Research Article

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Scaling law for velocity of domino toppling motion in curved paths

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Abstract: The arranged paths of dominoes have many shapes. The scaling law for the propagation speed of domino toppling has been extensively investigated. However, in all previous investigations the scaling law for the velocity of domino toppling motion in curved lines was not taken into account. In this study, the finite-element analysis (FEA) program ABAQUS was used to discuss the scaling law for the propagation speed of domino toppling motion in curved lines. It is shown that the domino propagation speed has a rising trend with increasing domino spacing in a straight line. It is also found that domino propagation speed is linearly proportional to the square root of domino separation. This research proved that the scaling law for the speed of domino toppling motion given by Sun [Scaling law for the propagation speed of domino toppling. AIP Adv. 2020;10(9):095124] is true. Moreover, the shape of domino arrangement paths has no influence on the scaling law for the propagation speed of dominoes, but can affect the coefficient of the scaling law for the velocity. Therefore, the amendatory function for the propagation speed of dominoes in curved lines was formulated by the FEA data. On one hand, the fitted amendatory function, φ_{revise} , provides the simple method for a domino player to quickly estimate the propagation speed of dominoes in curved lines; on the other hand, it is the rationale for the study of the domino effect.

Keywords: domino, toppling motion, scaling law, velocity, curved paths

1 Introduction

The game of dominoes attract many people because of its playfulness. Although dominoes is a simple game, it contains rich physical information. The mechanics of domino toppling motion have been extensively studied [1–15]. Shaw [2] studied the domino toppling time (T_N) and the number of dominoes (N) by combining theory with experiment. This study shows that the theoretical prediction is identical to that of the test; when $N > 6$, the N -versus- T_N curve is nearly linear. McLachlan et al. [3] presented a function for the propagation velocity, the spacing between dominoes, gravitational acceleration, and domino height when dominoes of zero thickness are spaced in a straight path. The functional relation is $v_{\text{McLachlan}} = v(h, \lambda, g) = \sqrt{gh}f\left(\frac{\lambda}{h}\right)$, where g is the gravitational acceleration, h is the domino height, λ is the spacing between dominoes, and $f(x)$ is an uncertain function of x . Bert [4] investigated the analytically formulated and numerical results of wave-propagation velocity in a row of falling dominoes based on an elliptic integral solution and on direct numerical integration. Szirtes and Rozsa [5] adopted dimensional analysis to investigate the propagation speed of dominoes. The main parameters are the domino thickness δ , domino separation λ , domino height h , gravitational acceleration g , and domino propagation speed $v = v(h, \lambda, \delta, g)$. The propagation speed of dominoes was determined by dimensional analysis to be $v_{\text{Szirtes}} = \sqrt{gh}f\left(\frac{\lambda}{h}, \frac{\delta}{h}\right)$. Larham [7] discussed the previous domino effect model and found that the proposed model diverges significantly from experimentally derived speed estimates over a significant range of domino spacing. Stronges' [8,9] study was focused on the wave of destabilizing collisions in the domino effect, in which the speed of the collision wavefront moving through the array is gradually close to the inherent velocity, which depends on the spacing and friction. VanLeeuwen [10] investigated the physics of a row of toppling dominoes in which the force between the falling dominoes, including the influence of friction, was analyzed, and the functional

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relationship between velocity and domino width, height, and spacing was derived. Shi et al. [12,13] studied the law of domino toppling with different masses and the same domino spacing by using the numerical model.

In 2020, Sun [15] obtained a new domino wave speed by using directional dimensional analysis, namely,

$$v_{\text{Sun}} = \lambda \sqrt{\frac{g}{h}} f\left(\frac{\delta}{\lambda}\right). \quad (1)$$

To obtain a simple explicit scaling law for the propagation speed of dominoes, the function $f\left(\frac{\delta}{\lambda}\right) \approx C\left(\frac{\delta}{\lambda}\right)^\alpha$ was deduced by curve fitting of the experimental data. Simultaneously, the index $\alpha \approx \frac{1}{2}$ was confirmed with Stronges' [8] experimental data, and the coefficient C is affected by the friction coefficient of domino surfaces. Finally, Sun [15] indicated that domino width has little influence on the propagation speed of domino toppling and proposed that the speed is $v_{\text{Domino}} \sim \lambda^{1/2} \sqrt{\frac{\delta g}{h}}$.

