Research Article

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Analysis of couple stress fluid flow with variable viscosity using two homotopy-based methods

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Abstract: In this article, the generalized plane Couette flow of Vogel's model of incompressible, non-isothermal, couple stress fluid flowing steadily between two parallel walls is investigated. The governing equations are reduced to ordinary differential equations. To investigate the non-linear coupled system of differential equations, the optimal homotopy asymptotic method with DJ polynomial and asymptotic homotopy perturbation method have been used. Important flow properties are presented and discussed. We have obtained expressions for velocity, average velocity, shear stress, volume flux and temperature. The results gained employing these techniques are in the form of infinite series; thus, the results can be easily calculated. Comparison of various results, obtained through the suggested approaches, is carried out and an excellent agreement is achieved.

Keywords: asymptotic homotopy perturbation method, optimal homotopy asymptotic method DJ polynomial, couple stress fluid

1 Introduction

In recent decades, the non-Newtonian fluids are very attractive due to their wide range of applications in

numerous industrial and engineering fields. The heat and mass transfer in non-Newtonian fluids are essential in substantial oils, greases and food processing [1–3]. Overall, non-Newtonian fluids have many engineering and natural applications and are involved in numerous organic circumstances. The non-Newtonian fluid has viscosity dependent on shear rate and strain rate, whereas Newtonian fluids do not have such viscosity. This diverse characteristic makes non-Newtonian fluids vital for countless practical applications [4–9]. In recent years, scientists have used their energy to establish the link between the viscosity and the flow state of the non-Newtonian fluid, as it is very important for numerous mathematical models to define such type of fluid [10–12].

In 1966, Stokes for the first time suggested the theory of couple stress fluids [13], which models a fluid medium. The notion of this theory arises because of the concept that for mechanical dealings how to model a fluid medium. This theory effectively describes the flow behavior of the fluids having a substructure such as lubricants with polymer additives, animal blood and liquid crystals [14]. The couple stress fluid acknowledged astonishing attention among the numerous models which are utilized to define the non-Newtonian behavior formed by certain fluid [15–17]. Aksoy [18] studied the entropy generation of couple stress fluid flow. In ref. [19], researchers studied the couple stress fluids in an inclined stretching cylinder using variable viscosity and thermal conductivity by means of analytical series solutions. Goswami et al. [20] have applied the homotopy perturbation transform method to approximate the non-linear fifth order KdV equations. Scientists in ref. [21] have studied the non-linear wave like equations with variable coefficients using the homotopy analysis transform method (HATM). Falade et al. [22] employed closed form solutions to examine the influence of variable viscosity on entropy in couple stress fluid flow. In viscous medium, the couple stress fluid model represents those fluids which contain haphazardly rigid and oriented particles. In classical Newtonian theory, the exact flow behavior of fluid cannot be predicted as the couple stress fluid model uses anti-symmetric stress

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tensor. The remarkable characteristic of couple stress model is that its solutions are similar to the Navier–Stokes equations. This model has been broadly used because of the mathematical simplicity as compared to other models formed for the fluid under consideration. Scientists in refs [23] have examined different couple stress fluid problems of flows past axisymmetric bodies. The governing equations of couple stress fluid flow are non-linear in nature and their orders are higher than that of Navier–Stokes equations. Consequently, to acquire the exact solution, it is extremely difficult. In many engineering applications, heat transfer flow is significant, for instance, radial diffusers, drag reduction, transpiration cooling, thrust bearing design and thermal revival of oil. Goswami et al. [24] have investigated the time fractional Kersten–Krasiloshchik coupled KdV–mKdV non-linear system using the homotopy perturbation sumudu transform method. Goswami et al. [25] have investigated the non-linear behavior of plasma using the homotopy perturbation sumudu transform method. Two different fractional equations, fractional modified equal width with different non-linearity and fractional equal width equations, have been investigated using this technique. Abbas and Marin [26] studied the analytical solutions of the thermoelastic interaction in a half-space. Itu et al. [27] utilized the finite element method to examine the composite plate. Gupta et al. [28] examined the effects of heat and mass transfer in the magneto-hydrodynamic three-dimensional flow consisting of Cu and Al_2O_3 water-based nanofluids with the help of optimal homotopy asymptotic method (OHAM). Researchers in ref. [29] applied OHAM to investigate the steady MHD-free convective boundary layer flow of nanofluids comprising incompressible, viscous and electrically conducting water-driven silver and titanium-oxide. These investigations study the competence of heat transfer of nanofluids in rubber sheets and cooling plants. Gupta et al. [30] investigated the effects of thermally developed Brownian motion and thermophoresis diffusion in non-Newtonian nanofluid using OHAM. The results of OHAM are compared with those of already published work. In the non-Newtonian mixtures, heat transfer has a dynamic role in handling and processing [31–33].

Different methods have been used to examine the flow problems in the literature. These techniques consist of numerical techniques, iterative techniques, homotopy-based techniques and perturbation techniques, which are the main techniques for examining the approximate solutions. Each technique has its own advantages and disadvantages. The discretization is used in numerical methods which affect the accuracy. The numerical techniques required ample amount of computational work

and plenty of time. In strong non-linear problems, these methods do not provide accurate results.

