Research Article

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Position-dependent finite symmetric mass harmonic like oscillator: Classical and quantum mechanical study

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Abstract: We formulated the oscillators with position-dependent finite symmetric decreasing and increasing mass. The classical phase portraits of the systems were studied by analytical approach (He's frequency formalism). We also study the quantum mechanical behaviour of the system and plot the quantum mechanical phase space for necessary comparison with the same obtained classically. The phase portrait in all the cases exhibited closed loop reflecting the stable system but the quantum phase portrait exhibited the inherent signature (cusp or kink) near origin associated with the mass. Although the systems possess periodic motion, the discrete eigenvalues do not possess any similarity with that of the simple harmonic oscillator having $m = 1$.

Keywords: position-dependent mass, harmonic oscillator, classical mechanics, quantum mechanics

1 Introduction

The study of non-linear vibration has become important in designing the flexible structures associated with aircraft, bridge, satellite, etc. [1]. This study can also be extended to acoustics, biology [2], and other branches of engineering such as electronics, robotics, and mechatronics [1]. It is therefore important to design the non-linear control vibration. For the purpose, one needs to consider the simple harmonic oscillator (SHO) with Hamiltonian [3,4]

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \quad (1)$$

as a model to design sustained non-linear vibration [2] as long as mass of the system is constant. Further, the closed contour of the phase space of the above system signifies the existence of the discrete energy in the system. At this point, we would like to state that the phase portrait of the operator [5]

$$H = \frac{p^2}{2} - \frac{x^2}{2} \quad (2)$$

is not formed closed orbit and hence it does not possess discrete energy states.

In recent years, systems with position-dependent mass (PDM) have attracted the attention of many researchers and scientists because of their importance in many branches of physics. These systems were first introduced in the theory of semiconductor physics [6–10], especially in the study of the electronic properties and band structure. Subsequently, the applicability of PDM systems can be found in many fields such as quantum mechanics [11,12], classical mechanics [13–15], nuclear physics [16], molecular physics [17], neutrino mass oscillations [18], and quantum information [19]. It is worth mentioning here that the PDM study can mostly be related to semiconductors as well as other solid state physics problem. Because of the wide range of applications of PDM, many efforts have been carried out in studying such systems [20–49].

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It is commonly seen that mass-dependent oscillators having $m(x)_{x \rightarrow \infty} \sim 0$. As an example of this, the $m(x)$ can be expressed as [24–28]

$$m(x) = \frac{m}{(1 + x^2)}. \quad (3)$$

In this context, we would like to say that at large distance mass becomes zero and the particle possesses infinite kinetic energy and zero potential energy at this point. Therefore, the particle behaves like a free particle and becomes unbound. An unbound particle is most probably unsuitable for any spectral observation. Similarly, considering another form of mass variation as

$$m(x) = m(1 + x^2) \quad (4)$$

for which $m(x)_{x \rightarrow \infty} \sim \infty$. It is meaning less to discuss the quantum mechanical behaviour of such a particle at this point. It is therefore desired to think of finite mass variation with distance. In this case, there are two choices: (i) decreasing or (ii) increasing. An example of controlled decreasing mass has been discussed in ref. [50]. The model example from the above literature is

$$m(x) = m \left(\frac{1 + \lambda + x^2}{1 + x^2} \right)^2. \quad (5a)$$

However, the authors [50] have considered a second model, where mass originates from zero and becomes constant at large distance. The proposed mass model [50] is

$$m(x) = m \tanh^2(\lambda x). \quad (5b)$$

A similar type of mass of the form [29,30]

$$m(x) = m \operatorname{sech}^2(ax) \quad (6a)$$

has recently been reported. In a very recent work, another form of decreasing mass

$$m(x) = e^{\left(\frac{x}{L}\right)^{a-1}} \quad (6b)$$

has also been reported [37].

The applicability of increased PDM has also been seen in some experiments. The deuteron–deuteron scattering has successfully explained by considering enhancement of electron mass [51]. The increased effective mass can be used on the basis of fluid to explain some phenomena in quantum field theory [52]. The enhancement of mass of quasiparticle in $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$ has been reported at quantum critical point which in turn affect the critical temperature of the superconducting state [53,54]. Effective mass of exciton in semiconductor coupled quantum well is predicted to be enhanced under the influence of

electromagnetic field [55]. Enhancement of mass in the quasiparticle led to the entanglement in Kondo problem [56]. In a similar manner, the enhancements of energy of electrons in quasicrystal [57], in hydrogen atom [58], and in quantum LC circuit [59] have also been reported. In addition, the enhancement of mass of electron in periodic lattice using nonlocal approach [60], implication of PDM for semiconductor as well as molecular physics [61], generation of massive photon in a magnetic material [62], etc. have also been predicted recently.

