Research Article

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Closed-form solutions for the Schrödinger wave equation with non-solvable potentials: A perturbation approach

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Abstract: To obtain closed-form solutions for the radial Schrödinger wave equation with non-solvable potential models, we use a simple, easy, and fast perturbation technique within the framework of the asymptotic iteration method (PAIM). We will show how the PAIM can be applied directly to find the analytical coefficients in the perturbation series, without using the base eigenfunctions of the unperturbed problem. As an example, the vector Coulomb ($-1/r$) and the harmonic oscillator ($-r^2$) plus linear ($-r$) scalar potential parts implemented with their counterpart spin-dependent terms are chosen to investigate the meson sectors including charm and beauty quarks. This approach is applicable in the same form to both the ground state and the excited bound states and can be easily applied to other strongly non-solvable potential problems. The procedure of this method and its results will provide a valuable hint for investigating tetraquark configuration.

Keywords: alternative perturbation method, quark–anti-quark structures, spin-dependent terms, Hadrons mass spectrum

1 Introduction

One of the challenging problems in non-relativistic quantum mechanics is to find closed-form solutions to the radial Schrödinger wave equation for non-solvable potentials that are used in different fields of physics. In particular, the perturbed Coulomb potentials are representing the most important potentials in many scientific models, especially in particle physics [1].

As an example, the experimental observations of the exotic states in the charm sector with $c\bar{c}c\bar{c}$ content at Belle [2] and BESIII [3], as well as the structures of the form $Q\bar{q}Q$ in ref. [4,5], have opened new interest to re-study the hadronic structures and their spectroscopy with different phenomenological potential models.

However, the lack of closed-form solutions to the radial Schrödinger wave equation with many potential models makes the study of such potentials one of the most popular theoretical laboratories for examining the validity of the various approximation techniques based on perturbative and non-perturbative approaches [6–11].

Many of the existing theoretical work in the literature can obtain the hadron masses by solving numerically the two-body radial Schrödinger wave equation with Cornell-inspired potentials, and then the respective spin-dependent interaction terms are added and treated by the standard perturbation technique [12]. Nevertheless, the computational schedule in those works is very complicated and can be hardly extended to study other potential models.

On this basis, the present work is aimed to solve the radial Schrödinger wave equation for heavy quark–anti-quark structures with the vector Coulomb potential plus linear and harmonic scalar potentials parts, implemented in the same form with their counterpart spin-dependent terms.

We will show how the PAIM can be used directly to find analytically the coefficients in the energy perturbation series as a solution to the radial Schrödinger wave equation, without the need of using the base eigenfunctions of the unperturbed problem. Furthermore, PIAM is unlike the other perturbation methods, with no constraints on the coupling constants or the quantum numbers involved in the phenomenological potential, and even it avoids sticking to numerical computations at...
2 Theoretical framework: Implementation of the spin
dependent terms within the phenomenological potential
parts

It is generally believed that the phenomenological poten-
tial models used in the hadronic spectroscopy with the
Coulomb potential perturbed by a term or terms involving
various powers of \( r \) seem to provide the clearest and most
accurate picture for meson states. The Coulomb potential
arises from the one-gluon exchange associated with a
Lorentz vector structure, and the various powers of \( r \)
parts in the potential are responsible for the confinement.

These parts are known as the static spin-independent parts. For our further discussion, we used a fairly flexible mixing type of such potentials. We took
the potential interaction parts as a sum of vector and scalar parts, plus the spin-dependent terms:

\[ V_{	ext{Qg}}(r) = V_{q}(r) + V_{s}(r) + V_{SD}(r), \]

with

\[ V_{q}(r) = -\frac{\Lambda}{r} + \beta(\sigma r + K r^2), \]

\[ V_{s}(r) = (1 - \beta)(\sigma r + K r^2), \]

and

\[ V_{SD}(r) = V_{SS}(r)\langle S_1 \cdot S_2 \rangle + V_{LS}(r)\langle L \cdot S \rangle + V_{T}(r)\langle S_{12} \rangle, \]

where, \( V_{q}(r) \), \( V_{s}(r) \), and \( V_{SD}(r) \) are the vector, scalar, and
spin-dependent parts, respectively. \( \Lambda, K, \sigma \) and \( \beta \) represent
the independent coupling constants of the potential.

In principle, the spin-dependent interaction terms are
usually added to account for the mass splitting between
states with different quantum numbers. Here, \( \beta \) is the mixing coefficient. Additionally, this potential has dis-
tinct features of strong interaction quantum chromody-
namics, the asymptotic freedom, and the confinement
parts.