In ref. [34], Daftardar-Gejji and Jafari used a new method for the solution of linear and non-linear functional equations. In ref. [35], the convergence of this technique has been confirmed. Afterward this technique was named Daftardar–Jafari method (DJM) [35]. Researchers in refs [35,36] used DJM in the OHAM, for the solution of non-linear differential equations and named this method as OHAM with DJ polynomials. In ref. [37], OHAM-DJ has been employed for the solution of Klein–Gordon equations for both linear and non-linear cases.

In 2019, Bushnaq et al. [38] proposed a new technique to study the non-linear fractional order partial differential equations, which is known as the asymptotic homotopy perturbation method (AHPM).

In this article, the couple stress fluid of generalized Couette flow of Vogel’s model in the channel has been studied. Employing the said techniques, the coupled system of differential equations is then explored.

This article consists of six sections. Section 2 consists of basic governing equations and problem formulation. Section 3 consists of basic ideas of the methods and in Section 4 solutions of the problem are given. Section 5 consists of results and discussion, whereas the conclusion of the article is given in the last section.

2 Basic equations and problem formulation

2.1 Basic equations

For an incompressible fluid, equations of balance of momentum, energy and mass conservation are as follows [39,40]:

$$\nabla \cdot \mathbf{V} = 0, \quad (2.1)$$

$$\rho \dot{\mathbf{V}} = \nabla \cdot \boldsymbol{\tau} - \eta \nabla^4 \mathbf{V} + \rho \mathbf{f}, \quad (2.2)$$

$$\rho c_p \dot{\theta} = \kappa \nabla^2 \theta + \tau L, \quad (2.3)$$

where $\mathbf{V}$ is the velocity vector, $\boldsymbol{\tau}$ is the Cauchy stress tensor, the temperature is denoted by θ , $\mathbf{f}$ is the body force per unit mass, ρ is the constant density, the thermal conductivity is denoted by κ , c_p is the specific heat, gradient of $\mathbf{V}$ is denoted by L and the couple stress parameter is represented by η . $\frac{D}{Dt}$ represents the material derivative and defined as follows:

$$\frac{D}{Dt}(\Psi) = \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) (\Psi). \quad (2.4)$$

The symbol τ represents the Cauchy stress tensor and is defined as

$$\tau = -pI + \mu A_1, \quad (2.5)$$

where the unit tensor is denoted by I , p is the dynamic pressure, μ represents the coefficient of viscosity and A_1 represents first Rivlin–Ericksen tensor and given as follows:

$$A_1 = L + L^T, \quad (2.6)$$

where the transpose of L is represented by L^T .

2.2 Problem formulation

Considering a non-isothermal and incompressible couple stress fluid of Vogel's model in the channel, the walls of the channel are $2d$ distant, the upper wall moves with $\mathbf{V}$ velocity and lower one is fixed. The lower and upper walls are kept at temperatures θ_0 and θ_1 , respectively. In the plane, the under consideration walls are placed in an orthogonal coordinate system (x, y) , at $y = -d$ and $y = d$, where x -axis is taken in the direction of fluid motion and the y -axis is orthogonal to the plates as shown in Figure 1. The viscosity μ is taken to be a function of $\theta(y)$, the value of pressure gradient is taken as zero, the velocity and temperature profiles are given as follows:

$$\mathbf{V} = \mathbf{V}(v, 0, 0), \quad v = v(y) \quad \text{and} \quad \theta = \theta(y). \quad (2.7)$$

Using these suppositions, equation (2.1) is identically fulfilled. If body force is taken as zero, the momentum and energy equations (2.2) and (2.3) become

Figure 1: The generalized plane Couette flow.

$$\mu \frac{d^2 v}{dy^2} + \frac{d\mu}{dy} \frac{dv}{dy} - \eta \frac{d^4 v}{dy^4} - A = 0, \quad (2.8)$$

$$\frac{d^2 \theta}{dy^2} + \frac{\mu}{\kappa} \left(\frac{dv}{dy} \right)^2 - \frac{\eta}{\kappa} \left(\frac{d^2 v}{dy^2} \right)^2 = 0. \quad (2.9)$$

Boundary conditions for equations (2.8) and (2.9) are as follows:

$$v(-d) = 0, \quad v(d) = V, \quad (2.10a)$$

$$v''(-d) = 0, \quad v''(d) = 0, \quad (2.10b)$$

$$\theta(-d) = \theta_0, \quad \theta(d) = \theta_1. \quad (2.11)$$

Equation (2.10a) represents no slip BCs. Equation (2.10b) indicates that at walls the couple stress vanishes. The non-dimensional parameters generally can be written as:

$$v^* = \frac{v}{V}, \quad \mu^* = \frac{\mu}{\mu_0}, \quad y^* = \frac{y}{d}, \quad \theta^* = \frac{\theta - \theta_0}{\theta_1 - \theta_0},$$

$$A^* = \frac{Ad^4}{V\eta}, \quad \lambda = \frac{\mu_0 V^2}{\kappa(\theta_1 - \theta_0)}, \quad B = d \sqrt{\frac{\mu_0}{\eta}}.$$

