Research Article

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Hopf bifurcation analysis for liquid-filled Gyrostat chaotic system and design of a novel technique to control slosh in spacecrafts

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Abstract: In this article, a fuel tank is coupled with gyrostat in a moving spacecraft to discuss its dynamical behaviour and bringing stability in velocity vectors. Parametric study is performed using Hopf bifurcation to find the bifurcation parameter for a considered mechanical model. Furthermore, a region is constructed in which negligible limit cycles appear around unstable spirals for angular momentum greater than bifurcation point. Based on local dynamical analysis, trajectories of angular velocities are observed with respect to damping constant, which is formulated in the form of bifurcation parameter. Moreover, a controller is designed in this article for considered dynamical system by achieving global stability, with the help of Lyapunov theory, into the spacecraft coupled with filled fuel tank, and their results are compared with effective spacecraft control strategies to observe the effectiveness of our proposed control technique. Finally, in presented research, numerical simulations are performed using MATLAB for validation of analytical results, which the authors have achieved for Hopf bifurcation and designed controller.

Keywords: autonomous dynamical system, Hopf bifurcation, active control, adaptive control, stability

1 Introduction

In 1852, Leon Foucault invented a device, used for investigating rotation of the earth, named as gyrostat. This device worked in a manner that plane of rotation depends on latitude of its location. Later on, gyrostat is used in aerial vehicle to detect path and attitude control. There are various shapes of gyrostat that has many advantages in aerial vehicles [1], satellites and robots [2]. The first governing equation for discussing dynamics of gyrostat was introduced by Wittenburg [3] in 1977, whereas Chatys and Koruba [4] used mini gyro as controller in mini aerial vehicles for the first time. A dual-spin spacecraft [5] is established in the presence of damping factor with the aid of Lie–Poisson structures. A gyrostat is a rigid body consisting of rotors to a main platform of the body. In Figure 1, one can observe that this frame is further categorized into three axisymmetric rotors, which are rotating about their respective axis with respect to body frame. Euler dynamical equations of rotational dynamics for a disturbed gyrostat system given by ref. [6] are as follows:

$$\begin{cases} I_x \dot{x} = (I_y - I_z)yz - h_z y + h_y z - \mu_x x + L_x \\ I_y \dot{y} = (I_z - I_x)zx + h_z x + \mu_y y + L_y \\ I_z \dot{z} = (I_x - I_y)xy - h_y x - \mu_z z + L_z. \end{cases} \quad (1)$$

Now, by shifting inertial terms I_x , I_y and I_z to the right side of system (1), we get:

$$\begin{cases} \dot{x} = \frac{(I_y - I_z)}{I_x} yz - \frac{h_z}{I_x} y + \frac{h_y}{I_x} z - \frac{\mu_x}{I_x} x + \frac{L_x}{I_x} \\ \dot{y} = \frac{(I_z - I_x)}{I_y} zx + \frac{h_z}{I_y} x + \frac{\mu_y}{I_y} y + \frac{L_y}{I_y} \\ \dot{z} = \frac{(I_x - I_y)}{I_z} xy - \frac{h_y}{I_z} x - \frac{\mu_z}{I_z} z + \frac{L_z}{I_z}. \end{cases} \quad (2)$$

Throughout this article, the authors have used the following parameter values for simplicity:

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Figure 1: Satellite attached with partially filled fuel tank and three rotors.

$$\left\{ \begin{array}{l} \mu_{\theta}^{\theta} = \frac{\mu_{\theta}}{I_{\theta}} \text{ for } \theta = x, y \text{ and } z \\ h_x^z = \frac{h_z}{I_x} \\ h_x^y = \frac{h_y}{I_x} \\ h_y^z = \frac{h_z}{I_y} \\ h_y^x = \frac{h_x}{I_y} \\ h_z^x = \frac{h_x}{I_z} \\ h_z^y = \frac{h_y}{I_z} \\ I_x^{yz} = \frac{I_y - I_z}{I_x} \\ I_y^{xz} = \frac{I_x - I_z}{I_y} \\ I_z^{xy} = \frac{I_x - I_y}{I_z} \\ L_x^x = \frac{L_{\theta}}{I_{\theta}} \text{ for } \theta = x, y, z. \end{array} \right. \quad (3)$$