From the literature, it is found that all works are focused on the mechanics of domino toppling motion in straight lines. Moreover, most research methods are mainly based on experimental and theoretical research, and the research methods are relatively complex. Compared with these other methods, the finite-element method is simple and easy to use to achieve more complex physical phenomena. Moreover, to the best of our knowledge, there is no study on domino toppling motion in curved paths. Thus, in the present work, the finite-element analysis (FEA) program ABAQUS was used to explore the mechanics of

domino toppling motion in curved paths. The different path shapes, i.e., straight, circular, and S-shaped paths, were investigated using FEA. The scaling law for the propagation speed of domino toppling in different curved paths was proposed by data fitting. When the curvature is zero, the scaling law obtained in the present work is consistent with the scaling law proposed by Sun [15].

2 Propagation speed of domino toppling in straight path

To study the law between the propagation speed of dominoes and domino spacing in the straight paths accurately, ABAQUS was used to simulate the numerical analysis for the propagation speed of 17 domino models with various domino separations. The details of the dominoes are shown in Table 1. In the FEA, discrete rigid bodies were used to simulate the ground and three-dimensional (3D) deformable bodies to simulate the dominoes. The Young's modulus of the domino is 2,000 MPa, the Poisson's ratio is 0.3, and the domino mass density is 1.4×10^{-9} kg/mm³. The gravitational acceleration is $g = 9,800$ mm/s² and the friction coefficient between dominoes and ground is 0.3. The friction coefficient between dominoes is identical to that between dominoes and ground. In this study, a dynamic explicit method was used to calculate the toppling of the dominoes. Notwithstanding substantial mesh types being provided in

Table 1: Domino parameters (length units, mm)

Height (h)	Width (w)	Thickness (δ)	Number (N)	Separation (λ)
50	20	8	60	5
50	20	8	60	7
50	20	8	60	9
50	20	8	60	11
50	20	8	60	13
50	20	8	60	15
50	20	8	60	17
50	20	8	60	19
50	20	8	60	21
50	20	8	60	23
50	20	8	60	25
50	20	8	60	27
50	20	8	60	29
50	20	8	60	31
50	20	8	60	33
50	20	8	60	35
50	20	8	60	37

Abaqus, the shear locking problem will arise with dynamic analysis. A further complication is this will cause non-convergence of the solution. To cope with this problem, the reduced integration linear shell elements (S4R) and integration linear solid elements (C3D8R) were used; this mesh type introduces an additional degree of freedom to enhance the element displacement gradient into a linear element. Doing so can well solve the non-convergence problem and greatly improve convergence and computational efficiency. The details of the FE model are shown in Table 1 and Figure 1.

The FEA results are shown in Figure 2. It can be seen from the curve, when the domino spacing is 0–19 mm, that the domino propagation speed exhibits a rising trend with increasing domino spacing in the straight path. However, the propagation speed of domino toppling decreases with increasing spacing as the domino separation exceeds 20 mm. Suffice it to say that the domino spacing is 19 mm and the domino propagation speed reaches the maximum. Moreover, when the domino spacing is 15–25 mm, there is little difference in the speed of domino toppling. All in all, when the domino spacing is between 0.3 and 0.5 h, the domino in the straight path propagation speed reaches the peak.

In this article, the focus is only on the part in which the propagation velocity of dominoes increases with increasing domino spacing. Thus, the data within the range of 0–19 mm will be redrawn to specifically investigate the law for the propagation speed of dominoes.

Using the function from ref. [15] to fit the curves shown in Figure 3, the explicit speed of domino toppling is obtained as follows:

$$v_{\text{straight}} = C_{\text{straight}} \lambda^{1/2} \sqrt{\frac{\delta g}{h}}, \quad (2)$$

where $C_{\text{straight}} = 4.1755$, δ is the domino thickness, λ is the domino spacing, h is the domino height, and g is the gravitational acceleration.

It is surprising to see that the function $v_{\text{Sun}} = C_{\text{Sun}} \lambda^{1/2} \sqrt{\frac{\delta g}{h}}$ from ref. [15] can fit the curves perfectly. There is little distinction, except for the constant coefficient C .

Figure 1: FE model.

Figure 2: Domino propagation speed for FEA.

The data fitting gives $C_{\text{straight}} = 4.1577$, but ref. [15] reports $C_{\text{Sun}} = 3.488$. As is pointed out in ref. [15], the coefficient C difference is due to different friction coefficients. When the domino spacing range is 0–19 mm, in general the law for propagation velocity and domino spacing satisfy the scaling law for the propagation speed of dominoes given in the literature [15]. In addition, verifying the FE models can accurately simulate the scaling law for the propagation speed of domino toppling.