In semiconductors and other problems related to solid state physics, we mainly deal with atoms whose masses cannot be infinite or zero. If it varies with distance, then it must be within the finite values. It is seen that the PDM in all the theoretical cases does not vary within the finite limit. However, in a very recent work, we demonstrated about the finite variation of mass under asymmetric condition [63]. Considering the above literature, we propose a new model position-dependent finite mass variation, i.e.

$$m(x)_{x \rightarrow 0} \rightarrow c_1; \quad m(x)_{x \rightarrow \infty} \rightarrow c_2 \quad (7)$$

for symmetric cases comprising of both increasing and decreasing PDM. In view of the importance of vibration and PDM, we focused our attention to study the vibration of a newly designed finite symmetric increasing and decreasing mass harmonic like oscillators using classical and quantum mechanical approaches. The purpose of this work is to choose the PDM that varies between two values avoiding the ambiguity situation as discussed above, i.e., avoiding the values of $m(x)_{x \rightarrow \infty} \sim 0$ and $m(x)_{x \rightarrow \infty} \sim \infty$.

The rest of the present work is organized as follows: In Section 2, a classical description of the harmonic like oscillator comprising of both symmetric decreasing and increasing mass is discussed by performing analytical calculations. In Section 3, quantum mechanical study is presented, and finally the discussions along with the important findings of the work are presented in Section 4.

2 Classical description of the system

Consider a particle whose Hamiltonian is given by

$$H = \frac{p_x^2}{2m(x)} + \frac{m(x)}{2}x^2. \quad (8)$$

The Lagrangian is related to the Hamiltonian as

$$H = \sum_i p_i \dot{q}_i - L. \quad (9)$$

In our case, we have just one generalized coordinate (x) and one generalized momentum (p_x). Therefore, we have

$$H = p_x \dot{x} - L$$

$$\frac{p_x^2}{2m(x)} + \frac{m(x)}{2} x^2 = p_x \dot{x} - L. \quad (10)$$

Solving equation (10), one can find the Lagrangian (L) as

$$L = p_x \dot{x} - \frac{p_x^2}{2m(x)} - \frac{m(x)}{2} x^2. \quad (11)$$

Using $\frac{dx}{dt} = \frac{\partial H}{\partial p_x} \Rightarrow \dot{x} = \frac{p_x}{m(x)}$, and $p_x = \dot{x}m(x)$, equation (11) becomes

$$L = \frac{m(x)}{2} (\dot{x}^2 - x^2). \quad (12)$$

Substituting equation (12) into the relation $\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$ with $q_i = x$, we obtained:

$$-xm(x) + \frac{(\dot{x}^2 - x^2)}{2} \frac{dm(x)}{dx} - \ddot{x}m(x) - \dot{x} \frac{dm(x)}{dt} = 0. \quad (13)$$

2.1 Decreasing PDM

Let the mass be specified as

$$m(x) = \frac{m}{2} (1 + e^{-\gamma|x|}). \quad (14)$$

Figure 1 shows the variation in $m(x)$ as stated above with x for $m = 1$ and $\gamma = 0.1, 0.5$, and 1 . In this case, the $m(x)$ is decreasing from 1 and progressively saturating to 0.5 as we move away from origin. In other words, the $m(x)$ varies between two finite limits, i.e., 1 and 0.5 . Further, the nature of variation in $m(x)$ is symmetric with respect to origin of the co-ordinate system. It is worth mentioning here that the value of constant parameter (γ) plays the major role in the variation in $m(x)$. For example: one will see that the $m(x)$ decreases very fast for $\gamma = 1$, whereas the same decreases very slow for $\gamma = 0.1$. Interested reader will see that the variation in $m(x)$ will look like a delta function for $\gamma > 1$. However, the variation in $m(x)$ in all the cases is confined between two limits, i.e., between 1 and 0.5 .

Figure 1: Variation in $m(x)$ as stated in equation (14) with x for $m = 1$ and $\gamma = 0.1, 0.5$, and 1 .