Hence, substituting parameter values (3) in system (2), we get the following system:

$$\left\{ \begin{array}{l} \dot{x} = -\mu_x^x x - h_x^z y + h_x^y z + I_x^{yz} yz + L_x^x \\ \dot{y} = h_y^z x + \mu_y^y y + I_y^{xz} zx + L_y^y \\ \dot{z} = -h_z^x x - \mu_z^z z + I_z^{xy} xy + L_z^z, \end{array} \right. \quad (4)$$

where $X = [x \ y \ z]^T$ is the angular velocity vector; I_x , I_y and I_z are the principal moments of inertia of the gyrostator in the body axis frame; h_x , h_y and h_z are constants of total angular momentum, whereas L_x , L_y and L_z are external torques applied on gyrostator. System (4) shows unpre-

dictability in trajectories for $\vec{T} = [0.85 \ 0.45 \ 0.2]^T$, $\vec{\mu} = [6 \ 6.43 \ 5.8]^T$, $\vec{h} = [0 \ 0.57416 \ 2.38]^T$ and $\vec{L} = [0 \ 0 \ 22.8]^T$.

A phenomenon, in which given dynamical system changes its structure with the variation in parameter values is famous as bifurcation [7]. Hopf bifurcation [8] is one of the important types of bifurcation, in which one can seek appearance and disappearance of oscillatory solutions for various values of parameter around bifurcation point. A variety of research exist for observing oscillatory solutions in dynamical systems. Recently, Marwan *et al.* have considered forced Chen system and discussed existence of Hopf bifurcation [9] for non-zero equilibria. In 2020, Marwan and Ahmad [10] discussed Bogdanov–Takens bifurcation for energy and resource transportation model between two cities in China. Existence of oscillatory solutions is not only limited to mechanical systems but also has a variety of impact on biological models. In 2019, Mondal and Samanta [11] utilized MATCONT for observing Hopf bifurcation in the predator–prey model, when predator is harvested at a constant rate. The same biological model is analysed in the presence of fear effect [12] for the existence of oscillatory solutions, whereas two different food chain models are considered by Sahoo and Samanta in their work [13], with the effect of delay parameter, to investigate switching of stability through Hopf bifurcation. In 2020, several bifurcations including generalized Hopf and codimension 2 bifurcations are studied in the predator–prey model under the assumption that prey population is infected by a disease [14]. Apart from importance of Hopf bifurcation in biological models, it has great impact on spacecraft models. A specific case of gyrostator-based satellite moving on circular path is examined for two different cases with the help of local bifurcation by Sontos and Melicio [15]. Similarly, various shaped gyrostats, such as spheroid gyrostator [16] and axial gyrostator [17], are studied for the existence of bifurcations. There are many ways to prove the existence of chaos in dynamical system, such as monotone, co-existing attractors, hidden attractors and self-existed attractors [18], but the basic way of obtaining chaos is bringing negligible changes in initial conditions and bifurcation parameters in given dynamical system. Qi and Yang [6] introduced chaotic gyrostator model for the first time and determined unpredictability of angular momentum for fixed parameter values. Qi *et al.* also discussed its dynamics with the aid of external and internal torques, forces and energies.

The term chaos has applications in numerous engineering-based systems, but unpredictability in gyrostator proceeds to disturbance in attitude control and communication loss. Therefore, controllability [19,20] in such systems seek attention of many researchers. Chai considered