To implement the spin-dependent terms within the
pre-assumed phenomenological potential parts, we based
on the Breit-Fermi one-gluon exchange Hamiltonian [13],
where the spin-dependent part for quark masses \( m_q \) and
\( m_{\bar{q}} \) have three distinctive types of radial-interaction parts
known as the spin–spin \( V_{SS}(r) \), the spin–orbit \( V_{LS}(r) \), and the
tensor part \( V_{T}(r) \), respectively:

\[ V_{SS}(r) = \frac{2}{3m_q m_{\bar{q}}} r V^2_{T}(r), \quad (5) \]

\[ V_{LS}(r) = \frac{1}{2m_q m_{\bar{q}}} \left[ \frac{1}{3} \frac{dV_{q}(r)}{dr} - \frac{dV_{s}(r)}{dr} \right], \quad (6) \]

\[ V_{T}(r) = \frac{1}{12m_q m_{\bar{q}}} \left[ \frac{1}{r} \frac{dV_{q}(r)}{dr} - \frac{dV_{s}(r)}{dr} \right]. \quad (7) \]

Performing the derivatives, simplifying the resulting
expressions, and collecting terms of the same powers
of \( r \), then we immediately obtain a phenomenological potential model implemented with the spin-dependent
terms:

\[ V_{Qg}(r) = -\frac{z}{r} + \sigma r + K r^2 + \frac{\lambda}{r^3} + g \delta \gamma(r) + \omega_0, \]

where we have used \( V^2\left(\frac{1}{r} \right) = -4\pi \delta \gamma(r) \), and

\[ z = \Lambda - \frac{2}{12m_q m_{\bar{q}}} \left[ 6(4\beta - 1)\langle L \cdot S \rangle + 16\beta\langle S_1 \cdot S_2 \rangle \right. \]

\[ + \left. \beta \langle S_{12} \rangle, \right. \]

\[ \lambda = \frac{3\Lambda}{12m_q m_{\bar{q}}} \left[ 6\langle L \cdot S \rangle + \langle S_{12} \rangle \right], \]

and

\[ \omega_0 = \frac{K}{m_q m_{\bar{q}}} - [(4\beta - 1)\langle L \cdot S \rangle + 4\beta\langle S_1 \cdot S_2 \rangle] ; \]

\[ g = \frac{8\pi\Lambda}{3m_q m_{\bar{q}}}. \]
Here it should be noted that the obtained phenomenological potential in equation (8) is elaborated in such a way to be comfortable with the application of the PAIM. Furthermore, when $\ell = 0$ this leads to $\tilde{z} = \Lambda - \frac{4q_{\text{f}}}{m_{\text{f}}m_{\text{o}}} \langle S_1 \cdot S_2 \rangle$, $\lambda = 0$, and $\tilde{\omega}_0 = \frac{\hbar k}{m_{\text{f}}m_{\text{o}}} (S_1 \cdot S_2)$. In this case, the phenomenological potential model in equation (8) is reduced to a potential model, which is responsible for generating the S-wave states:

$$V_{QQ}(r) = -\frac{\tilde{z}}{r} + \sigma r + Kr^2 + g\delta^3(r) + \tilde{\omega}_0,$$  \hspace{1cm} (12)

while, when $\ell \neq 0$ then $\delta^3(r) = 0$, since only S-wave states have a non-zero value of the wave function at the origin. For this case, the phenomenological potential model in equation (8) is reduced to a potential that is responsible for generating the P-wave, D-wave, etc. states:

$$V_{QQ}(r) = -\frac{\tilde{z}}{r} + \sigma r + Kr^2 + \frac{\lambda}{r^3} + \omega_0.$$  \hspace{1cm} (13)

In principle, from now on, and for our future discussion, we will only solve the radial Schrödinger wave equation with the potential model of equation (13) for $\ell \neq 0$, which represents the most general case.

The expectation values of the spin–spin interaction operator in equation (4) are calculated in terms of the spin quantum numbers: $\langle S_1 \cdot S_2 \rangle = \left\langle \frac{1}{2}(S_1^2 - S_1^2 - S_2^2) \right\rangle$, where $S_1$ and $S_2$ are the spins of the two quarks $m_1$ and $m_2$, respectively, and $S$ stands for the total spin of the structure under consideration. Since each quark (anti-quark) has spin $\frac{1}{2}$, thus they can couple into singlets $S = 0$ with $\langle S_1 \cdot S_2 \rangle = -\frac{1}{4}$ or triplet $S = 1$ with $\langle S_1 \cdot S_2 \rangle = \frac{1}{4}$.

