_2(i \cos(a_x + y_2(kx + tw + vy + \lambda z)) + \cos(y_2(kx + tw + vy + \lambda z) - ia_6))}
\]  

(12)
Figure 3: This figure for wave solutions (12) using parameters: $k = -4, \lambda = 2, \delta = 1, a_2 = 2, a_4 = 1, a_6 = 2, b_2 = 2, z = 1, y = 2, v = 2, b_0 = 1$.

(a) and (b) Plotted as wave solutions in three dimensions, (c) and (d) showed wave solutions in two dimensions and (e) and (f) a contour solutions.
Figure 4: This figure for wave solutions (15) using parameters: $a_1 = \frac{2}{\sqrt{3}}, a_3 = \frac{2}{\sqrt{3}}$. (a) and (b) Represented wave solutions in three dimensions, (c) and (d) showed wave solutions in two dimensions and (e) and (f) a contour solutions.
Figure 5: This figure for wave solutions (18) using parameters: $k = 1, z = 1, y = 1, \rho_1 = 1, p = 1, \lambda = 1, \mu = \frac{1}{3}, v = 1, q_2 = 1, q_4 = 1, q_6 = -i, b_2 = 1, b_4 = 1$, (a) and (b) Represented wave solutions in three dimensions, (c) and (d) showed wave solutions in two dimensions and (e) and (f) a contour solutions.
2.2 Type (II): Interactional phenomena and exponential form

The hypothesis of a double exponential function takes the form:

\[ f(\xi) = b_1 e^{a_1 \xi + a_2} + b_2 e^{a_3 \xi + a_4}, \]  

(13)

where the real constants \( a_j \) (\( j = 1, \ldots, 6 \)) are determined later. By substituting (13) into (5) with the help of symbolic accounts having all the coefficients of all powers of exponential functions zero, we have a system of algebraic equations. By resolving this system, we get

\[ a_1 = \frac{2}{\sqrt{3}}, \quad a_3 = \frac{2}{\sqrt{3}}, \quad \mu = -6\sqrt{3}v^2 + 12v^2 - 11\sqrt{3} + 22, \]  

(14)

substitution from (13), (14) and (4), we obtain

\[ u_2(x, y, z, t) = \frac{2}{3\sqrt{3}} \left( e^{a_1 b_1 + a_3 b_2} e^{-\frac{2\sqrt{3}(v + vy + \lambda)}{\sqrt{3}}} \right), \]  

(15)

Figure 4

2.3 Type (III): Homoclinic breather approach

The Homoclinic breather approach function takes the form:

\[ f(\xi) = e^{\xi(\rho)} + b_0 \cos(\xi p_1) + b_1 e^{\xi p}, \]  

(16)

where the real constants \( a_j \) (\( j = 1, \ldots, 6 \)) are determined later. Substituting (16) into (4) and by resolving the algebraic equations, we obtain:

\[ a_5 = \frac{1}{\sqrt{3}} \frac{\mu}{\lambda^2 p_1}, \quad a_1 = \frac{3\mu}{\lambda^2 p}, \quad a_3 = -\frac{3\mu}{\lambda^2 p}, \quad a_7 = -\frac{9\mu^2}{\lambda^2}, \]  

(17)

substitution from (16) and (17) using (4). By taking \( y_3 = \frac{\mu}{\lambda^2} \), the wave solution takes the form:

\[ u_3(x, y, z, t) = \frac{2}{b_0 e^{a_3 p + 3\sqrt{3}(v + vy + \lambda) + \lambda} \cosh(\sqrt{3} y_3 (kx + tw + vy + \lambda z - \lambda a_3 p_1) - 3b_1 e^{a_1 p_1}) + 1}{b_0 e^{a_3 p + 3\sqrt{3}(v + vy + \lambda) + \lambda} \cosh(\sqrt{3} y_3 (kx + tw + vy + \lambda z - \lambda a_3 p_1) - 3b_1 e^{a_1 p_1})}. \]  

(18)

Figure 5

3 Conclusion

In this work, the multi-wave solution method has been used successfully to obtain solution the (3 + 1) dimensions (Z–K) equation by helping of logarithmic transformation and symbolic computation. This methods is important from an applied and theoretical point of view, and an additional family of the solutions has been found, which is the best thing about this method. Novel solutions include the generalized solutions to (2 + 1)-dimensional (Z–K) equation and represent their graphs by determining the appropriate values of the parameters involved.

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References


Iqbal M, Seadawy AR. Khalil OH, Lu D. Propagation of long internal waves in density stratified ocean for the (2+1)-dimensional nonlinear Nizhnik-Veselov dynamical equation. Results Phys. 2020;16:102838.