six degrees of freedom vehicle [21] and designed its attitude control. In 2012, Aslanov and Yudintsev considered free gyrostat chaotic system [22] along with three-dimensional inertial components and asymmetric rotors. A chaotic gyrostat dynamical system is synchronized with identical drive system with the help of newly designed controller [23], named as variable substitution and feedback controller (VSFC) for achieving global stability. In 2018, Chai *et al.* [24] introduced receding horizon control for spacecraft path planning. They have also decreased computational cost with the aid of two nested gradient methods by embedding in control scheme. A new three-layer hybrid control [25] is designed for space vehicles based on stochastic algorithm to enhance global stability. Abtahi derived a gyrostat model [26] based on the Hamiltonian method for orbital motion of spacecraft with the aid of extended Deprit transformation and showed its chaotic effect using the Melnikov technique. Doroshin [27] generalized case of V. A. Stekloff's rigid body motion and investigated its angular motion, whereas Wang *et al.* [28] worked on its controllability using Fuzzy controller based on fuzzy logic. Several researchers worked on attitude dynamics and control of spacecraft models [29–31].

We have studied dynamical systems derived for spacecrafts using Hamiltonian mechanics [32] and devices attached within it [33,34]. We have noticed that several biological models are taken into account for oscillatory solutions [35,36] in the aforementioned citations, but impact of seeking local oscillatory solutions using Andronov–Hopf bifurcation for spacecraft with attached fuel tank is never discussed. Moreover, the review work of Chai [37], in which authors have discussed path planning and controllability of spacecraft models using different techniques in detail, has motivated us to work on the design of controller for bringing stability in sloshing effect which arises inside coupled tank and leads to havoc in dynamics of gyrostat.

This article is organized as follows: in Section 2.1, the existence of Hopf bifurcation is shown using singularity condition. Moreover, Section 3 consists of our proposed controller; M_{chaotic} and its algorithm, which is applied to gyrostat chaotic system in Section 4. Finally, numerical simulation, discussion of Hopf bifurcation and M_{chaotic} controller are provided in Section 5.

2 Dynamical analysis

2.1 Hopf bifurcation in gyro chaotic system

A physical phenomenon in which trajectory of any non-linear dynamical system changes with the change in

parameter values involved in system is famous as bifurcation. Similar to the concept of *clear blue sky* given in ref. [38], limit cycles around equilibrium points can be created, destroyed or changes its stability and such type of concept is famous as Hopf bifurcation. In this section, we have discussed Hopf bifurcation and its impact on our considered dynamical system.

A spherical filled fuel tank is attached with original gyro chaotic system [6] under the consideration of circular liquid movement given in system (4), which is chaotic for the aforementioned parameter values. Jacobian matrix is the fundamental step in tools related to local and global bifurcations. Therefore, Jacobian matrix for system (4) is as follows:

$$J = \begin{pmatrix} -\mu_x^x & -h_x^z + I_x^{yz}z^* & h_x^y + I_x^{yz}y^* \\ h_y^z + I_y^{xz}z^* & \mu_y^y & I_y^{xz}x^* \\ -h_z^y + I_z^{xy}y^* & I_z^{xy}x^* & -\mu_z^z \end{pmatrix}. \quad (5)$$

System (4) has five equilibrium points including origin under the absence of external torques:

$$\begin{aligned} E_1 &= [0, 0, 0], & E_2 &= [-10.6, 13.1, -8.5], \\ E_3 &= [13.5, 23.9, 21.1], & E_4 &= [-12.6, -25.5, 23.5], \\ E_5 &= [9.7, -13.2, -9.8]. \end{aligned}$$

Hence, Jacobian matrix (5) of system (4) at E_1 is as follows:

$$J = \begin{pmatrix} -\mu_x^x & -h_x^z & h_x^y \\ h_y^z & \mu_y^y & 0 \\ -h_z^y & 0 & -\mu_z^z \end{pmatrix}. \quad (6)$$

Now, for discussing its dynamics around origin; O , characteristic polynomial equation is derived using eigenvalues for equation (6);

$$\lambda^3 + \delta_{1a}\lambda^2 + \delta_{2a}\lambda + \delta_{3a} = 0, \quad (7)$$

where

$$\begin{cases} \delta_{1a} = \mu_x^x - \mu_y^y + \mu_z^z \\ \delta_{2a} = h_x^z h_y^z - \mu_x^x \mu_y^y + \mu_x^x \mu_z^z + h_x^y h_z^y - \mu_y^y \mu_z^z \\ \delta_{3a} = -\mu_x^x \mu_y^y \mu_z^z + h_x^z h_y^z \mu_z^z - h_x^y \mu_y^y h_z^y. \end{cases} \quad (8)$$