On the other hand, the expectation values of the spin–orbit interaction operator are calculated in terms of the total angular momentum quantum number $J$, defined by the vector sum $J = L + S$, $\langle L \cdot S \rangle = \left\langle \frac{1}{2}(J^2 - L^2 - S^2) \right\rangle$. For S-wave states ($\ell = 0$) and/or for the singlet state case $S = 0$, the spin–orbit term $\langle L \cdot S \rangle$ is always zero. However, for the triplet state cases $S = 1$ with $\ell \neq 0$, we have $J = \ell + 1, \ell, \ell - 1$, which, in general, can yield a non-zero value. Similarly, the expectation values of the tensor operator are calculated in terms of the total spin $S$, and the orbital angular momentum quantum number $\ell$, and it is given by ref. [13]:

$$\langle S_{12} \rangle = \left\langle \frac{4}{(2\ell + 1)(2\ell - 3)} \left[ S^2 \Omega^2 - \frac{3}{2} L \cdot S - 3(L \cdot S)^2 \right] \right\rangle.$$  \hspace{1cm} (14)

The results of this operator are equal to zero only for $S = 0$ or $\ell = 0$, and its expectation values are non-zero for the triplet states $S = 1$ with $\ell \neq 0$, and $J = \ell + 1, \ell, \ell - 1$:

$$\langle S_{12} \rangle = \begin{cases} -\frac{2\ell}{(2\ell + 3)}, & \text{if } J = \ell + 1, \\ -2, & \text{if } J = \ell, \\ -\frac{2(\ell + 1)}{(2\ell - 1)}, & \text{if } J = \ell - 1. \end{cases}$$  \hspace{1cm} (15)

### 3 Formalism of the PAIM for the potential $V_{QQ}(r)$ for $\ell \neq 0$

The radial Schrödinger wave equation that describes the internal structure between any quark–antiquark with the phenomenological potential model in equation (13) for $\ell \neq 0$ (in atomic units, $\hbar = c = 1$) is as follows:

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{2r^2} + \mu \left( -\frac{z}{r} + \sigma r + Kr^2 + \frac{\lambda}{r^3} \right) \right] R_{nt}(r) = \mu(E_{nt} - \omega_0) R_{nt}(r),$$  \hspace{1cm} (16)

where $\mu = \frac{m_1m_2}{m_1 + m_2}$ is the reduced mass of the quark–antiquark system, $n_t$ is the radial quantum number, and for states of definite orbital angular momentum $l$ the radial wavefunction is $\psi_{ll}(r) = r^{-l} R_{nt}(r)$.

The solution of equation (16) within the framework of PAIM is started with the change of variables by letting $r = t^2$, and further we removed the first derivative by employing the ansatz $\Phi_{nt}(t) = R_{nt}(t) \exp(-p(t)/2)$, provided that $p(t) = \frac{1}{2}$, which in turn implies a Schrödinger-like radial equation:

$$\frac{d^2\Phi_{nt}(t)}{dt^2} - \left[ \frac{\ell(\ell + 1)}{t^2} - 8\mu(E_{nt} - \omega_0) \right] t^2 \Phi_{nt}(t) = 0,$$  \hspace{1cm} (17)

with $\ell(\ell + 1) = 4\ell(\ell + 1) + \frac{3}{4}$. For further simplifications to equation (17), we employed the transformation $\Phi_{nt}(t) = e^{\ell(\ell + 1)/4} f_{nt}(t)$, simplifying out the mathematics, collecting the terms of equal powers in $t$, and choosing an arbitrary parameter $a = \sqrt{8\mu K}$ in such a way to remove the terms of higher $t$ powers. This procedure leads to another second-order homogeneous linear differential equation of the form [11,14]:

$$f''_{nt}(t) + \lambda_{nt}(t) f'_{nt}(t) + S_{nt}(t) f_{nt}(t),$$  \hspace{1cm} (18)

where
\[ \lambda_0(t, y) = 2 \left( \alpha t^3 - \frac{(t + 1)}{t} \right) , \]
\[ s_0(t, y) = \left( -\epsilon_0 t^2 + 8\mu \left( \alpha t^4 + \frac{\lambda}{t^5} - z \right) \right) , \quad (19) \]

with \( \epsilon_0 = 8\mu (E_{0,t} - \omega_0) - (2 \xi + 5) \alpha \), and \( y \) is an artificially introduced perturbation expansion parameter to be equal to one at the end of the calculations.