Thus, equations (2.8) and (2.9) with BCs (2.10a)–(2.11) and dropping the asterisks can be written as follows:

$$\frac{d^4 v}{dy^4} - B^2 \mu \frac{d^2 v}{dy^2} - B^2 \frac{d\mu}{dy} \frac{dv}{dy} + A = 0, \quad (2.12)$$

$$\frac{d^2 \theta}{dy^2} + \lambda \mu \left(\frac{dv}{dy} \right)^2 + \frac{\lambda}{B^2} \left(\frac{d^2 v}{dy^2} \right)^2 = 0, \quad (2.13)$$

$$v(1) = 1, \quad v''(1) = 0, \quad v(-1) = 0, \quad v''(-1) = 0, \quad (2.14)$$

$$\theta(1) = 1, \quad \theta(-1) = 0. \quad (2.15)$$

The dimensionless form of Vogel's viscosity model [41,42] is as follows:

$$\mu = \mu_* \exp \left(\frac{A_0}{B_0 + \theta} - \theta_w \right). \quad (2.16)$$

After using Taylor series expansion [43,44], it becomes

$$\mu = \alpha^2 \left(1 - \frac{A_0}{B_0 + \theta} \right). \quad (2.17)$$

Here these are the viscosity parameter for Vogel's model, $\alpha^2 = \mu_* \exp \left(\frac{A_0}{B_0} - \theta_w \right)$, B_0 and A_0 . Let us assume $A_0 = \varepsilon d$, where ε is a small parameter.

3 Description of the methods

A brief introduction of OHAM-DJ and AHPM is given in this section.

3.1 Basic idea of OHAM-DJ

Consider the following differential equation:

$$L(v(x)) + g(x) + M(v(x)) = 0, \quad B\left(v, \frac{dv}{dx}\right) = 0, \quad (3.1)$$

where $g(x)$ is the known function, B is the boundary operator, L represents the linear operator, v is the required function and M is a non-linear term.

Using OHAM we get [45]

$$\begin{aligned} (1 - q)[L(v(x, q)) + g(x)] \\ = H(q)[L(v(x, q)) + g(x)] + M(v(x, q)), \\ B\left(v(x, q), \frac{dv(x, q)}{dx}\right) = 0. \end{aligned} \quad (3.2)$$

Here the embedding parameter is $q \in [0, 1]$, the auxiliary function is denoted by $H(q)$, such that for $q \neq 0$, $H(q) \neq 0$ and for $q = 0$ that is $H(0) = 0$, obvious when $q = 0$ and $q = 1$ it gives

$$v(x, 0) = v_0(x), \quad v(x, 1) = v(x). \quad (3.3)$$

The result $v(x, q)$ differs from $v_0(x) \rightarrow v(x)$ as q differs from $0 \rightarrow 1$ the result $v_0(x)$ can be obtained by substituting $q = 0$ in equation (3.1),

$$L(v_0(x)) + g(x) = 0, \quad B\left(v_0, \frac{dv_0}{dx}\right) = 0. \quad (3.4)$$

Here auxiliary function is given as follows:

$$H(q) = qb_1 + q^2b_2 + q^3b_3 + \dots, \quad (3.5)$$

where b_i , $i = 1, 2, 3, \dots$ are constants to be determined. Equation (3.2) can be written as follows:

$$v(x, q, b_i) = v_0(x) + \sum_{j=1}^n v_j(x, b_i)q^j, \quad i = 1, 2, \dots \quad (3.6)$$

We can decompose the non-linear part $M(v(x, q))$ as follows:

$$\begin{aligned} M = M(v_0) + q[M(v_0 + v_1) - M(v_0)] \\ + q^2[M(v_0 + v_1 + v_2) - M(v_0 + v_1)] + \dots, \end{aligned} \quad (3.7)$$

where DJ polynomials are represented as

$$\begin{aligned} M(v_0), [M(v_0 + v_1) - M(v_0)], \\ [M(v_0 + v_1 + v_2) - M(v_0 + v_1)], \dots \end{aligned}$$

The convergence of the aforementioned polynomials has been given in ref. [35]. Simply the aforementioned polynomials can be presented as:

$$\begin{aligned} M_0 &= M(v_0), \\ M_1 &= M(v_0 + v_1) - M(v_0), \\ M_2 &= M(v_0 + v_1 + v_2) - M(v_0 + v_1), \end{aligned}$$

consequently, we have

$$M_n = \left(M \sum_{j=0}^n v_j\right) - \left(M \sum_{j=0}^{n-1} v_j\right). \quad (3.8)$$

$$M = M_0 + \sum_{j=0}^n q^j M_j. \quad (3.9)$$

Substituting (3.5), (3.6), (3.7) and (3.9) in (3.2) and comparing similar powers of q we have

$$L(v_0(x)) + g(x) = 0, \quad B\left(v_0, \frac{dv_0}{dx}\right) = 0, \quad (3.10)$$

$$L(v_1(x)) = b_1 M_0(v_0(x)), \quad B\left(v_1, \frac{dv_1}{dx}\right) = 0, \quad (3.11)$$

$$\begin{aligned} L(v_j(x) - v_{j-1}(x)) \\ = b_j M_0(v_0(x)) + \sum_{k=1}^{j-1} b_k [L(v_{j-k}(x)) \\ + M_{j-k}(v_0(x), v_1(x), \dots, v_{j-1}(x))], \\ B\left(v_j, \frac{dv_j}{dx}\right) = 0, \quad j = 1, 2, \dots \end{aligned} \quad (3.12)$$