For Hopf bifurcation the basic condition is $\delta_{1a}\delta_{2a} = \delta_{3a}$;

$$\begin{aligned} (\mu_x^x - \mu_y^y + \mu_z^z)(h_x^z h_y^z - \mu_x^x \mu_y^y + \mu_x^x \mu_z^z + h_x^y h_z^y - \mu_y^y \mu_z^z) \\ = -\mu_x^x \mu_y^y \mu_z^z + h_x^z h_y^z \mu_z^z - h_x^y \mu_y^y h_z^y. \end{aligned} \quad (9)$$

Simplifying equation (9), we get:

$$\begin{aligned} h_x^y[(\mu_x^x + \mu_z^z)h_z^y] &= (\mu_y^y - \mu_x^x)(h_x^z h_y^z + \mu_x^x \mu_z^z + \mu_y^y \mu_z^z \\ &\quad + \mu_z^z) + \mu_y^y(\mu_z^z h_x^z - \mu_x^x \mu_y^y). \end{aligned} \quad (10)$$

It is tedious to discuss Hopf bifurcation from equation (10). Therefore, using $\mu_y^y = \mu_x^x$ into equation (9) yields:

$$h_x^y = \frac{\mu_x^x(\mu_z^z h_x^z - \mu_x^{x2})}{(\mu_x^x + \mu_z^z)h_z^y}. \quad (11)$$

Substituting $\mu_z^z = \mu_x^x$ and $h_z^y = 1$ into equation (11) we get equation (10) in the simplified form:

$$h_x^y = \frac{\mu_x^x}{2}(h_x^z - \mu_x^x). \quad (12)$$

Hence, several cases arise for selection of h_x^z , but we have chosen $h_x^z = 2\mu_x^x$ such that h_x^y become positive;

$$h_y = \frac{\mu_x^2}{2I_x}. \quad (13)$$

Substituting equations (11)–(13) into (7) and (8), we get:

$$\lambda^3 + \delta_{1b}\lambda^2 + \delta_{2b}\lambda + \delta_{3b} = 0, \quad (14)$$

where

$$\begin{cases} \delta_{1b} = \mu_x^x \\ \delta_{2b} = \frac{\mu_x^x(4 - \mu_x^x)}{2} \\ \delta_{3b} = 2\mu_x^{x2} - \frac{3\mu_x^{x3}}{2}. \end{cases} \quad (15)$$

Substituting $\lambda = i\omega$ [39] into (14) for achieving fundamental condition of Hopf bifurcation:

$$\omega = \sqrt{\delta_{2b}} \text{ and } \delta_{1b}\delta_{2b} = \delta_{3b}, \quad (16)$$

where δ_{2b} is given in equation (15). Therefore, for Hopf bifurcation μ_x^x must be greater than 4. Phase portrait for verification of equations (11)–(16) is given in Section 5.1.

3 Algorithm and limitations of M_{chaotic} controller

In this section, we used the concept of Lyapunov stability based on the classical control theory to design algorithm for M_{chaotic} controller. This controller works in a manner that estimated parameters start approaching towards their original values. Then, unpredictable trajectories start moving to their desired values as t approaches to infinity. It is important to mention that we have selected stable equilibria of

system (4), for the sake of convergence, instead of any desired values.

Algorithm 1. Algorithm for M_{chaotic} controller

1. Suppose a closed loop system with $M = M_A + M_W$:

$$\dot{y} = g(y, M) \quad (17)$$

for an open loop system:

$$\dot{x} = f(x). \quad (18)$$

2. Add a controller; M to the right side of open loop system (18), where, number of controllers depends on the nature of chaotic system.
3. Derive error dynamical system:

$$\dot{e} = \dot{y} - \dot{x}, \quad (19)$$

using error term between the state variables of systems (18) and (17).