It should be noted here that the choice of the ansatz in equation (19) is motivated, that is, when we switch \( y \) off, equation (19) is reduced to an exactly solvable eigenvalue problem within the framework of AIM [14,15]. On the other hand, if we switch \( y \) on, a straightforward application of AIM to equation (19) does not give any reasonable results, where it is observed that the sequence appears not to converge, a result that violates the principle behind the AIM.

Accordingly, to have a closed-form solution for equation (19) within the framework of PAIM, we proposed \( s_0(t, y) \) be expressed as a sum of two parts:
\[ s_0(t, y) = s_0^\ell(t) + y s_0^y(t) , \quad (20) \]

with \( s_0^\ell(t) = -\epsilon_0 t^2 \) which has an exact solution within AIM, and \( s_0^y(t) = 8\mu \left( \alpha t^4 + \frac{\lambda}{t^5} - z \right) \) to be solved using PAIM.

Following the systematic procedure of PAIM, the energy eigenvalues \( E_{n,t} \) are solved by imposing the asymptotic aspect of the method for sufficiently large iteration number \( i \):
\[ \delta_i(t, y) = s_i(t, y) \lambda_{i+1}(t, y) - s_{i+1}(t, y) \lambda_i(t, y) = 0 , \quad (21) \]

with
\[ \lambda_i(t, y) = \lambda_{i+1}(t, y) + s_{i+1}(t, y) + \lambda_0(t, y) \lambda_{i+1}(t, y) \quad (22) \]

and
\[ s_i(t, y) = s_{i+1}(t, y) + s_0(t, y) \lambda_{i+1}(t, y) \quad (23) \]

As a step further, to obtain the energy eigenvalues for \( s_0^y(t) \), we expand \( \delta_i(t, y) \) around \( y = 0 \):
\[ \delta_i(t, y) = \delta_i(t, 0) + \frac{y}{1!} \frac{\partial \delta_i(t, y)}{\partial y} \bigg|_{y=0} + \frac{y^2}{2!} \frac{\partial^2 \delta_i(t, y)}{\partial y^2} \bigg|_{y=0} + \ldots \quad (24) \]

and according to the procedure of PAIM, \( \delta_i(t, y) \) must be zero; if this to be true for every \( y \) value, then every term in the series must be zero. That is to say:
\[ \delta_i^{(j)}(t, y) = \frac{y^j}{j!} \frac{\partial^j \delta_i(t, y)}{\partial y^j} \bigg|_{y=0} = 0 , \quad j = 0, 1, 2, \ldots \quad (25) \]

At this stage, it is also convenient to expand \( \epsilon_{n,t} \) in terms of \( y \):
\[ \epsilon_{n,t} = \epsilon_{n,t}^{(0)} + \epsilon_{n,t}^{(1)} + \epsilon_{n,t}^{(2)} + \epsilon_{n,t}^{(3)} + \ldots \quad (26) \]

A quantitative estimation for \( \epsilon_{n,t} \) expansion terms are obtained by comparing the terms of equation (24) with the same order of \( y \) as the terms of equation (26). Therefore, it is obvious from the procedure of AIM that the roots of \( \delta_1^{(0)}(t, 0) = 0 \) give the leading contribution energy terms \( \epsilon_{n,t}^{(0)} \). Likewise, the roots of \( \delta_1^{(1)}(t, y) = 0 \) give the first correction terms \( \epsilon_{n,t}^{(1)} \) and so on.

To obtain the analytical leading energy term \( \epsilon_{n,t}^{(0)} \), one should simply switch off \( y \) in equation (19), and for further simplifications, we recall the mathematical definitions of equations (21)–(23). Then the roots of \( \delta_1^{(0)}(t, 0) = 0 \) leads to an exactly solvable eigenvalue in terms of \( \alpha \):
\[ \epsilon_{n,t}^{(0)} = 8n_0 \alpha , \quad n_0 = 0, 1, 2, \ldots \quad (27) \]

It should be addressed here that for each iteration the expression \( \delta_1^{(0)}(t, 0) = 0 \) depends on \( t \), and the calculated eigenvalues \( \epsilon_{n,t}^{(0)} \) using this condition should, in principle, be independent of the choice of \( t \). Nevertheless, the choice of \( t \) is critical only to the stability of the AIM process. In this formalism, it is observed that the starting value for \( t \) is the value at which \( \lambda_0(t, 0) = 0 \), so that \( t = t_0 = \left( \frac{t+1}{2} \right)^{1/4} \), and the sequence appears to converge when the number of iterations \( i \geq 25 \), as shown in Table 1.