The system of equations (3.4), (3.11) and (3.12) can be evaluated for $v_j(x)$, $j \geq 0$, easily. The solution of equation (3.6), i.e., convergence, is completely dependent on $b_1, b_2, b_3, \dots$. If at $q = 1$ it converges, subsequently equation (3.6) gives

$$v(x, b_k) = v_0(x) + \sum_{j=1}^n v_j(x, b_k). \quad (3.13)$$

In general, the solution result of equation (3.1) is given as

$$v^n(x, b_l) = v_0(x) + \sum_{j=1}^n v_j(x, b_l), \quad l = 1, 2, \dots, n. \quad (3.14)$$

After substituting equation (3.14) in equation (3.1) the residual becomes:

$$\begin{aligned} R(x, b_l) &= L(v^n(x, b_l)) + g(x) + N(v^n(x, b_l)), \\ l &= 1, 2, \dots, n. \end{aligned} \quad (3.15)$$

If $R(x, b_k) = 0$, i.e., the residual is equal to zero, the exact solution $v^n(x, c_k)$ is obtained. However, if $R(x, c_k) \neq 0$, it can be minimized as follows:

$$J(b_k) = \int_a^e R^2(x, b_k) dx, \quad (3.16)$$

where a, e are constants, depending on problem under consideration. Also, constants $b_1, b_2, b_3, \dots$ can be obtained using the following equation:

$$\frac{\partial J}{\partial b_i} = 0, \quad i = 1, 2, \dots, n. \quad (3.17)$$

After obtaining these constants, the approximate solution can be obtained from equation (3.14).

3.2 Basic idea of AHPM

In this subsection, the asymptotic homotopy perturbation method is illustrated.

$$L(v(x)) + g(x) + N(v(x)) = 0, \quad (3.18)$$

where $g(x)$ is the known function, $v(x)$ is the unknown function, the linear operator is represented by L and the non-linear term is $N(v(x))$. Now construct homotopy $\Psi(x, q) : \Omega \times [0, 1] \rightarrow R$ [38] such that

$$L(\Psi(x, q)) + g(x) - q[N(\Psi(x, q))] = 0, \quad (3.19)$$

where the embedding parameter is $q \in [0, 1]$. Equation (3.19) is an alternate form of the deformation equation OHAM proposed by Marinca *et al.* [45] as follows:

$$(1 - q)[L(\Psi(x, q)) + g(x)] - H(q)[L(\Psi(x, q)) + g(x) + N(\Psi(x, q))] = 0, \quad (3.20)$$

obviously as $q = 0$ and $q = 1$ it gives

$$\Psi(v(x, 0)) = v_0(x), \quad \Psi(v(x, 1)) = v(x).$$

Consider $\Psi(v(x, q))$ as

$$\Psi(v(x, q)) = v_0(x) + \sum_{j=1}^{\infty} v_j(x)q^j, \quad (3.21)$$

$N(\Psi(x, q))$ can be expanded as

$$N(\Psi(x, q)) = A_1 N_0 + \sum_{j=1}^{\infty} \left(\sum_{n=0}^j A_{j+1-n} N_n \right) q^j, \quad (3.22)$$

$$A_1 + A_2 + \dots = -1,$$

where

$$A_j = A_j(x, b_j), \quad j = 1, 2, 3, \dots \quad (3.23)$$

Substituting (3.21) and (3.22) in (3.19) and then equating like powers of q , it gives

$$q^0 : L(v_0(x)) + g(x) = 0,$$

$$\begin{aligned} v_0 = & 0.04166(12.01 + 12y - 0.012y^2 + 0.002y^4) + 0.00039682(-1.29812 \times 10^{-6} + 1.40524 \times 10^{-16}y \\ & + 1.55775 \times 10^{-6}y^2 - 2.12684 \times 10^{-16}y^3 - 2.59625 \times 10^{-7}y^4 + 7.97566 \times 10^{-17}y^5 + 2.21413 \times 10^{-14}y^6 \\ & - 7.59587 \times 10^{-18}y^7), \end{aligned} \quad (4.1)$$

$$\begin{aligned} \Theta_0 = & (1 + y)/2 + 0.003444466(0.0180172 - 0.0000117521y - 0.0215166y^2 + 0.0000128713y^3 + 0.00437431y^4 \\ & - 1.11915 \times 10^{-6}y^5 - 0.000874862y^6 - 6.399 \times 10^{-13}y^7 - 1.33233 \times 10^{-10}y^8 + 6.22125 \times 10^{-14}y^9). \end{aligned} \quad (4.2)$$

$$q^1 : L(v_1(x)) = A_1 N_0,$$

$$q^2 : L(v_2(x)) = A_2 N_0 + A_1 N_1,$$

$$q^3 : L(v_3(x)) = A_3 N_0 + A_2 N_1 + A_1 N_2,$$

generally

$$q^l : L(v_l(x)) = \sum_{j=0}^{l-1} A_{l-j} N_j.$$

Equation (3.19) at $q = 1$ converges to the exact solution of equation (3.18), i.e.,

$$v(x, b_k) = v_0(x) + \sum_{j=1}^{\infty} v_j(x, b_k), \quad k = 1, 2, 3, \dots \quad (3.24)$$