4. As $M = M_A + M_W$ given in step 1. Therefore, select M_A on the basis of active control technique [18], in which all terms, excluding error terms, are eliminated.
5. For M_W , consider the remaining parameters involved in error dynamical system (19) as unknown or anonymous.
6. Select a positive definite quadratic energy function in terms of error trajectories and estimated parameters:

$$V(e) = \sum_{i=1}^n e_i^2 + e_{\Phi_i}^2 \quad (20)$$

for finding anonymous parameters using updated parameter law, where Φ_i shows estimated parameters.

7. If $\dot{V}$ is less than zero, then we are done, else go to step 1 again.
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Algorithm 1 is utilized in Section (4) for Gyrostat chaotic system (4) to achieve global stability in its velocity vectors. We also have the following limitation for our designed controller:

- a: Chaos in dynamical systems occurs due to negligible change in initial solution or bifurcation parameter and sometimes due to both reasons. But, if chaos occurs due to initial conditions only, then our designed controller fails to be implemented in such case.
- b: If estimated parameters existed in any chaotic system are in complicated way, then it may be difficult to design a suitable controller.

4 M_{chaotic} controller for gyro chaotic system

Theorem 4.1. An open loop system (4) is controlled at $t = 1$, with control input $M_i = A_i + W_i$ for $i = 1, 2, 3$, where $A_i = S\mathbf{x}^* + N(e_i) + \kappa$ with $A_i = (A_1, A_2, A_3)^T$,

$$\mathbf{S} = \begin{pmatrix} \mu_x^x & h_x^z & -h_x^y \\ -h_y^z & \mu_y^y & 0 \\ h_z^y & 0 & \mu_z^z \end{pmatrix}, \quad (21)$$

$$\mathbf{N}(e_i) = \begin{pmatrix} -I_x^{yz}(e_2 e_3 + e_2 z^* + e_3 y^*) \\ -I_y^{xz}(e_1 e_3 + e_1 z^* + e_3 x^*) \\ -I_z^{xy}(e_1 e_2 + e_1 y^* + e_2 x^*) \end{pmatrix},$$

(a)

(b)

(c)

(d)

Figure 2: Parametric study of gyrostat chaotic system for seeking oscillatory solutions: (a) $h_x^y \ll B_H$, (b) $h_x^y < B_H$, (c) $h_x^y = B_H$, (d) $h_x^y > B_H$.

$$\kappa = \begin{pmatrix} -I_x^{yz}y^*z^* - L_x^x \\ -I_y^{xz}x^*z^* - L_y^y \\ -I_z^{xy}x^*y^* - L_z^z \end{pmatrix}, \quad (22)$$

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \bar{\mu}_x^x(t) - \rho & \bar{h}_x^z(t) & -\bar{h}_x^y(t) \\ -\bar{h}_y^z(t) & -\bar{\mu}_y^y(t) - \rho & 0 \\ \bar{h}_z^y(t) & 0 & \bar{\mu}_z^z(t) - \rho \end{pmatrix}$$

and suitable updated law (33).

Proof. We begin proof of Theorem 4.1 by adding time-dependant control inputs; $M_i(t)$ with all equations given in system (4):

$$\begin{cases} \dot{x} = -\mu_x^x x - h_x^z y + h_x^y z + I_x^{yz} yz + L_x^x + M_1(t) \\ \dot{y} = h_y^z x + \mu_y^y y + I_y^{xz} zx + L_y^y + M_2(t) \\ \dot{z} = -h_z^y x - \mu_z^z z + I_z^{xy} xy + L_z^z + M_3(t). \end{cases} \quad (23)$$

First step for finding control input is derivation of error dynamical systems using error terms:

$$\vec{e}_i = \vec{x} - \vec{x}^* \text{ for } i = 1, 2, 3 \text{ and } \vec{x} = (x, y, z). \quad (24)$$