To find the other analytical perturbed expansion terms, one should simply go to equation (19) switch \( y \) on, therein, replace \( \epsilon_{n,t} \) with \( \epsilon_{n,t}^{(0)} + \epsilon_{n,t}^{(1)} \), and terminate the iterations by imposing the condition \( \delta_1^{(1)}(t, y) = 0 \). The first root of the resulting equation gives \( \epsilon_{n,t}^{(1)} \). Proceeding exactly in the same way, one can obtain the second-order closed-form perturbed expansion terms and so on. The analytical expressions of this procedure are displayed in Table 1. Of course, in Table 1, \( z \) and \( \lambda \) are changeable according to the total spin and total angular momentum of the structure state.

Consequently, the final closed-form solution for the energy eigenvalues \( E_{n,t} \) in conjunction with equation (19) within PAIM framework is as follows:
\[ E_{n,t} = \frac{1}{8\mu} \left( 8n_0 \alpha + \alpha(2\xi + 5) + \xi \epsilon_{n,t}^{(1)} + \xi^2 \epsilon_{n,t}^{(2)} + \xi^3 \epsilon_{n,t}^{(3)} + \ldots \right) + \omega_0 , \quad (28) \]

where \( \xi = 8\mu y \).

A further interesting quantity in this respect is the transformed eigenfunctions \( f_{n,t}(t) \). Knowledge of \( f_{n,t}(t) \) will help to construct the radial eigenfunctions \( R_{n,t}(t) \). Inspired by equation (18), given that \( g_{n}(t, y) = \frac{s_{n}(t, y)}{\lambda(t, y)} \), and following again exactly the same systematic procedure of PAIM explained above, by expanding this time \( g(t, y) \):

\[ e_{n,t} = e_{n,t}^{(0)} + e_{n,t}^{(1)} + e_{n,t}^{(2)} + e_{n,t}^{(3)} + \ldots \quad (26) \]
Having solved the radial Schrödinger wave equation with the proposed phenomenological potential model, the corresponding mass of a particular quark–antiquark $QQ$ structure within this model can be written as:

$$M(QQ) = m_Q + m_{ar{Q}} + E_{n,t},$$

where the energy eigenvalues $E_{n,t}$ are given in equation (28). Typically, to obtain numerically the meson masses, this approach needs various input parameters that one should fix at this stage. For the quark masses, we used $m_b = (4.992 \pm 0.001)$ GeV, and $m_c = (1.55 \pm 0.001)$ GeV. Furthermore, the PAM approach contains four auxiliary parameters with their working regions to be determined, namely, the Coulomb gluon strength parameter $\Lambda$, the linear and harmonic confining parts $\sigma$ and $K$, and the mixing parameters $\beta$. Our analysis shows that $\Lambda, \sigma, K,$ and $\beta = 0.25$ are varied in the following regions, respectively, $\Lambda = 0.73 \pm 0.01$, $\sigma = 0.018 \pm 0.001$ GeV$^2$, $K = 0.026 \pm 0.001$ GeV$^3$, and $\beta = 0.25 \pm 0.01$.

Few P-wave mass spectrums of the charmonium $cc$ are calculated and listed in Table 2. The calculated masses are found to be in good agreement with the recently observed ones. Moreover, with the same input parameters, in a unified way, we extend this study and we predicted the P-wave bottonium $bb$ mass spectrums. In Table 3, we present our final findings on the P-wave mass spectrums of bottonium $bb$. In the tables, we reported also on the predictions of the other methods together with the experimental results. A direct comparison of our results

<table>
<thead>
<tr>
<th>$n_t$</th>
<th>$\ell$</th>
<th>$E^{(1)}_{n,t}$</th>
<th>$E^{(2)}_{n,t}$</th>
<th>$E^{(3)}_{n,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$-0.797z$ $\sqrt{2}$</td>
<td>$-0.0195z^2 - 0.0546z^2$</td>
<td>$-0.00070z^3 + 0.00389z^3$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-0.645z$ $\sqrt{2}$</td>
<td>$-0.0077z^2 - 0.0492z^2$</td>
<td>$-0.00000z^3 + 0.00018z^3$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$-0.592z$ $\sqrt{2}$</td>
<td>$-0.0043z^2 - 0.0482z^2$</td>
<td>$-0.00005z^3 + 0.00013z^3$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$-0.543z$ $\sqrt{2}$</td>
<td>$-0.0028z^2 - 0.0470z^2$</td>
<td>$-0.00005z^3 - 0.00958z^3$</td>
</tr>
</tbody>
</table>

Table 1: Calculated analytical expressions of the energy expansion coefficients by means of this work with $i \geq 25$ iterations for different values of $n_t$ and $\ell$.