After substituting equation (3.24) in equation (3.18), we obtain the residual,

$$\begin{aligned} R(x, b_k) = & L(v(x, b_k)) + g(x) + N(v(x, b_k)), \\ & k = 1, 2, \dots, n. \end{aligned} \quad (3.25)$$

If the residual is equal to zero, i.e., $R(x, b_k) = 0$, the exact solution $v(x, b_k)$ is achieved. However, if $R(x, b_k) \neq 0$, it can be minimized as follows:

$$J(b_k) = \int_a^e R^2(x, b_k) dx, \quad (3.26)$$

where a, e are constants, which are dependent on the consider problem. Also, constants $b_1, b_2, b_3, \dots$, can be obtained from the following equation:

$$\frac{\partial J}{\partial b_n} = 0, \quad n = 1, 2, \dots, m. \quad (3.27)$$

After getting the values of the aforementioned constants and using these values in equation (3.24) one can obtain the approximate solution.

4 Solution of the problem

The solutions of velocity profile (v_0) and temperature distributions (Θ_0) using OHAM-DJ up to first order are as follows:

The solutions of velocity profile (v_A) and temperature distributions (θ_A) up to first order using AHPM are as follows:

$$v_A = 0.041666(12.01 + 12y - 0.012y^2 + 0.002y^4) + 1.01587 \times 10^{-9}(-1 + y^2)(0.007(61 - 14y^2 + y^4) + 0.0003(-105(-5 + y^2) - 0.002(427 + y(74 + y(-98 + y(-38 + y(7 + 4y))))))), \quad (4.3)$$

$$\theta_A = 0.5(1 + y) + 0.00344466(-1 + y^2)(-4.50(0.000336 \times (11.0 - 4.0y^2 + y^4) + 2.56 \times 10^{-6}(1260.0 + 0.168y(-9.0 + y^2) + 4 \times 10^{-6}(159.0 + 159.0y^2 - 51.0y^4 + 5.0y^6))) + 7.68 \times 10^{-10} \times (3780.0(3.0 + y) + 0.504(1 + y)^2(-13.0 + y(-1.0 + 2.0y))) \quad (4.4)$$

4.1 Volume flux on plates

The dimensionless volume flux is given as follows:

$$Q = \int_{-1}^1 v \, dy. \quad (4.5)$$

Using (4.1) and (4.3) in (4.5) we get

$$Q_0 = 1 - 0.266666A + 1.33443 \times 10^{-7}AB^2\alpha^2 - \frac{8.24205 \times 10^{-8}B^2d\alpha^2\epsilon}{B_0^2} - \frac{6.67213 \times 10^{-7}AB^2d\alpha^2\epsilon}{B_0^2}, \quad (4.6)$$

$$Q_A = 1 - 0.266666A + 0.10794AB^2\alpha^2 - \frac{0.0666666B^2d\alpha^2\epsilon}{B_0^2} - \frac{0.0539682AB^2d\alpha^2\epsilon}{B_0^2}. \quad (4.7)$$

4.2 Average velocity

Here $\bar{v}$ represents the average velocity defined as follows:

$$\bar{v} = \frac{Q}{d}. \quad (4.8)$$

In dimensionless form, equation (4.8) matches with equations (4.6) and (4.7).

4.3 Shear stress

In dimensionless form, the shear stress is given as follows:

$$\tau_p = -\mu D[v, y], y = 1. \quad (4.9)$$

The negative sign in equation (4.9) is because the upper wall is facing the negative y -direction of the coordinate system [46].

$$\tau_{p0} = -\mu \left(\frac{1}{24}(12 + 8A) + \frac{0.0010385B^2d\alpha^2\epsilon + 0.000929703AB^2d\alpha^2\epsilon - 0.00415399AB_0^2 - 0.0016616AB^2\alpha^2B_0^2}{10080 B_0^2} \right). \quad (4.10)$$

$$\tau_{pA} = -\mu \left(\frac{1}{24}(12 + 12A) + \frac{0.000198412B^2\alpha^2((420 + 376A)d\epsilon - 672AB_0^2)}{B_0^2} \right). \quad (4.11)$$