Substituting equation (14) in equation (13), we have

$$\frac{m}{2} (1 + e^{-\gamma|x|}) \ddot{x} + \frac{(\dot{x}^2 - x^2)}{2} \frac{m}{2} (\gamma e^{-\gamma|x|}) \frac{d|x|}{dx} + \frac{m}{2} (x + x e^{-\gamma|x|}) - \frac{\dot{x}m}{2} (\gamma e^{-\gamma|x|}) \frac{d|x|}{dt} = 0. \quad (15)$$

The equation (15) can further be simplified to

$$\ddot{x} + x + (\dot{x}^2 - x^2) \left(\frac{\gamma e^{-\gamma|x|}}{2(1 + e^{-\gamma|x|})} \right) \frac{d|x|}{dx} + -\dot{x}^2 \left(\frac{\gamma e^{-\gamma|x|}}{2(1 + e^{-\gamma|x|})} \right) = 0, \quad (16a)$$

which leads to

$$\ddot{x} + x - \left(\frac{\gamma e^{-\gamma x}}{2(1 + e^{-\gamma x})} \right) (\dot{x}^2 + x^2) = 0; \quad x \geq 0 \quad (16b)$$

and

$$\ddot{x} + x + 3\dot{x}^2 \left(\frac{\gamma e^{-\gamma x}}{2(1 + e^{-\gamma x})} \right) - x^2 \left(\frac{\gamma e^{-\gamma x}}{2(1 + e^{-\gamma x})} \right) = 0; \quad (16c)$$

$$x < 0.$$

This is essentially a modified harmonic oscillator that reflects the amplitude dependence frequency. In other words, vibration of particle is controlled by amplitude or *vice versa*. Further, it can be regarded as a self-controlled vibration. It is worth mentioning here that one can also be able to derive equation (16b and 16c) following the work of El-Nabulsi and others [64–70]. When the value of $\gamma = 0$, the above equation (16b and 16c) will be reduced to

$$\ddot{x} + x = 0, \quad (17)$$

which is the equation of motion for a free harmonic oscillator.

2.1.1 Analytical calculation on decreasing PDM

Let us rewrite the equation of motion (equation 16(b)) as

$$\ddot{x} + x + F\left(x, \frac{dx}{dt}\right) = 0, \quad (18)$$

where

$$F\left(x, \frac{dx}{dt}\right) = -Q(x^2 + \dot{x}^2) \quad (19a)$$

with

$$Q = \frac{\gamma}{2(1 + e^{\gamma x})}. \quad (19b)$$

The equation (18) has the similarity with the Kryloff–Bogoliuboff autonomous system, which is expressed as [5]

$$\ddot{x} + x + \varepsilon f\left(x, \frac{dx}{dt}\right) = 0 \quad (20)$$

with the solution is of the form

$$x = a(t)\sin(\omega t + \phi). \quad (21)$$

It should be stated here that

$$F \neq \varepsilon f. \quad (22)$$

In our analytical approach, we assume a similar type of solution as

$$x = A \cos \omega t \quad (23)$$

in which the frequency of oscillation (ω) is a function of A . Here, we also consider the same boundary condition as considered in numerical study, i.e., $x(t=0) = 1$ and $\dot{x}(t=0) = 0$ to study classical dynamics using analytical approach. We use widely used ancient Chinese method based on He's frequency formalism [63,71–73] to derive the ω for the equation (18) using the following condition:

$$\omega^2 = \frac{\omega_1^2 R_2(0) - \omega_2^2 R_1(0)}{R_2(0) - R_1(0)}, \quad (24a)$$

where

$$R_1(0) = -\left(\frac{A^2}{2}\right)\left(\frac{\gamma e^{-\gamma A}}{1 + e^{-\gamma A}}\right) \quad (24b)$$

and

$$R_2(0) = (1 - \omega_2^2)A - \left(\frac{A^2}{2}\right)\left(\frac{\gamma e^{-\gamma A}}{1 + e^{-\gamma A}}\right). \quad (24c)$$

To obtain the expression of the value of ω for the system using the He's approach [63,71–73], one needs to replace $\omega_1 = 1$ and $\omega_2 = \omega$ in the final expression of equation 24(a). Based on our initial boundary condition (i.e., $x(t=0) = 1$ and $\dot{x}(t=0) = 0$), the two equations (16b and 16c) would effectively give rise to same expression of frequency as

$$\omega = \sqrt{1 - \frac{A}{2}\left(\frac{\gamma e^{-\gamma A}}{1 + e^{-\gamma A}}\right)}. \quad (25)$$

Figure 2 showed the variation in frequency (ω) with respect to amplitude (A). Initially, the ω decreased from 1 to a certain value (~ 0.92) and then the same showed increasing trend, which finally saturated to 1 with increasing A . Like the $m(x)$, the $\omega(A)$ also varies between two finite values. The analytical results on the variation in $x(t)$ with respect to t , $p(t)$ with respect to t , and trajectory of phase space (p_x vs x) for $\gamma = 0.1, 0.5$, and 1 with $A = 1$ are shown in Figure 3.