3 Propagation speed of domino toppling in circular path

Using accurate FE models, domino models were designed with a circular path. The radius of the circular path is

Figure 3: Domino propagation speed for $\lambda = 0$ –19 mm.

Figure 4: Details of dominoes in circular path.

150–250 mm and the domino spacing range is 0–19 mm. Owing to the fact that the speed direction of the domino with the circular path will change, only the speed value is explored. Details of the FE models and dominoes are shown in Figure 4 and Table 2.

Table 2 shows that the radius increased from 150 to 250 mm and domino spacing gradually increased from 7.7 to 18.18 mm. Using the FE results, the scatter diagram between the propagation speed of domino toppling and domino spacing is plotted. Figure 5 indicates that the domino propagation speed has a rising trend with increasing radius in the circular path when the radius is 150–250 mm (domino spacing is 7.7–18.18 mm). Compared to the FE model in which the dominoes are in the straight path, the propagation speed of domino toppling in the circular path is lower than in the straight model at the same domino spacing. It is found that domino propagation speed in the circular path is linearly proportional to the square root of domino separation by using the data fitting. The equation is expressed as follows:

$$v_{\text{circle}} = C_{\text{circle}} \lambda^{1/2} \sqrt{\frac{\delta g}{h}}, \quad (3)$$

where $C_{\text{circle}} = 3.289$. It can be seen that formula (3) is similar to the law provided in the literature [15], except for the value of the coefficient C . Therefore, it is known that the shape of the domino arrangement paths has little influence on the scaling law for the propagation speed of domino toppling.

Figure 5: Domino propagation speed with circular path.

4 Propagation speed of domino toppling in S-shaped path

The results of the domino FE models with straight and circular paths demonstrated that the shape of domino arrangement paths has little influence on the scaling law for the propagation speed of domino toppling, but it can change the value of the velocity. Therefore, domino FE models with different curvature paths were designed. The details of the geometric parameters of dominoes are shown in Table 3. To change the curvature, the spacing must be first fixed as 19 mm of each domino, and then the dominoes moved in the width direction and the dominoes arranged in an S shape. There are many different curvatures because of different distances (Δw). The curvature calculation formula is $K = 1/R$. The model of a domino with an S-shaped path is displayed in Figure 6.

Table 3 shows seven FE models with various curvatures that were designed. The range of Δw is 2–14 mm. FE analysis shows that the domino is not successively toppling when $\Delta w = 12$ and 14 mm. The failure modes are shown in Figures 7 and 8.

Table 2: Parameters of domino in circular path (length units, mm)

Height (h)	Width (w)	Thickness (δ)	Radius (R)	Separation (λ)
50	20	8	150	7.7
50	20	8	170	9.7
50	20	8	190	11.7
50	20	8	210	14
50	20	8	230	16.09
50	20	8	250	18.18

Table 3: Parameters of dominoes in S-shaped path

h	w	δ	λ	Δw	K
50	20	8	19	2	0.00096
50	20	8	19	4	0.00188
50	20	8	19	6	0.00274
50	20	8	19	8	0.00352
50	20	8	19	10	0.00420
50	20	8	19	12	0.00478
50	20	8	19	14	0.00525

From Figure 7, when $\Delta w = 12$ mm, only 37 dominoes topple. With increasing Δw , the number of toppling dominoes decreases. When $\Delta w = 14$ mm, just 13 dominoes topple. This is demonstrated when Δw exceeds half the width of the dominoes, and the dominoes do not successfully topple completely. Owing to the fact that 60 dominoes in the two models have not been successfully toppled, the focus is on the propagation speed of dominoes toppling with $\Delta w = 2 - 10$ mm. The domino propagation velocities are shown in Table 4.

As shown in Table 4, it can be clearly seen that the propagation speed of dominoes gradually decreases as curvature increases. In addition, with increasing path radius, the propagation speed of dominoes shows an increasing trend. This is the same law as seen in the FE models with a circular path. To study the effect of curvature on the propagation speed of domino toppling,

Figure 6: Parameters of dominoes in S-shaped path. **Figure 7:** Failure mode for $\Delta w = 12$ mm.

dimensionless methods were used to deal with v_{S-num} , i.e., dividing the propagation speed of domino toppling by $v_{straight}$. Using the data from FE results, the modified formula of propagation speed is proposed, and the equation is expressed as follows:

$$\varphi_{revise} = C_{straight} \left[\alpha \left(\frac{h}{R} \right)^3 + \beta \left(\frac{h}{R} \right)^2 + \gamma \left(\frac{h}{R} \right) + 1 \right], \quad (4)$$

$$h = 50 \text{ mm}, \quad R = (240 - 1,050 \text{ mm}),$$

where $\alpha = -109.38$, $\beta = 27.54$, and $\gamma = -2.36$.