This procedure allows us to find the coefficients of the transformed eigenfunctions:

$$q(t, y) = q(t, 0) + \int \frac{\partial q(t, y)}{\partial y} \bigg|_{y=0} + \frac{y^2}{2!} \frac{\partial^2 q(t, y)}{\partial y^2} \bigg|_{y=0} + \ldots \quad (29)$$

and finally this will lead to a general solution to the transformed second-order differential equation (18):

$$f_{n,t}(t) = N \prod_{j=0}^{\infty} f_{n,t}^{(j)}(t), \quad (31)$$

where $N$ is an integration constant.
with the experimental results reveals that they are in good agreement.

Evidently, our numerical analysis shows that the contribution of the non-perturbative terms is about 90%, while the maximum contribution of the perturbative terms is about 10% of the total sum.

Finally, although in this work we have tested with PIAM the P-wave mass spectrum of the potential model of equation (1), one can easily find the analytical and numerical solutions for other potential models using the same procedure. It is also important to emphasize that the PAIM described in this article and its analytical expressions are much more useful than pure numerical calculations.

## 5 Conclusion

The present work is focused on the theoretical implementation of the PAIM and to show how it is simple and easy to solve the radial Schrödinger wave equation with non-solvable potential models. Although we aimed not to make the fittings with the available experimental we only emphasized on the method and how it works, and we believe that the interested readers if necessary can do better fitting. Also, we aimed to show how it is an easy task to use the PAIM without having to worry about the ranges of the couplings in the introduced potential model. Moreover, the degree of precision of the results can also be drastically improved by raising the perturbative order in the energy expansion series one step more when it is necessary without any technical difficulties. We believe that this method and its results will provide a valuable future hint for investigating tetraquark configuration.

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**Conflict of interest:** Authors state no conflict of interest.

## References


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**Table 2:** P-wave mass spectrum for the charmonium $c\bar{c}^{2S+1}J_f$ in (GeV) by means of this work

<table>
<thead>
<tr>
<th>Meson</th>
<th>$^{2S+1}J_f$</th>
<th>$^{1}P_0$</th>
<th>$^{1}P_1$</th>
<th>$^{1}P_2$</th>
<th>$^{1}P_3$</th>
<th>$^{1}P_4$</th>
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<td>$^{1}P_0$</td>
<td>$^{1}P_1$</td>
<td>$^{1}P_2$</td>
<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
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<tr>
<td>$\chi_{c1}(1P)$</td>
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<td>$^{1}P_1$</td>
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<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
<td></td>
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<tr>
<td>$\chi_{c2}(1P)$</td>
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<td>$^{1}P_1$</td>
<td>$^{1}P_2$</td>
<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
<td></td>
</tr>
<tr>
<td>$\eta_{c}(1P)$</td>
<td>$^{1}P_0$</td>
<td>$^{1}P_1$</td>
<td>$^{1}P_2$</td>
<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
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</table>

**Table 3:** P-wave mass spectrum for the bottomonium $b\bar{b}^{2S+1}J_f$ in (GeV) by means of this work

<table>
<thead>
<tr>
<th>Meson</th>
<th>$^{2S+1}J_f$</th>
<th>$^{1}P_0$</th>
<th>$^{1}P_1$</th>
<th>$^{1}P_2$</th>
<th>$^{1}P_3$</th>
<th>$^{1}P_4$</th>
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<td>$^{1}P_2$</td>
<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
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<tr>
<td>$\chi_{b1}(1P)$</td>
<td>$^{1}P_0$</td>
<td>$^{1}P_1$</td>
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<td>$\chi_{b2}(1P)$</td>
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<td>$\eta_{b}(1P)$</td>
<td>$^{1}P_0$</td>
<td>$^{1}P_1$</td>
<td>$^{1}P_2$</td>
<td>$^{1}P_3$</td>
<td>$^{1}P_4$</td>
<td></td>
</tr>
</tbody>
</table>


Barakat T. The asymptotic iteration method for the eigenenergies of the Schrödinger equation with the potential $V(r) = -Z/r + gr + \lambda r^2$. J Phys A. 2006;39:823.