Hence, error dynamical system of equation (23) is achieved by differentiating equation (24):

$$\begin{aligned} \dot{e}_1 &= -\mu_x^x e_1 - h_x^z e_2 + h_x^y e_3 - \mu_x^x x^* - h_x^z y^* + h_x^y z^* \\ &\quad + I_x^{yz} yz + L_x^x + M_1(t) \\ \dot{e}_2 &= h_y^z e_1 + \mu_y^y e_2 + h_y^z x^* + \mu_y^y y^* + I_y^{xz} zx + L_y^y + M_2(t) \\ \dot{e}_3 &= -h_z^y e_1 - \mu_z^z e_3 - h_z^y x^* - \mu_z^z z^* + I_z^{xy} xy + L_z^z \\ &\quad + M_3(t), \end{aligned} \quad (25)$$

where

$$M_i = A_i + W_i \text{ for } i = 1, 2, 3.$$

Here, control input M_i is divided into two parts. First, we find A_i using the basic active control strategy [18]:

$$\begin{cases} A_1 = \mu_x^x x^* + h_x^z y^* - h_x^y z^* - I_x^{yz} yz - L_x^x \\ A_2 = -h_y^z x^* - \mu_y^y y^* - I_y^{xz} zx - L_y^y \\ A_3 = h_z^y x^* + \mu_z^z z^* - I_z^{xy} xy - L_z^z. \end{cases} \quad (26)$$

Hence, we get $\mathbf{S}$, $\mathbf{N}(e_i)$ and κ mentioned in equation (21). Therefore, substituting equation (26) into equation (25), we get the simplified form of error dynamical system:

$$\begin{cases} \dot{e}_1 = -\mu_x^x e_1 - h_x^z e_2 + h_x^y e_3 + W_1 \\ \dot{e}_2 = h_y^z e_1 + \mu_y^y e_2 + W_2 \\ \dot{e}_3 = -h_z^y e_1 - \mu_z^z e_3 + W_3. \end{cases} \quad (27)$$

Second portion of our proposed controller include W_i for $i = 1, 2, 3$, which can be obtained on the assumption that parameters remaining in equation (27) are anonymous. Therefore, we have introduced estimated parameters and negative eigenvalue ρ for W_i :

$$\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \bar{\mu}_x^x(t) - \rho & \bar{h}_x^z(t) & -\bar{h}_x^y(t) \\ -\bar{h}_y^z(t) & -\bar{\mu}_y^y(t) - \rho & 0 \\ \bar{h}_z^y(t) & 0 & \bar{\mu}_z^z(t) - \rho \end{pmatrix}. \quad (28)$$

Hence, substituting equation (28) into equation (27) yields:

$$\begin{cases} \dot{e}_1 = -(e_{\mu_x^x} + \rho)e_1 - e_{h_x^z}e_2 + e_{h_x^y}e_3 \\ \dot{e}_2 = e_{h_y^z}e_1 + (e_{\mu_y^y} - \rho)e_2 \\ \dot{e}_3 = -e_{h_z^y}e_1 - (e_{\mu_z^z} + \rho)e_3, \end{cases} \quad (29)$$

where e_i , $i = 1, 2, 3$ are error terms between state variable and desired point, whereas $e_k = k - \bar{k}(t)$ for $k = \mu_x^x$, h_x^z , h_x^y , h_y^z , μ_y^y , h_z^y and μ_z^z . Finally, we need to find anonymous time-dependant parameter values involved in equation (29). Therefore, we use quadratic energy-type function:

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_{\mu_x^x}^2 + e_{h_x^z}^2 + e_{h_x^y}^2 + e_{h_y^z}^2 + e_{\mu_y^y}^2 + e_{h_z^y}^2 + e_{\mu_z^z}^2), \quad (30)$$

and must be dissipative. Hence, for verification we differentiate V along trajectories to get:

$$\begin{aligned} \dot{V} &= (e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 + e_{\mu_x^x} \dot{e}_{\mu_x^x} + e_{h_x^z} \dot{e}_{h_x^z} + e_{h_x^y} \dot{e}_{h_x^y} \\ &\quad + e_{h_y^z} \dot{e}_{h_y^z} + e_{\mu_y^y} \dot{e}_{\mu_y^y} + e_{h_z^y} \dot{e}_{h_z^y} + e_{\mu_z^z} \dot{e}_{\mu_z^z}). \end{aligned} \quad (31)$$

Figure 3: Time history of system (4) indicating damping oscillation.