5 Results and discussion

In this article, the variation of velocity profile and temperature distributions of couple stress fluid of Vogel's model has been investigated in the channel employing OHAM-DJ and AHPM, using different parameters, for example, A , α , λ , ϵ , B_0 , d and B . In Tables 1 and 2, solutions of the velocity profile, temperature distributions and their residuals are given using OHAM-DJ and AHPM, respectively. In Tables 3–6, the comparison of velocity and temperature distributions of both techniques is given using different parameters. In Figures 2 and 3,

velocity profile of the fluid is plotted using both techniques and found in good agreement for different parameters. There is an increase in velocity as it moves from the fixed wall in the direction of the moving wall. The temperature distributions are plotted in Figures 4 and 5, and from these graphs it has been noted that temperature of the fluid increases, as the values of parameter λ increase. λ is actually a dimensionless number known as the Brinkman number and is generally denoted by B_r . As can be seen in Figure 4, the increase in the value of Brinkman number increases the temperature of the fluid owing to viscous heating of the fluid. Figure 5 also

Table 1: Comparison of velocity and its residual for $B_0 = 0.5$, $A = -0.002$, $d = 0.1$, $\lambda = 1$, $B = 0.08$, $\alpha = 0.02$ and $\varepsilon = 0.003$ employing OHAM-DJ and AHPM

y	AHPM v_A	Residual v_A	OHAM-DJ v_0	Residual v_0
-1.0	-8.89046×10^{-18}	1.72772×10^{-15}	-8.89046×10^{-18}	-1.70359×10^{-9}
-0.9	0.0500663	1.62814×10^{-15}	0.0500663	-1.21724×10^{-9}
-0.8	0.100131	1.49601×10^{-15}	0.100131	-7.82161×10^{-10}
-0.7	0.150192	1.33917×10^{-15}	0.150192	-3.98354×10^{-10}
-0.6	0.200247	1.16551×10^{-15}	0.200247	-6.58013×10^{-11}
-0.5	0.250297	9.82793×10^{-16}	0.250297	2.15508×10^{-10}
-0.4	0.300339	7.98479×10^{-16}	0.300339	4.45587×10^{-10}
-0.3	0.350372	6.19572×10^{-16}	0.350372	6.24448×10^{-10}
-0.2	0.400397	4.52434×10^{-16}	0.400397	7.52102×10^{-10}
-0.1	0.450412	3.02622×10^{-16}	0.450412	8.28563×10^{-10}
0	0.500417	1.74713×10^{-16}	0.500417	8.53843×10^{-10}
0.1	0.550412	7.21359×10^{-17}	0.550412	8.27953×10^{-10}
0.2	0.600397	-2.99572×10^{-18}	0.600397	7.50906×10^{-10}
0.3	0.650372	-5.00545×10^{-17}	0.650372	6.22715×10^{-10}
0.4	0.700339	-7.00662×10^{-17}	0.700339	4.43392×10^{-10}
0.5	0.750297	-6.58776×10^{-17}	0.750297	2.12948×10^{-10}
0.6	0.800247	-4.23234×10^{-17}	0.800247	-6.8603×10^{-11}
0.7	0.850192	-6.39308×10^{-18}	0.850192	-4.0125×10^{-10}
0.8	0.900131	3.26019×10^{-17}	0.900131	-7.84979×10^{-10}
0.9	0.950066	6.28607×10^{-17}	0.950066	-1.21978×10^{-9}
1.0	1	6.99251×10^{-17}	1	-1.70564×10^{-9}

Table 2: Comparison of temperature and its residual for $B_0 = 0.5$, $A = -0.002$, $d = 0.1$, $\lambda = 1$, $B = 0.08$, $\alpha = 0.02$ and $\varepsilon = 0.003$ employing OHAM-DJ and AHPM

y	AHPM θ_A	Residual θ_A	OHAM-DJ θ_0	Residual θ_0
-1.0	0	2.54478×10^{-12}	-3.55959×10^{-21}	4.20966×10^{-5}
-0.9	0.0500178	7.63736×10^{-13}	0.0500103	3.98326×10^{-5}
-0.8	0.100035	-2.85964×10^{-13}	0.10002	3.39727×10^{-5}
-0.7	0.15005	-7.75371×10^{-13}	0.150029	2.57953×10^{-5}
-0.6	0.200064	-8.61272×10^{-13}	0.200037	1.64282×10^{-5}
-0.5	0.250077	-6.84026×10^{-13}	0.250044	6.8488×10^{-6}
-0.4	0.300087	-3.65691×10^{-13}	0.300051	-2.11589×10^{-6}
-0.3	0.350096	-8.42891×10^{-15}	0.350056	-9.7892×10^{-6}
-0.2	0.400102	3.06806×10^{-13}	0.400059	-1.56449×10^{-5}
-0.1	0.450106	5.21299×10^{-13}	0.450061	-1.93071×10^{-5}
0	0.500107	5.99325×10^{-13}	0.500062	-2.05504×10^{-5}
0.1	0.550106	5.28613×10^{-13}	0.550061	-1.92996×10^{-5}
0.2	0.600102	3.20533×10^{-13}	0.600059	-1.56302×10^{-5}
0.3	0.650096	9.9797×10^{-15}	0.650056	-9.76779×10^{-6}
0.4	0.700087	-3.45025×10^{-13}	0.700051	-2.08852×10^{-6}
0.5	0.750077	-6.64029×10^{-13}	0.750044	6.88113×10^{-6}
0.6	0.800064	-8.45143×10^{-13}	0.800037	1.64643×10^{-5}
0.7	0.85005	-7.66335×10^{-13}	0.850029	2.58336×10^{-5}
0.8	0.900035	-2.87013×10^{-13}	0.90002	3.40114×10^{-5}
0.9	0.950018	7.50078×10^{-13}	0.95001	3.98696×10^{-5}
1.0	1	2.51669×10^{-12}	1	4.21299×10^{-5}