2.2 Increasing PDM

Here, we consider an increasing position-dependent mass, which is expressed as

Figure 2: Variation in frequency (ω) with amplitude (A) for $\gamma = 0.1, 0.5$, and 1.

Figure 3: Variation in (a) x with respect to time t , (b) p_x with respect to t , and (c) p_x with respect to x obtained from analytical calculation for $\gamma = 0.1, 0.5$, and 1 with $A = 1$.

$$m(x) = \frac{m}{1 + e^{-\lambda|x|}}. \quad (26)$$

Figure 4 shows the variation in $m(x)$ as stated in (equation (26)) with x for $m = 1$ and $\lambda = 0.1, 0.5$, and 1 . In this case, the $m(x)$ is increasing from 0.5 and progressively saturating to 1 as we move away from origin. In this case, the mass also exhibits symmetric nature like the decreasing mass case. Here, we find an interesting feature on larger value of λ , i.e., the variation in $m(x)$ looks like a volcanic type feature around small values of x .

2.2.1 Analytical calculation on increasing PDM

Substituting equation (26) in equation (13) and simplifying, we have

Figure 4: Variation in $m(x)$ as stated in (equation (28)) with x for $m = 1$ and $\lambda = 0.1, 0.5$, and 1 .

$$\ddot{x} + x - \left(\frac{\lambda e^{-\lambda x}}{2(1 + e^{-\lambda x})} \right) (\dot{x}^2 + x^2) = 0; \quad x \geq 0, \quad (27a)$$

$$\ddot{x} + x + 3\dot{x}^2 \left(\frac{\lambda e^{-\lambda x}}{2(1 + e^{-\lambda x})} \right) - x^2 \left(\frac{\lambda e^{-\lambda x}}{2(1 + e^{-\lambda x})} \right) = 0; \quad (27b)$$

$$x < 0.$$

In this case, we also consider the formalism discussed above (in Section 2.1.1) to derive the frequency of oscillation for the solution to equations (27a and 27b) using the He's formalism [63,71–73] as

$$\omega = \sqrt{1 - \frac{A}{2} \left(\frac{\lambda e^{-\lambda A}}{1 + e^{-\lambda A}} \right)}. \quad (28)$$

The analytical results on the variation in $x(t)$ with respect to t , $p(t)$ with respect to t , and trajectory of phase space (p_x vs x) for and $\lambda = 0.1, 0.5$, and 1 with $A = 1$ are

shown in Figure 5. The phase portrait exhibits egg structure like closed loop. The shrinking of loop of phase portrait in one side while expanding the other side is evident with increasing λ values (Figure 5(c)) for increasing mass case.

3 Quantum mechanical study

In this case, we solve the eigenvalue relation using matrix diagonalization method [22,63,74–77] as

$$H|\psi\rangle = E|\psi\rangle, \quad (29)$$

where

$$|\psi\rangle = \sum_n A_n |n\rangle. \quad (30)$$

Figure 5: Variation in (a) x with respect to time t , (b) p_x with respect to t , and (c) p_x with respect to x obtained from analytical calculation for $\lambda = 0.1, 0.5$, and 1 with $A = 1$.

Table 1: First five eigenvalues of Hamiltonian (equation (8)) with PDM (equation (14)) with different values of γ and $m = 1$

Energy level (n)	γ	Eigenvalue (E_n)
0	0.1	0.5
1		1.4725
2		2.4458
3		3.4060
4		4.3667
0	0.5	0.5
1		1.3846
2		2.2946
3		3.1659
4		4.0635
0	1	0.5
1		1.3281
2		2.2570
3		3.1322
4		4.0761

Table 2: First five eigenvalues of Hamiltonian (equation (8)) with PDM (equation (26)) with different values of λ and $m = 1$

Energy level (n)	λ	Eigenvalue (E_n)
0	0.1	0.5
1		1.536
2		2.5699
3		3.6203
4		4.6675
0	0.5	0.5
1		1.6353
2		2.7172
3		3.8568
4		4.9540
0	1	0.5
1		1.6922
2		2.7526
3		3.9041
4		4.9618

Figure 6: Phase trajectories of the Hamiltonian (equation (8)) with PDM (equation (14), i.e., decreasing mass case) considering $E = H$ with $m = 1$ for (a) $\gamma = 0.1$, (b) $\gamma = 0.5$, and (c) $\gamma = 1$.