Figure 4: Existence of Hopf bifurcation from both ends: (a) $\mu_x < B_H$, (b) $\mu_x = B_H$, (c) $\mu_x > B_H$.

Rearranging and making groups of all variables involved in equation (31) yield:

$$\begin{aligned} \dot{V} = & -e_1^2 - e_2^2 - e_3^2 - e_{\mu_x}(e_1^2 + \dot{\mu}_x^x) - e_{h_z}(e_1e_2 + \dot{h}_x^z) \\ & - e_{h_y}(-e_1e_3 + \dot{h}_x^y) - e_{h_z}(-e_1e_2 + \dot{h}_y^z) \\ & - e_{\mu_y}(e_2^2 + \dot{\mu}_y^y) - e_{h_z}(e_1e_3 + \dot{h}_z^y) - e_{\mu_z}(e_3^2 + \dot{\mu}_z^z). \end{aligned} \quad (32)$$

In view of equation (32), one can achieve the following dynamical system of estimated parameters [10]:

$$\begin{cases} \dot{\mu}_x^x = -e_1^2 + k_1e_{\mu_x^x} \\ \dot{h}_x^z = -e_1e_2 + k_2e_{h_x^z} \\ \dot{h}_x^y = e_1e_3 + k_3e_{h_x^y} \\ \dot{h}_y^z = e_1e_2 + k_4e_{h_y^z} \\ \dot{\mu}_y^y = e_2^2 + k_5e_{\mu_y^y} \\ \dot{h}_z^y = -e_1e_3 + k_6e_{h_z^y} \\ \dot{\mu}_z^z = -e_3^2 + k_7e_{\mu_z^z}, \end{cases} \quad (33)$$

which is our desired result. In Section 5.2, we have briefly discussed analytical results given in Theorem 4.1 that how chaotic trajectories start moving towards their stable equilibria as estimated parameters start approaching their original values in chaotic way instead of linear. $\square$

5 Results and discussion

In this section, we have verified analytical results given in Sections 2–4 with the aid of simulations. Moreover, we have divided this portion into two sub-sections.

5.1 Hopf bifurcation

Gyrostat is a rigid body, used for detecting path and helps in stabilizing movement of spacecrafts [6]. Here, we have proved the existence of Hopf bifurcation for two parameters involved in system (4).

In Figure 2(a)–(d), we have noticed the existence of oscillation which changes its shape by adjusting parameter values given in equation (13). We have selected different values to observe trajectories for confirmation of Hopf bifurcation. In Figure 2(a) and (b), one can observe if damping constant is less than Hopf bifurcation point; $h_y = \frac{\mu_x^2}{2k_x}$, it shows if disturbance in torques of gyro and components of total angular momentum along y-axis are small enough. Then, there exists a stable spiral which starts shrinking into origin with selection of value equals to Hopf bifurcation point, which can be seen in Figure 2(c). These trajectories change into an unstable spiral, given in Figure 2(d), surrounding stable limit cycle for parameter value greater than h_y . Hence, qualitative behaviour of such trajectories confirms the existence of supercritical Hopf bifurcation.

Further working on studying various aspects of dynamical behaviour of the gyro chaotic system, in our another paper, it leads us to the point that there also exists Hopf bifurcation for our considered system, with different parametric values.

In Figure 3, one can observe damping of oscillation in trajectories of gyrostat chaotic system. From Figure 4(a)–(c), the existence of subcritical Hopf bifurcation is confirmed. In Figure 4(a), we have selected μ_x as bifurcation parameter. For value less than bifurcation point, unstable spiral can be observed in Figure 4(a), which vanishes in Figure 4(b) and emerges again with stable spiral in Figure 4(c).

Figure 5: (a)–(c) Phase portraits and (d) time history of system (4): (a) xy -axis, (b) yz -axis, (c) xz -axis, (d) time history.