Table 3: Comparison of velocity and temperature solutions for parameters $B_0 = 0.9$, $A = 3$, $d = 0.01$, $\lambda = 0.01$, $B = 0.3$, $\alpha = 0.4$ and $\varepsilon = 0.2$ using OHAM-DJ and AHPM

y	AHPM v_A	OHAM-DJ v_0	Difference	AHPM θ_A	OHAM-DJ θ_0	Difference
-1.0	0	4.55756×10^{-23}	4.55756×10^{-23}	0	2.22454×10^{-18}	2.22454×10^{-18}
-0.8182	-0.087001	-0.088032	0.001031	0.11515	0.104931	0.010219
-0.6162	-0.16427	-0.166344	0.002074	0.241835	0.220793	0.021042
-0.4141	-0.204207	-0.207116	0.002909	0.364628	0.334424	0.030204
-0.2121	-0.19411	-0.197562	0.003452	0.480421	0.443983	0.036438
-0.0101	-0.12632	-0.129973	0.003653	0.587027	0.548227	0.0388
0.0101	-0.11622	-0.119873	0.003653	0.597134	0.558331	0.038803
0.2121	0.01799	0.0145376	0.0034524	0.692666	0.656167	0.036499
0.4141	0.209893	0.206984	0.002909	0.77897	0.748664	0.030306
0.6162	0.451929	0.449856	0.002073	0.858298	0.837145	0.021153
0.8182	0.731199	0.730168	0.001031	0.93353	0.923235	0.010295
1.0	1	1	0	1	1	0

Table 4: Comparison of velocity and temperature solutions for $B_0 = 0.7$, $A = 1$, $B = 0.4$, $\alpha = 0.2$, $d = 0.04$, $\lambda = 0.02$ and $\varepsilon = 0.3$ using AHPM and OHAM-DJ

y	AHPM v_A	OHAM-DJ v_0	Difference	AHPM θ_A	OHAM-D θ_0	Difference
-1.0	0	-7.44114×10^{-24}	7.44114×10^{-24}	0	-8.77124×10^{-19}	8.77124×10^{-19}
-0.9192	0.0136198	0.0135528	6.7×10^{-5}	0.041757	0.0411852	0.0005718
-0.7172	0.0508636	0.0506365	0.0002271	0.146095	0.144116	0.001979
-0.5152	0.0978535	0.097489	0.0003645	0.25012	0.246867	0.003253
-0.3131	0.15969	0.159224	0.000466	0.35351	0.349271	0.004239
-0.1111	0.239716	0.239196	0.00052	0.455819	0.451029	0.00479
0.1111	0.350815	0.350296	0.000519	0.566932	0.562136	0.004796
0.3131	0.472789	0.472324	0.000465	0.666643	0.66239	0.004253
0.5152	0.613053	0.612689	0.000364	0.765364	0.762092	0.003272
0.7172	0.768063	0.767836	0.000227	0.863333	0.861339	0.001994
0.9192	0.93282	0.932753	6.7×10^{-5}	0.960972	0.960394	0.000578
1.0	1	1	0	1	1	0

Table 5: Comparison of velocity and temperature solutions for $B_0 = 0.6$, $A = 1$, $d = 0.02$, $\lambda = 0.07$, $B = 0.2$, $\varepsilon = 0.2$ and $\alpha = 0.9$ using AHPM and OHAM-DJ

y	AHPM v_A	OHAM-DJ v_0	Difference	AHPM θ_A	OHAM-DJ θ_0	Difference
-1.0	0	-2.84754×10^{-23}	2.84754×10^{-23}	0	9.49779×10^{-18}	9.49779×10^{-18}
-0.8586	0.0246198	0.0240213	0.0005985	0.104983	0.0905368	0.0144462
-0.6566	0.0647981	0.0634032	0.0013949	0.253391	0.218967	0.034424
-0.4545	0.116333	0.114281	0.002052	0.395902	0.344007	0.051895
-0.2525	0.183688	0.181187	0.002501	0.526996	0.46242	0.064576
-0.0505	0.269757	0.267054	0.002703	0.642305	0.5717	0.070605
0.0505	0.320256	0.317554	0.002702	0.69323	0.622445	0.070785
0.2525	0.436187	0.433687	0.0025	0.781474	0.716064	0.06541
0.4545	0.570831	0.568781	0.00205	0.85337	0.800225	0.053145
0.6566	0.721397	0.720003	0.001394	0.912986	0.8773	0.035686
0.8586	0.883219	0.882621	0.000598	0.965328	0.950146	0.015182
1.0	1	1	0	1	1	0