Figure 7: Phase trajectories of the Hamiltonian (equation (8)) with PDM (equation (26), i.e., increasing mass case) considering $E = H$ with $m = 1$ for (a) $\lambda = 0.1$, (b) $\lambda = 0.5$, and (c) $\lambda = 1$.

Here, $|n\rangle$ satisfies the exact eigenvalue relation as

$$H_0 |n\rangle = (p^2 + x^2)n = (2n + 1)|n\rangle. \quad (31)$$

Using the above-mentioned procedure, one can get the recursion relation satisfied by A_n as

$$\sum_{k=2,4,6,\dots} P_n^k A_{n-k} + Q_n A_n + R_n^k A_{n+k} = 0, \quad (32a)$$

where

$$P_n^k = \langle n | H | n - k \rangle, \quad (32b)$$

$$R_n^k = \langle n | H | n + k \rangle, \quad (32c)$$

$$Q_n = \langle n | H | n \rangle - E. \quad (32d)$$

The energy eigenvalues of the Hamiltonian (equation (8)) with PDM (equation (14)) are obtained following the above-mentioned procedure for different values of γ and

$m = 1$ (Table 1). We also calculated the energy eigenvalues of the Hamiltonian (equation (8)) with PDM (equation (26)) with different values of λ and $m = 1$ (Table 2). It is to be noted here that the Hamiltonian is used as

$$H = p \frac{1}{2m(x)} p + \frac{1}{2} m(x) \omega^2 x^2 \quad (33)$$

for our quantum mechanical calculation following the work of Wilkes and Muljarov [55].

To have more information about the system, we plot quantum mechanical phase trajectories of the system (i) with PDM (equation (14)) with different values of γ and $m = 1$ (Figure 6) and (ii) with PDM (equation (26)) with different values of λ and $m = 1$ (Figure 7) considering $E_n = E = H$ of respective states. The formation of closed orbit (Figures 6 and 7) clearly indicated the stable behaviour of the system. Comparing the quantum phase

portrait of both the case, one can see that the appearance of cusp in decreasing mass case and kink in increasing mass case near origin which is significant for higher values of γ and λ , respectively. This feature could be because of the inherent property associated with corresponding mass.

4 Discussion and conclusion

The Harmonic like oscillator under the influence of decreasing and increasing PDM has been studied. In the case of increasing mass, $m(x)$ increased from 0.5 to 1 and in case of decreasing mass, the same decreased from 1 to 0.5. The purpose of selecting finite mass is that a mechanical or an oscillating particle can possess neither zero mass nor infinite mass at any point of motion. Particle with zero mass will act like a photon and is non-interacting in nature. Similarly the particle with infinite mass can hardly be physically viable. The equations of the investigated systems are compared well with the Kryloff–Bogoliuboff autonomous system [5]. The classical solution of the system for each case has been obtained analytically using He's method where the frequency of oscillation is basically depends on amplitude. It is worth mentioning here that the frequency of oscillation (equations (25) and (28)) for both the cases are less than equal to 1 (i.e., $\omega \leq 1$) for any value of A , justifying the low frequency oscillation. We believe that the analytical approach as used in the present study is of appropriate. The existence of closed curve in classical phase portrait signifies that the system will admit discrete eigenvalues microscopically (quantum mechanically). It should be born in the mind that if the classical phase portrait is not closed, the quantum state of system will not possess discrete eigenvalue [5]. Further, in quantum mechanical study, the nature of closed phase portraits in two different cases clearly reflect the signature of mass involved in the system. It is worth mentioning here that the quantum phase portrait is the fingerprint of the PDM system reflecting the association of the nature of mass variation in the system even though the mass varies between 1 and 0.5 in both the cases. The discrete eigenvalues of the present system do not possess any similarity with that of the SHO having $m = 1$. The close view reveals that the difference between any two adjacent energy levels is no longer a constant factor, i.e., $E_{i+1} - E_i \neq E_i - E_{i-1}$. This could be useful to establish a common link between classical and quantum mechanics [78].

In conclusion, the present model analysis can be extended to PDM that varies between any finite values of $m(x) \sim m_1$ to m_2 and hence to study the dynamics of the system classically as well as quantum mechanically. We believe that the present analysis is new to the literature and the suggested method can also be used for any non-singular PDM system.

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