5.2 M_{chaotic} controller

We have already discussed analytical results for our proposed control technique in Sections 3 and 4. Therefore, we have considered system (4), for applying M_{chaotic} controller, which is chaotic for the aforementioned fixed parameter values. Figure 5(a)–(c) is phase portraits and Figure 5(d) is time history for the existence of chaos in gyrostator system. An attached spherical cylinder with spacecraft can be seen in Figure 1. Therefore,

negligible disturbance in movement of spacecraft can bring sloshing around walls inside fuel tank. Due to this effect, one can observe frequent unstable changes in centre of mass and exert pressure on walls of attached tank. Hence, gyro device is attached within spacecrafts for continuous stability and path planning for spacecrafts. Moreover, chaos in gyrostator can lead to create disconnection between satellite and spacecraft and can lose its position in space before reaching to its desired destination.

Figure 6: Controlled state variables of gyro chaotic system.

Figure 7: Chaotic updated law (33).

Therefore, we have introduced new control function for surpassing unpredictability in system (4). However, many researchers have proved that chaotic trajectories start converging towards desired point as estimated parameters approach their original values for time approaches to infinity. But, after introducing control technique given in this article for chaotic systems, we have observed that the same result of controllability is achieved with chaotic updated law. This means, when the control inputs designed in this article are applied to chaotic system (4) using Algorithm 1, then randomness in trajectories, which can be seen in Figure 5(d), is controlled and can be observed in Figure 6 at $t = 1.2$.

In Figure 6, we can see that solutions of angular velocities begin with oscillations but with advancement in time, our proposed control inputs start its function at $t = 0.5$. Hence, chaos in gyrostat dynamical system is controlled. Interestingly, Figure 7 is not any other chaotic system, but is numerical simulation of updated law given in equation (33). Moreover, the results achieved in Figure 7 are negation of the fundamental updated law, especially, used in adaptive control technique. Hence, in our case, as estimated parameters start approaching towards their original values in random way. Then, angular velocities of all coordinates start moving in controlled way after some oscillation.

The authors have included almost all phase portraits Figure 8(a)–(l) for chaotic updated law to clarify their new findings. Moreover, in comparison with various

other techniques, it has been observed that, in 2019, Marwan *et al.* [40] used three controllers for double convection model in which adaptive controller is activated at $t = 2$, sliding control technique works at $t = 5.5$ and backstepping technique acquire the same result for $t = 6$. Similarly, in the work of Moysis *et al.* [18] active controller is used for hyperjerk system and chaos is surpassed at $t = 3$. But, controller designed in our work is activated at $t = 1$ (can be seen in Figure 6), which shows our proposed controller give better results in comparison.

6 Conclusion

In this work, normal form theory was utilized to show oscillatory solutions in mechanical model coupled with fuel filled tank. It was observed that Hopf bifurcation was not only limited to biological models but had great impact in spacecraft models. Furthermore, appearance and disappearance of negligible limit cycle around stable spiral were observed with changes in parameter values. On the other hand, a new controller based on active control technique and estimated parameter law was designed to bring stability in the considered model. Moreover, these results were also compared with other latest techniques and observed that our proposed technique took less time in comparison.

Figure 8: Chaotic behaviour of updated law for controlling chaos in system (4): (a) 3D Phase portrait of $\mu_x^x - h_x^z - h_x^y$, (b) 3D phase portrait of $h_y^z - \mu_y^y - h_z^y$, (c) 3D Phase portrait of $\mu_y^y - h_z^y - \mu_z^z$, (d) $\mu_x^x h_x^z$ -axis, (e) $\mu_x^x h_x^z$ -axis, (f) $\mu_y^y h_z^y$ -axis, (g) $h_x^z h_x^y$ -axis, (h) $\mu_y^y h_z^y$ -axis, (i) $h_z^y \mu_z^z$ -axis, (j) $\mu_x^x h_x^y$ -axis, (k) $h_y^z h_z^y$ -axis, and (l) $\mu_y^y \mu_z^z$ -axis.

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