Table 6: Comparison of velocity and temperature solutions for $A = 1.6$, $B_0 = 0.5$, $\alpha = 0.2$, $B = 0.3$, $\lambda = 0.02$, $\varepsilon = 0.4$ and $d = 0.05$ using OHAM-DJ and AHPM

y	AHPM v_A	OHAM-DJ v_0	Difference	AHPM θ_A	OHAM-DJ θ_0	Difference
-1.0	7.40149×10^{-17}	7.40148×10^{-17}	0	0	1.32524×10^{-18}	1.32524×10^{-18}
-0.9596	-0.00130032	-0.00132923	2.891×10^{-5}	0.0232696	0.0219761	0.0012935
-0.7576	-0.00434255	-0.00451191	0.0001694	0.139478	0.131776	0.007702
-0.5556	0.00628332	0.00599079	0.0002925	0.254575	0.240933	0.013642
-0.3535	0.039247	0.0388609	0.0003861	0.366958	0.34854	0.018418
-0.1515	0.100504	0.100063	0.000441	0.474888	0.45355	0.021338
0.1515	0.252003	0.251563	0.00044	0.626415	0.605065	0.02135
0.3535	0.392745	0.392361	0.000384	0.720513	0.702072	0.018441
0.5556	0.561881	0.561591	0.00029	0.810241	0.796571	0.01367
0.7576	0.753256	0.753088	0.000168	0.897132	0.889407	0.007725
0.9596	0.958299	0.958271	2.8×10^{-5}	0.982882	0.981583	0.001299
1.0	1	1	0	1	1	0

Figure 2: The velocity for $B = 0.2$, $d = 0.1$, $\alpha = 2$, $\varepsilon = 0.002$ and $B_0 = 1$ using OHAM-DJ. **Figure 4:** The temperature distributions for $A = 2$, $d = 2$, $\alpha = 0.01$, $B_0 = 0.6$, $\varepsilon = 0.002$ and $B = 0.9$ using OHAM-DJ. **Figure 3:** The velocity for $B = 0.2$, $d = 0.1$, $\alpha = 2$, $\varepsilon = 0.002$ and $B_0 = 1$ using AHPM. $B = 0.2$, $d = 0.1$, $\alpha = 2$, $\varepsilon = 0.002$ and $B_0 = 1$. **Figure 5:** The temperature distributions for $A = 2$, $d = 2$, $\alpha = 0.01$, $B_0 = 0.6$, $\varepsilon = 0.002$ and $B = 0.9$ using AHPM.

Figure 6: The volume flux for $\varepsilon = 0.003$, $B_0 = 2$, $\alpha = 0.2$ and $d = 0.1$ using OHAM-DJ.

Figure 9: The shear stress on the plates for $B = 2$, $\alpha = 0.4$, $\varepsilon = 0.0002$, $B_0 = 1$ and $d = 0.2$ using AHPM.

Figure 7: The volume flux for $\varepsilon = 0.003$, $B_0 = 2$, $\alpha = 0.2$ and $d = 0.1$ using AHPM.

Figure 8: The shear stress on the plates for $B = 2$, $\alpha = 0.4$, $\varepsilon = 0.0002$, $B_0 = 1$ and $d = 0.2$ using OHAM-DJ.

illustrates the direct relationship between the temperature distribution and the dimensionless parameter λ . This implies that more and more heat is produced from the

viscous heating of the fluid over the heat transfer from the heated wall to the fluid as illustrated in these two figures. In Figures 6 and 7, the volume flux is plotted using both techniques for different parameters and an excellent agreement is found. In Figures 8 and 9, the share stress is plotted for both techniques and an excellent agreement is found. It has been noted that shear stress τ_p and parameter A are inversely related.

6 Conclusion

A fully developed time independent flow with heat transfer and couple stresses has been investigated taking the effect of variable viscosity into consideration. The strongly non-linear and coupled differential equations are investigated using AHPM and OHAM-DJ to obtain the approximate solutions to estimate both the velocity field and temperature distribution. By applying both the techniques, the analytical expressions for velocity, temperature profile, average velocity, volumetric flow rate and shear stresses on the plates have been obtained. The results obtained by both the techniques for different parameters are compared numerically as well as graphically. The results gained using these techniques are in the form of infinite series; thus, the results can be easily calculated. The convergence and effectiveness of both these techniques are crystal clear from the tables and figures of this work. The results obtained using both methods are in good agreement and as a result it will be more appealing for the researchers to apply the proposed methods to different problems arising in fluid dynamics.

Nomenclature

f	body force
τ	Cauchy stress tensor
ρ	constant density
η	couple stress parameter
μ	dimensional coefficient of viscosity
θ	dimensional temperature
v	dimensional velocity
μ^*	dimensionless coefficient of viscosity
θ^*	dimensionless temperature
v^*	dimensionless velocity
p	dynamic pressure
A_1	first Rivlin–Ericksen tensor
L	gradient of V
θ_0	lower plate temperature
$\frac{D}{Dt}$	material derivative
μ_0	reference viscosity
c_p	specific heat
κ	thermal conductivity
I	unit tensor
θ_1	upper plate temperature
V	velocity vector

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