Therefore, the equation for the propagation speed of domino toppling with curved paths is expressed as follows (Figure 9):

$$v_s = C_{straight} \lambda^{1/2} \left[\alpha \left(\frac{h}{R} \right)^3 + \beta \left(\frac{h}{R} \right)^2 + \gamma \left(\frac{h}{R} \right) + 1 \right] \sqrt{\frac{\delta g}{h}}, \quad (5)$$

$$\lambda = (0 - 19 \text{ mm}), \quad R = (240 - 1,050 \text{ mm}),$$

$$h = 50 \text{ mm}.$$

5 Equation verification

To validate formula (5), the propagation speeds of domino toppling obtained from this equation were then compared with those obtained from FE analysis. The results are

Figure 8: Failure mode for $\Delta w = 14$ mm.

Table 4: Propagation velocities of dominoes in S-shaped path

h	δ	Δw	R	K	$v_{S\text{-num.}}$	φ
50	8	2	1046.17	0.0009559	618.109	0.906
50	8	4	532.08	0.0018794	589.327	0.864
50	8	6	364.72	0.0027418	574.262	0.842
50	8	8	284.04	0.0035206	485.807	0.712
50	8	10	238.03	0.0042011	393.164	0.577

1. Propagation speed of dominoes with S-shaped path by FEA is $v_{S\text{-numerical}}$.

2. $\varphi = v_{S\text{-numerical}} / v_{\text{straight}}$ when $\lambda = 19 \text{ mm}$ and $v_{\text{straight}} = 681.927 \text{ mm/s}$.

Figure 9: Modified formula of domino toppling motion.

shown in Table 5. As illustrated by the results presented in Table 5, the propagation speeds calculated from equation (5) show a strong similarity to the FE results. The errors of the three model results are within 6%. It can be shown that the propagation speed of domino toppling in the curved paths was realized using equation (5).

6 Conclusion

The propagation speeds of dominoes toppling in differently shaped paths were studied numerically and the results are presented in this article. From the results obtained, the law for domino propagation velocity and spacing satisfies the scaling law for the propagation speed of dominoes given in the literature [15]. This confirms that the scaling law for the speed of domino-toppling motion given in the literature [15] is true. When the domino spacing is between 0.3 and 0.5 h in the straight path, the speed of domino toppling reaches the

Table 5: Comparisons of FEA results with those from equation (5) (length units, mm; time, s)

h	δ	λ	R	K	v_s	$v_{S\text{-num.}}$	Error (%)
50	8	7	314.17	0.003183	383.30	383.8	0.1
50	8	9	406.17	0.002462	455.52	438.4	3.9
50	8	18.18	250	0.004	529.46	557.76	5.1

maximum. In the “S”-shaped path, if Δw exceeds half the width of the dominoes, the dominoes do not successively topple completely. By analyzing the finite-element models with a circular path, the result highlights that the domino propagation speed in the circular path is linearly proportional to the square root of domino separation. Compared with one another, for the finite-element models in the straight path the same scaling law for the propagation speed of domino toppling applies, but the value for speed in the circular path is lower than that of the straight models at the same domino spacing. It was revealed that over a specific range of domino spacing the shape of domino arrangement paths has no effect on the scaling law of the propagation speed of domino toppling, but can change the value of the domino propagation speed. The modified formula φ_{revise} for propagation speed was proposed. Furthermore, the equation for the propagation speed of domino toppling with curved paths over a specific range of domino spacing was obtained; the function is:

$$v_s = \varphi v_{\text{straight}} = \varphi_{\text{revise}} \lambda^{1/2} \sqrt{\frac{\delta g}{h}}.$$

Conflict of interest: Authors state no conflict of interest.

Data availability statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix

Table A1: Summary of notations

φ	Velocity of domino-toppling formula correction coefficient
g	Gravitational acceleration
h	Domino height
w	Domino width
λ	Spacing between dominoes
δ	Domino thickness
v	Domino propagation speed
α	Index
C	Coefficient
R	Bending radius
K	Curvature
N	Number of domino
$f(x)$	Uncertain function of x
Δw	Deviation of domino in width direction